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Original article

Topological aspects of extended Sierpiński structures with help of underlying networks

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ARTICLE INFO

Article history:

Received 6 January 2022

Revised 15 February 2022

Accepted 23 May 2022

Available online 28 May 2022

Keywords:

Zagreb indices

Forgotten index

Extended Sierpiński networks

Extremal networks

ABSTRACT

Sierpiński networks are the most studied networks of fractal nature with applications in various fields of science. A generalized Sierpiński network is obtained by copying the base network, resulting in the self-similar network. The extended Sierpiński networks are obtained by introducing a new vertex in a generalized Sierpiński network and attaching this vertex with the extreme vertices. Certain network invariants are used to find thermodynamic properties, physio-chemical properties, and biological activities of chemical compounds. These network invariants play a dynamic role in QSAR/QSPR study. In this paper, we discussed Zagreb indices and forgotten topological index for extended Sierpiński networks by using any base network \mathcal{H} . Moreover, for the studied topological indices, we attained some bounds using different parameters i.e. order, size, maximum and minimum degrees of vertices in network \mathcal{H} .

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1. Introduction

Sierpiński networks are the most studied networks similar to fractals. A fractal is a geometric structure that is self-similar throughout its structure. Fractal models are tremendously common since, nature is full of fractals, for example, plants, canals, coastlines, mountains, clouds, seashells, and tornadoes. Fractals help to study and comprehend key scientific ideas, such as how bacteria grow, freezing water patterns, and brain waves. Sierpiński and Sierpiński type networks are considered in fractal theory (TePLYAEV, 1998). Klavžar and Milutinović showed that the Sierpiński networks are similar structure to the Tower of Hanoi (Klavžar and Milutinović, 1997). The Sierpiński networks have many attractive properties for instance coding and metric properties and play an important role in numerous areas of science i.e. dynamic systems, probability, psychology, biology, chemical graph theory, computer networking and physical sciences. For more detail see (Alquran et al., 2020; Naseem et al., 2021; Klavžar et al., 2002; Romik, 2006; Vecchia and Sanges, 1988).

The networks studied in this article assumed to be finite and simple. A network/graph $\mathcal{H} = (V, E)$ is a collection of set of vertices $V(\mathcal{H})$ and set of edges $E(\mathcal{H})$. The order of graph \mathcal{H} is the cardinality of its vertices, while cardinality of edges is called size and frequently denoted by p and q respectively. The degree of vertex v is known as the number of edges connected to that particular vertex and denoted by d_v . A graph \mathcal{H} is known as complete if every two vertices are incident to each other. $\delta(\mathcal{H})$ and $\Delta(\mathcal{H})$ represent the minimum and maximum degree of a vertex in graph \mathcal{H} . If $\delta(\mathcal{H}) = \Delta(\mathcal{H}) = l$, then \mathcal{H} is a l -regular graph. The path, star, cycle and complete graph of order p are represented by P_p, S_p, C_p and K_p .

In mathematical chemistry, chemical graph theory, and pharmaceutical industry, topological invariants are very important. The physio-chemical properties of chemical structures can be forecasted by using topological invariants. From the last few decades, several topological indices were established and examined in literature (Todeschini and Consonni, 2000), which are applied to attain the facts of numerous characteristics of organic materials which depend on their molecular structures. Wiener a chemist in 1947 introduced the first topological index in order to determine the boiling points of paraffins (Wiener, 1947).

Gutman et al. in Gutman and Trinajstić (1972) and Gutman et al. (1975) introduced the Zagreb indices, which are stated as

$$M_1(\mathcal{H}) = \sum_{r \in V(\mathcal{H})} d_{\mathcal{H}}(r)^2 = \sum_{rs \in E(\mathcal{H})} (d_{\mathcal{H}}(r) + d_{\mathcal{H}}(s))$$

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Peer review under responsibility of King Saud University.



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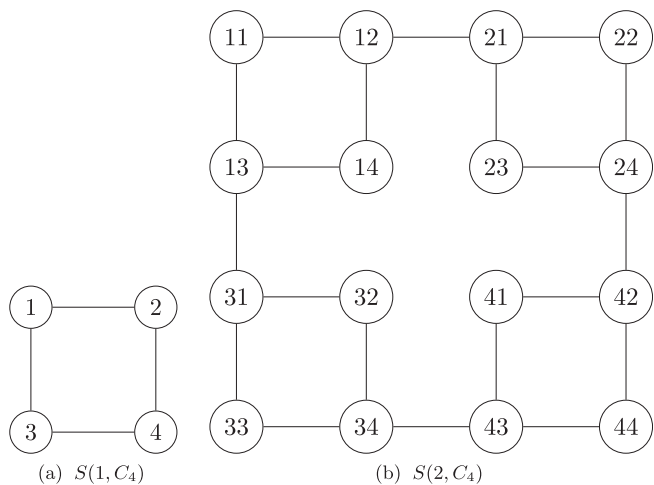


Fig. 1. Sierpinski graphs $S(1, C_4)$ and $S(2, C_4)$.

$$M_2(\mathcal{H}) = \sum_{rs \in E(\mathcal{H})} (d_{\mathcal{H}}(r)d_{\mathcal{H}}(s))$$

Furtula and Gutman (2015) proposed forgotten topological index, stated as

$$F(\mathcal{H}) = \sum_{r \in V(\mathcal{H})} d_{\mathcal{H}}(r)^3 = \sum_{rs \in E(\mathcal{H})} (d_{\mathcal{H}}(r)^2 + d_{\mathcal{H}}(s)^2)$$

For more detail on topological indices see Liu et al. (2019), Havare (2021), Akhter and Imran (2017), An and Das (2018), Che and Chen (2016), Cristea and Steinsky (2013), Gutman (2013), Horoldagva and Das (2015), Hua and Das (2013), Horoldagva et al. (2016), and Yoon and Kim (2006).

The generalized Sierpiński graph of dimension t is represented by $S(\mathcal{H}, t)$ is a graph with vertex set V^t , where $V = V(\mathcal{H})$. The vertex set V^t is the set of all words $v_1 v_2 \dots v_t$ of length t , where $v_p \in V, 1 \leq p \leq t$, two vertices u, w linked by an edge in $S(\mathcal{H}, t)$ if and only if there is $i \in \{1, 2, \dots, t\}$ such that

- $u_j = w_j$ if $j < i$.
- $u_i \neq w_i$ and $u_i, w_i \in E(\mathcal{H})$.
- $u_j = w_i$ and $u_i = w_j$ if $j > i$.

From above definition, it is clear that, if $uw \in E(S(\mathcal{H}, t))$ then $rs \in E(\mathcal{H})$ and a word z such that $u = zrs \dots s$ and $w = zsr \dots r$. A vertex of the form $uu \dots u$ is known as extreme vertex and denoted by \bar{w} . For a graph \mathcal{H} of order $p, S(\mathcal{H}, t)$ has p extreme vertices. Moreover, extreme vertices have same degree in $S(\mathcal{H}, t)$ as in base graph $\mathcal{H}, d_{S(\mathcal{H}, t)}(u) + 1 = d_{S(\mathcal{H}, t)}(wuu \dots u)$ and $d_{\mathcal{H}}(w) + 1 = d_{S(\mathcal{H}, t)}(uww \dots w)$. Fig. 1 and Fig. 2 represents the generalized and extended Sierpiński graphs respectively, where extended Sierpiński graph is obtained by involving a new vertex x in generalized Sierpiński graph and joining it with extreme vertices. Extended Sierpiński graph is represented by $ES(\mathcal{H}, t)$.

For $v \in V, d_{S(\mathcal{H}, t)}(v) \in \{d_{\mathcal{H}}(v), d_{\mathcal{H}}(v) + 1\}$, here $d_{\mathcal{H}}(v)$ represents the degree of v in \mathcal{H} . For our convenience $d_{\mathcal{H}}(v)$ is represented by d_v in this article. Let $|d_r, d_s|_{ES(\mathcal{H}, t)}$ is the number of copies of $\{r, s\}$ edge with degrees d_r and d_s in $ES(\mathcal{H}, t)$. For $r, s \in V(\mathcal{H}), \triangleleft(r, s)$ represents the triangles of \mathcal{H} having r and s as its vertices, while $\triangleleft(\mathcal{H})$ represents the number of triangles in \mathcal{H} . For $rs \in E(\mathcal{H})$, we have $|N_r \cap N_s| = \triangleleft(r, s), |N_r \cup N_s| = d_r + d_s - \triangleleft(r, s)$ and $|N_r - N_s| = d_r - \triangleleft(r, s)$. We used the function $\phi_p(t) = 1 + p + p^2 + \dots + p^{t-1} = \frac{p^t - 1}{p - 1}$ for a graph of order p . Imran and Jamil (2020) calculate the constraints of generalized Sierpiński

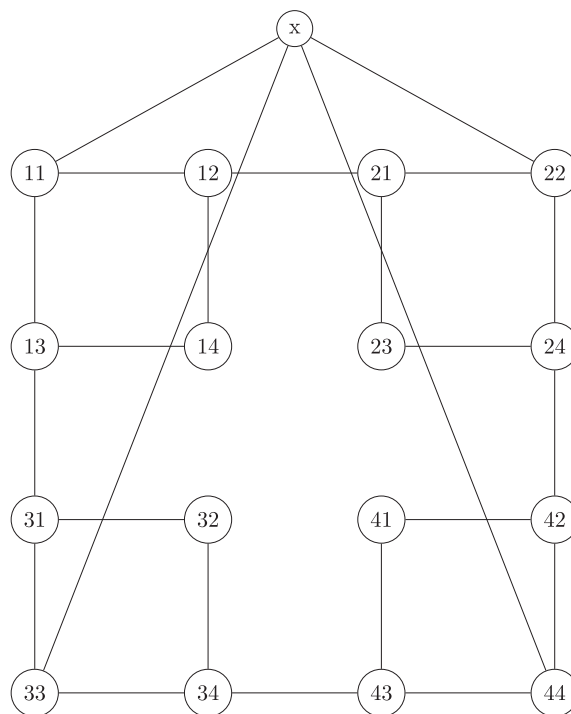


Fig. 2. Extended Sierpiński graph $ES(2, C_4)$.

graphs. We will establish the results for topological properties of extended Sierpiński graph with any base graph \mathcal{H} . For these topological indices we will obtain some sharp bounds in terms of numerous parameters. In this article, we will select the first Zagreb, second Zagreb and forgotten indices to investigate the invariants of $ES(\mathcal{H}, t)$ graphs. Following lemmas are helpful in finding the main results of the paper.

Lemma 1.1. Zhou (2004) Let \mathcal{H} be a graph without triangle having order p , size $q > 0$. Then $M_1(\mathcal{H}) \leq pq$ and equality holds if and only if \mathcal{H} is a complete bipartite graph.

Lemma 1.2. Zhou (2004) Let \mathcal{H} be a graph without triangle having size $q > 0$. Then $M_2(\mathcal{H}) \leq q^2$ and equality holds if and only if \mathcal{H} is a union of a complete bipartite graph and isolated vertices.

Lemma 1.3. Das (2003) Let p and $q > 0$ be vertices and edges respectively of a graph \mathcal{H} . Then $\frac{4q^2}{p} \leq M_1(\mathcal{H}) \leq (\frac{2q}{p-1} + p - 2)$ and left equality holds if and only if \mathcal{H} is a regular graph and right equality holds if and only if \mathcal{H} is $K_p, K_{1,p-1}$ or $K_{1 \cup p-1}$.

Lemma 1.4. Zhou (2004) Let $q > 0$ be a size of a graph \mathcal{H} . Then $M_2(\mathcal{H}) \leq q(\sqrt{2q} + \frac{1}{4} - \frac{1}{2})^2$ and equality holds if and only if \mathcal{H} is a union of a complete and isolated vertices.

Lemma 1.5. Estrada-Moreno and Rodríguez-Velázquez (2019) Let p be the order of a graph \mathcal{H} , for any edge rs and integer $t \geq 2$, we have

1. $|d_r, d_s|_{S(\mathcal{H}, t)} = p^{t-2}(p - d_r - d_s + \triangleleft(r, s))$
2. $|d_r + 1, d_s|_{S(\mathcal{H}, t)} = p^{t-2}(d_r - \triangleleft(r, s)) - \phi_p(t - 2)d_s$
3. $|d_r, d_s + 1|_{S(\mathcal{H}, t)} = p^{t-2}(d_s - \triangleleft(r, s)) - \phi_p(t - 2)d_r$
4. $|d_r + 1, d_s + 1|_{S(\mathcal{H}, t)} = p^{t-2}(\triangleleft(r, s) + 1) + \phi_p(t - 2)(d_r + d_s + 1)$.

2. Main results

In this part of paper, we obtained the Zagreb and forgotten topological indices for extended Sierpiński graph by considering any arbitrary graph \mathcal{H} . Furthermore, we also compute some bounds for $ES(\mathcal{H}, t)$. Here d_w represents the degree of extreme vertices and d_x is the degree of new vertex which is introduced in generalized Sierpiński graph in order to obtain $ES(\mathcal{H}, t)$ throughout this article. By using Lemma 1.5 we can deduce the following result for extended Sierpiński graph.

Lemma 2.1. Let p be the order of a graph \mathcal{H} , for any edge rs and integer $t \geq 2$, we have

1. $|d_r, d_s|_{ES(\mathcal{H}, t)} = p^{t-2}(p - d_r - d_s + \triangleleft(r, s))$
2. $|d_r + 1, d_s|_{ES(\mathcal{H}, t)} = p^{t-2}(d_r - \triangleleft(r, s)) - \phi_p(t - 2)d_s - 1$
3. $|d_r, d_s + 1|_{ES(\mathcal{H}, t)} = p^{t-2}(d_s - \triangleleft(r, s)) - \phi_p(t - 2)d_r - 1$
4. $|d_r + 1, d_s + 1|_{ES(\mathcal{H}, t)} = p^{t-2}(\triangleleft(r, s) + 1) + \phi_p(t - 2)(d_r + d_s + 1) + 2$
5. $|d_w, d_x|_{ES(\mathcal{H}, t)} = p$.

Theorem 2.2. Let p and q be vertices and edges of a graph \mathcal{H} . Then first Zagreb index of extended Sierpiński graph $ES(\mathcal{H}, t)$ of the graph \mathcal{H} of dimension $t \geq 2$ is $M_1(ES(\mathcal{H}, t)) = (\phi_p(t) + \phi_p(t - 1))M_1(\mathcal{H}) + 2q(1 + \phi_p(t - 1)) + p(d_w + d_x)$.

Proof. Let p and q be vertices and edges of a graph \mathcal{H} . Then the first Zagreb index of $ES(\mathcal{H}, t)$ can be defined as

$$M_1(ES(\mathcal{H}, t)) = \sum_{rs \in E(\mathcal{H})} \sum_{i,j=0}^1 |d_r + i, d_s + j|(d_r + i + d_s + j) + |d_w, d_x|_{ES(\mathcal{H}, t)}(d_w + d_x)$$

Now, by using Lemma 2.1 we have

$$\begin{aligned} M_1(ES(\mathcal{H}, t)) &= \sum_{rs \in E(\mathcal{H})} [p^{t-2}(p - d_r - d_s + \triangleleft(r, s))(d_r + d_s) \\ &\quad + (p^{t-2}(d_r - \triangleleft(r, s)) - \phi_p(t - 2)d_s - 1)(d_r + d_s + 1) \\ &\quad + (p^{t-2}(d_s - \triangleleft(r, s)) - \phi_p(t - 2)d_r - 1)(d_r + d_s + 1) \\ &\quad + (p^{t-2}(\triangleleft(r, s) + 1) + \phi_p(t - 2)(d_r + d_s + 1) + 2)(d_r + d_s + 2)] \\ &\quad + p(d_w + d_x) \\ &= \sum_{rs \in E(\mathcal{H})} (p^{t-1} + 2p^{t-2} + 2\phi_p(t - 2))(d_r + d_s) \\ &\quad + \sum_{rs \in E(\mathcal{H})} 2(1 + p^{t-2} + \phi_p(t - 2)) + p(d_w + d_x) \\ &= (\phi_p(t) + \phi_p(t - 1))M_1(\mathcal{H}) + 2q(1 + \phi_p(t - 1)) + p(d_w + d_x). \end{aligned}$$

□

From Lemma 1.3 we obtained the next result.

Corollary 2.3. Let p and $q > 0$ be vertices and edges respectively of a graph \mathcal{H} . Then $(\phi_p(t) + \phi_p(t - 1))\frac{4q^2}{p} + 2q(1 + \phi_p(t - 1)) + p(d_w + d_x) \leq M_1(ES(\mathcal{H}, t)) \leq \frac{\phi_p(t)}{p-1}(2q + p^2 - 3p + 2) + \frac{\phi_p(t-1)}{p-1}(2pq + p^2 - 3p + 2) + 2q(1 + \phi_p(t - 1)) + p(d_w + d_x)$.

The lower bound is obtained if \mathcal{H} is isomorphic to a regular graph and upper bound is obtained if \mathcal{H} is isomorphic to $K_p, K_{1,p-1}$ or $K_{1 \cup p-1}$. Lemma 1.1 gives the result for the upper bound of $ES(\mathcal{H}, t)$.

Corollary 2.4. Let \mathcal{H} be a graph without triangle having order p , size $q > 0$ and $t \geq 1$. Then $M_1(ES(\mathcal{H}, t)) \leq (\phi_p(t) + \phi_p(t - 1))pq + 2q(1 + \phi_p(t - 1)) + p(d_w + d_x)$.

Corollary 2.5. Let P_p, S_p, C_p and K_p be path, star, cycle and complete graphs of order p . Then the first Zagreb index for extended Sierpiński graph with dimension $t \geq 1$ of these graphs is given as

1. $M_1(ES(P_p, t)) = \phi_p(t)(4p - 6) + \phi_p(t - 1)(6p - 8) + p^2 + 5p - 4$
2. $M_1(ES(S_p, t)) = \phi_p(t)(p^2 - p) + \phi_p(t - 1)(p^2 + p - 2) + p^2 + 5p - 4$
3. $M_1(ES(C_p, t)) = \phi_p(t)(4p) + \phi_p(t - 1)(6p) + p^2 + 5p$
4. $M_1(ES(K_p, t)) = \phi_p(t)p(p - 1)^2 + \phi_p(t - 1)p^2(p - 1) + 3p^2 - p$.

Proof. From Theorem 2.2, we have

$$M_1(ES(\mathcal{H}, t)) = (\phi_p(t) + \phi_p(t - 1))M_1(\mathcal{H}) + 2q(1 + \phi_p(t - 1)) + p(d_w + d_x).$$

Now, by replacing the value of $M_1(\mathcal{H}), q$ and $p(d_w + d_x)$ by taking path, star, cycle and complete graph as a base graph in above equation, then we will obtain

$$\begin{aligned} M_1(ES(P_p, t)) &= (\phi_p(t) + \phi_p(t - 1))(2(3) + 4(p - 3)) \\ &\quad + 2(p - 1)(1 + \phi_p(t - 1)) + 2(p + 2) + (p - 2)(3 + p) \\ &= \phi_p(t)(4p - 6) + \phi_p(t - 1)(6p - 8) + p^2 + 5p - 4 \end{aligned}$$

$$\begin{aligned} M_1(ES(S_p, t)) &= (\phi_p(t) + \phi_p(t - 1))(p - 1)(p) \\ &\quad + 2(p - 1)(1 + \phi_p(t - 1)) + (p + p) + (p - 1)(2 + p) \\ &= \phi_p(t)(p^2 - p) + \phi_p(t - 1)(p^2 + p - 2) + p^2 + 5p - 4 \end{aligned}$$

$$\begin{aligned} M_1(ES(C_p, t)) &= (\phi_p(t) + \phi_p(t - 1))(4p) + 2(p)(1 + \phi_p(t - 1)) + p(p + 3) \\ &= \phi_p(t)(4p) + \phi_p(t - 1)(6p) + p^2 + 5p. \end{aligned}$$

$$\begin{aligned} M_1(ES(K_p, t)) &= (\phi_p(t) + \phi_p(t - 1))p(p - 1)^2 \\ &\quad + p(p - 1)(1 + \phi_p(t - 1)) + p(2p) \\ &= \phi_p(t)p(p - 1)^2 + \phi_p(t - 1)p^2(p - 1) + 3p^2 - p \end{aligned}$$

□

Theorem 2.6. Let \mathcal{H} is a base graph with minimum and maximum degree δ and Δ respectively. Then for extended Sierpiński graphs, we have $2q\delta(\phi_p(t) + \phi_p(t - 1)) + 2q(1 + \phi_p(t - 1)) + p(d_w + d_x) \leq M_1(ES(\mathcal{H}, t)) \leq 2q\Delta(\phi_p(t) + \phi_p(t - 1)) + 2q(1 + \phi_p(t - 1)) + p(d_w + d_x)$ left equality holds if $\mathcal{H} \cong \delta$ -regular graph and right equality holds if $\mathcal{H} \cong \Delta$ -regular graph.

Proof. Let \mathcal{H} be a base graph having order p and size q . The first Zagreb index of $ES(\mathcal{H}, t)$ can be stated as

$$M_1(ES(\mathcal{H}, t)) = \sum_{rs \in E(\mathcal{H})} \sum_{i,j=0}^1 |d_r + i, d_s + j|(d_r + i + d_s + j) + |d_w, d_x|_{ES(\mathcal{H}, t)}(d_w + d_x)$$

Now, by using Lemma 2.1 we have,

$$\begin{aligned} M_1(ES(\mathcal{H}, t)) &= \sum_{rs \in E(\mathcal{H})} [p^{t-2}(p - d_r - d_s + \triangleleft(r, s))(d_r + d_s) \\ &\quad + (p^{t-2}(d_r - \triangleleft(r, s)) - \phi_p(t - 2)d_s - 1)(d_r + d_s + 1) \\ &\quad + (p^{t-2}(d_s - \triangleleft(r, s)) - \phi_p(t - 2)d_r - 1)(d_r + d_s + 1) \\ &\quad + (p^{t-2}(\triangleleft(r, s) + 1) + \phi_p(t - 2)(d_r + d_s + 1) + 2)(d_r + d_s + 2)] \\ &\quad + p(d_w + d_x) \end{aligned}$$

Since, $\delta(\mathcal{H}) = \delta$ is the minimum degree in graph \mathcal{H} , then we obtained

$$\begin{aligned}
 M_1(ES(\mathcal{H}, t)) &\geq \sum_{rs \in E(\mathcal{H})} [p^{t-2}(p - 2\delta + \triangleleft(r, s))2\delta \\
 &+ (2p^{t-2}(\delta - \triangleleft(r, s)) - 2\phi_p(t-2)\delta - 1)(2\delta + 1) \\
 &+ (p^{t-2}(\triangleleft(r, s) + 1) + \phi_p(t-2)(2\delta + 1) + 2)(2\delta + 2)] \\
 &+ p(d_w + d_x) \\
 &= 2q\delta(\phi_p(t) + \phi_p(t-1)) + 2q(1 + \phi_p(t-1)) + p(d_w + d_x)
 \end{aligned}$$

and equality holds if $\mathcal{H} \cong \delta$ -regular graph

Since, $\Delta(\mathcal{H}) = \Delta$ is the maximum degree in \mathcal{H} , then inequality becomes

$$\begin{aligned}
 M_1(ES(\mathcal{H}, t)) &\leq \sum_{rs \in E(\mathcal{H})} [p^{t-2}(p - 2\Delta + \triangleleft(r, s))2\Delta \\
 &+ (2p^{t-2}(\Delta - \triangleleft(r, s)) - 2\phi_p(t-2)\Delta - 1)(2\Delta + 1) \\
 &+ (p^{t-2}(\triangleleft(r, s) + 1) + \phi_p(t-2)(2\Delta + 1) + 2)(2\Delta + 2)] \\
 &+ p(d_w + d_x) \\
 &\leq 2q\Delta(\phi_p(t) + \phi_p(t-1)) + 2q(1 + \phi_p(t-1)) + p(d_w + d_x)
 \end{aligned}$$

and equality holds if $\mathcal{H} \cong \Delta$ -regular graph. \square

Corollary 2.7. Let $p \geq 3$ be the vertices of a regular graph \mathcal{H} . Then for extended Sierpiński graph, we have $\phi_p(t)(4p) + \phi_p(t-1)(6p) + p^2 + 5p \leq M_1(ES(\mathcal{H}, t)) \leq \phi_p(t)p(p-1)^2 + \phi_p(t-1)p^2(p-1) + 3p^2 - p$ the left equality holds if $\mathcal{H} \cong C_p$ and right equality holds if $\mathcal{H} \cong K_p$.

Now, in next theorem we compute the formula of second zagreb index for extended Sierpiński graph.

Theorem 2.8. Let $ES(\mathcal{H}, t)$ be extended Sierpiński graph with dimension $t \geq 2$, where \mathcal{H} having p vertices and q edges. Then second Zagreb index of $ES(\mathcal{H}, t)$ is $M_2(ES(\mathcal{H}, t)) = (\phi_p(t) + 2\phi_p(t-1))M_2(\mathcal{H}) + (\phi_p(t-1) + \phi_p(t-2) + 1)M_1(\mathcal{H}) + q(2 + \phi_p(t-1)) + p(d_w \times d_x) + p^{t-2} \sum_{rs \in E(\mathcal{H})} \triangleleft(r, s)$.

Proof. Let \mathcal{H} be a graph having p vertices and q edges. The second Zagreb index of $ES(\mathcal{H}, t)$ can be stated as

$$\begin{aligned}
 M_2(ES(\mathcal{H}, t)) &= \sum_{rs \in E(\mathcal{H})} \sum_{i,j=0}^1 |d_r + i, d_s + j|(d_r + i)(d_s + j) \\
 &+ |d_w, d_x|_{ES(\mathcal{H}, t)}(d_w \times d_x)
 \end{aligned}$$

Now, by using Lemma 2.1 we have

$$\begin{aligned}
 M_2(ES(\mathcal{H}, t)) &= \sum_{rs \in E(\mathcal{H})} [p^{t-2}(p - d_r - d_s + \triangleleft(r, s))(d_r \times d_s) \\
 &+ (p^{t-2}(d_r - \triangleleft(r, s)) - \phi_p(t-2)d_s - 1)(d_r + 1)(d_s) \\
 &+ (p^{t-2}(d_s - \triangleleft(r, s)) - \phi_p(t-2)d_r - 1)(d_r)(d_s + 1) \\
 &+ (p^{t-2}(\triangleleft(r, s) + 1) + \phi_p(t-2)(d_r + d_s + 1) + 2)(d_r + 1)(d_s + 1)] \\
 &+ p(d_w \times d_x) \\
 &= \sum_{rs \in E(\mathcal{H})} [(p^{t-1} + p^{t-2} + \phi_p(t-2) + 2\phi_p(t-2) + 2p^{t-2})(d_r \times d_s)] \\
 &+ \sum_{rs \in E(\mathcal{H})} [(2 + p^{t-2} + \phi_p(t-2)) + (p^{t-2} + 2\phi_p(t-2) + 1)(d_r + d_s)] \\
 &+ p^{t-2} \sum_{rs \in E(\mathcal{H})} \triangleleft(r, s) + p(d_w \times d_x) \\
 &= (\phi_p(t) + 2\phi_p(t-1))M_2(\mathcal{H}) + (\phi_p(t-1) + \phi_p(t-2) + 1)M_1(\mathcal{H}) \\
 &+ q(2 + \phi_p(t-1)) + p(d_w \times d_x) + p^{t-2} \sum_{rs \in E(\mathcal{H})} \triangleleft(r, s).
 \end{aligned}$$

\square

Corollary 2.9. Let P_p, S_p, C_p and K_p be path, star, cycle and complete graphs of order p . Then second Zagreb index for extended Sierpiński graph with dimension $t \geq 1$ of these graphs is given as

$$\begin{aligned}
 M_2(ES(P_p, t)) &= \phi_p(t)(4p - 8) + \phi_p(t-1)(13p - 23) + \phi_p(t-2) \\
 &(4p - 6) + 3p^2 + 4p - 8
 \end{aligned}$$

$$\begin{aligned}
 M_2(ES(S_p, t)) &= \phi_p(t)(p - 1)^2 + \phi_p(t-1)(3p^2 - 4p + 1) + \\
 &\phi_p(t-2)(p^2 - p) + 4p^2 - p - 2; p \geq 4 \\
 M_2(ES(C_p, t)) &= (\phi_p(t) + \phi_p(t-2))(4p) + \phi_p(t-1)(13p) + 3p^2 + 6p \\
 M_2(ES(K_p, t)) &= \phi_p(t) \frac{p(p-1)^3}{2} + \frac{1}{2} \phi_p(t-1)(2p^4 - 4p^3 + 3p^2 - p) + \\
 &\phi_p(t-2)p(p-1)^2 + 2p^3 - p^2 + p^{t-2} \left(\frac{p^3 - 3p^2 + 2p}{2} \right).
 \end{aligned}$$

Proof. From Theorem 2.8, we have

$$\begin{aligned}
 M_2(ES(\mathcal{H}, t)) &= (\phi_p(t) + 2\phi_p(t-1))M_2(\mathcal{H}) + (\phi_p(t-1) + \phi_p(t-2) + 1)M_1(\mathcal{H}) \\
 &+ q(2 + \phi_p(t-1)) + p(d_w \times d_x) + p^{t-2} \sum_{rs \in E(\mathcal{H})} \triangleleft(r, s)
 \end{aligned}$$

Now, by replacing the value of $M_1(\mathcal{H}), M_2(\mathcal{H}), q$ and $p(d_w \times d_x)$ and $\triangleleft(r, s)$ by taking path, star, cycle and complete graph as a base graph in above equation, then we will obtain

$$\begin{aligned}
 M_2(ES(P_p, t)) &= (\phi_p(t) + 2\phi_p(t-1))(4 + (p-3)4) + (\phi_p(t-1) + \phi_p(t-2) + 1)(6 + 4(p-3)) \\
 &+ (p-1)(2 + \phi_p(t-1)) + 4p + (p-2)3p \\
 &= \phi_p(t)(4p - 8) + \phi_p(t-1)(13p - 23) + \phi_p(t-2)(4p - 6) + 3p^2 + 4p - 8
 \end{aligned}$$

$$\begin{aligned}
 M_2(ES(S_p, t)) &= (\phi_p(t) + 2\phi_p(t-1))(p-1)^2 + (\phi_p(t-1) + \phi_p(t-2) + 1)(p(p-1)) \\
 &+ (p-1)(2 + \phi_p(t-1)) + p^2 + (p-1)2p \\
 &= \phi_p(t)(p-1)^2 + \phi_p(t-1)(3p^2 - 4p + 1) + \phi_p(t-2)(p^2 - p) + 4p^2 - p - 2
 \end{aligned}$$

$$\begin{aligned}
 M_2(ES(C_p, t)) &= (\phi_p(t) + 2\phi_p(t-1))(4p) + (\phi_p(t-1) + \phi_p(t-2) + 1)(4p) \\
 &+ (p)(2 + \phi_p(t-1)) + p(p-1) \\
 &= (\phi_p(t) + \phi_p(t-2))(4p) + \phi_p(t-1)(13p) + 3p^2 + 6p
 \end{aligned}$$

$$\begin{aligned}
 M_2(ES(K_p, t)) &= (\phi_p(t) + 2\phi_p(t-1)) \frac{p(p-1)^3}{2} + (\phi_p(t-1) + \phi_p(t-2) + 1)(p(p-1)^2) \\
 &+ (2 + \phi_p(t-1)) \frac{p(p-1)}{2} + p^3 + 3p^{t-2} \triangleleft(\mathcal{H}) \\
 &= (\phi_p(t) + 2\phi_p(t-1)) \frac{p(p-1)^3}{2} + (\phi_p(t-1) + \phi_p(t-2) + 1)(p(p-1)^2) \\
 &+ (2 + \phi_p(t-1)) \frac{p(p-1)}{2} + p^3 + p^{t-2} \left(\frac{p^3 - 3p^2 + 2p}{2} \right) \\
 &= \phi_p(t) \frac{p(p-1)^3}{2} + \frac{1}{2} \phi_p(t-1)(2p^4 - 4p^3 + 3p^2 - p) + \phi_p(t-2)p(p-1)^2 \\
 &+ 2p^3 - p^2 + p^{t-2} \left(\frac{p^3 - 3p^2 + 2p}{2} \right)
 \end{aligned}$$

\square

Theorem 2.10. Let \mathcal{H} is a base graph with minimum and maximum degree δ and Δ respectively. Then for extended Sierpiński graph, we have $\mu(p, \delta) + p^{t-2} \sum_{rs \in E(\mathcal{H})} \triangleleft(r, s) + p(d_w \times d_x) \leq M_2(ES(\mathcal{H}, t)) \leq \mu(p, \Delta) + p^{t-2} \sum_{rs \in E(\mathcal{H})} \triangleleft(r, s) + p(d_w \times d_x)$ where $\mu(p, \delta) = q\delta^2(\phi_p(t) + 2\phi_p(t-1)) + 2q\delta(\phi_p(t-1) + \phi_p(t-2) + 1) + q(\phi_p(t-1) + 2)$ left equality holds if $\mathcal{H} \cong \delta$ -regular graph and right and only if $\mathcal{H} \cong \Delta$ -regular graph.

Proof. Let p and q are the order and size respectively of a graph \mathcal{H} . Then second Zagreb index of $ES(\mathcal{H}, t)$ can be stated as

$$\begin{aligned}
 M_2(ES(\mathcal{H}, t)) &= \sum_{rs \in E(\mathcal{H})} \sum_{i,j=0}^1 |d_r + i, d_s + j|(d_r + i)(d_s + j) \\
 &+ |d_w, d_x|_{ES(\mathcal{H}, t)}(d_w \times d_x)
 \end{aligned}$$

Now, by using Lemma 2.1 we have

$$\begin{aligned}
 M_2(ES(\mathcal{H}, t)) &= \sum_{rs \in E(\mathcal{H})} [p^{t-2}(p - d_r - d_s + \triangleleft(r, s))(d_r \times d_s) \\
 &+ (p^{t-2}(d_r - \triangleleft(r, s)) - \phi_p(t-2)d_s - 1)(d_r + 1)(d_s) \\
 &+ (p^{t-2}(d_s - \triangleleft(r, s)) - \phi_p(t-2)d_r - 1)(d_r)(d_s + 1) \\
 &+ (p^{t-2}(\triangleleft(r, s) + 1) + \phi_p(t-2)(d_r + d_s + 1) + 2)(d_r + 1)(d_s + 1)] \\
 &+ p(d_w \times d_x)
 \end{aligned}$$

Since, $\delta(\mathcal{H}) = \delta$ is the minimum degree of \mathcal{H} . Then we obtained

$$M_2(ES(\mathcal{H}, t)) \geq \sum_{rs \in E(\mathcal{H})} [p^{t-2}(p - 2\delta + \triangleleft(r, s))\delta^2 + 2(p^{t-2}(\delta - \triangleleft(r, s)) - 2\phi_p(t-2)\delta - 1)(\delta^2 + \delta) + (p^{t-2}(\triangleleft(r, s) + 1) + \phi_p(t-2)(2\delta + 1) + 2)(\delta^2 + 2\delta + 1)] + p(d_w \times d_x) = q\delta^2(\phi_p(t) + 2\phi_p(t-1)) + 2q\delta(\phi_p(t-1) + \phi_p(t-2) + 1) + q(\phi_p(t-1) + 2) + p(d_w \times d_x) + p^{t-2} \sum_{rs \in E(\mathcal{H})} \triangleleft(r, s)$$

and equality holds if $\mathcal{H} \cong \delta$ -regular graph.

As $\Delta(\mathcal{H}) = \Delta$ is the maximum degree of \mathcal{H} . Then we obtained

$$M_2(ES(\mathcal{H}, t)) \leq \sum_{rs \in E(\mathcal{H})} [p^{t-2}(p - 2\Delta + \triangleleft(r, s))\Delta^2 + (2p^{t-2}(\Delta - \triangleleft(r, s)) - 2\phi_p(t-2)\Delta - 1)(\Delta^2 + \Delta) + (p^{t-2}(\triangleleft(r, s) + 1) + \phi_p(t-2)(2\Delta + 1) + 2)(\Delta^2 + 2\Delta + 1)] + p(d_w \times d_x) = q\Delta^2(\phi_p(t) + 2\phi_p(t-1)) + 2q\Delta(\phi_p(t-1) + \phi_p(t-2) + 1) + q(\phi_p(t-1) + 2) + p(d_w \times d_x) + p^{t-2} \sum_{rs \in E(\mathcal{H})} \triangleleft(r, s)$$

and equality holds if $\mathcal{H} \cong \Delta$ -regular graph. \square

If \mathcal{H} is without triangle, then above result becomes as follow.

Corollary 2.11. Let \mathcal{H} be a graph without triangle. Then $\mu(p, \delta) + p(d_w \times d_x) \leq M_2(ES(\mathcal{H}, t)) \leq \mu(p, \Delta) + p(d_w \times d_x)$.

Corollary 2.12. Let $p \geq 4$ be the order of a connected regular graph \mathcal{H} . Then $(\phi_p(t) + \phi_p(t-2))(4p) + \phi_p(t-1)(13p) + 3p^2 + 6p \leq M_2(ES(G, t)) \leq \phi_p(t) \frac{p(p-1)^3}{2} + \frac{1}{2} \phi_p(t-1)(2p^4 - 4p^3 + 3p^2 - p) + \phi_p(t-2)p(p-1)^2 + 2p^3 - p^2 + p^{t-2} \left(\frac{p^3 - 3p^2 + 2p}{2} \right)$. The left equality holds if $\mathcal{H} \cong C_p$ and right equality holds if $\mathcal{H} \cong K_p$.

The following theorem gives the exact formula of forgotten index of $ES(\mathcal{H}, t)$.

Theorem 2.13. Let $ES(\mathcal{H}, t)$ be extended Sierpiński graph of dimension $t \geq 2$ of base graph \mathcal{H} with p vertices and q edges. Then the forgotten topological index of $ES(\mathcal{H}, t)$ is $F(ES(\mathcal{H}, t)) = (3\phi_p(t-1) + 2)M_1(\mathcal{H}) + (\phi_p(t) + 2\phi_p(t-1))F(\mathcal{H}) + 2q(\phi_p(t-1) + 1) + p(d_w^2 + d_x^2)$.

Proof. Let p and q be order and size of a graph \mathcal{H} . Then forgotten topological index of $ES(\mathcal{H}, t)$ can be defined as

$$F(ES(\mathcal{H}, t)) = \sum_{rs \in E(\mathcal{H})} \sum_{i,j=0}^1 |d_r + i, d_s + j| (d_r + i)^2 + (d_s + j)^2 + |d_w, d_x|_{ES(\mathcal{H}, t)} (d_w^2 + d_x^2)$$

Now, by using Lemma 2.1 we have

$$F(ES(\mathcal{H}, t)) = \sum_{rs \in E(\mathcal{H})} [p^{t-2}(p - d_r - d_s + \triangleleft(r, s))(d_r^2 + d_s^2) + (p^{t-2}(d_r - \triangleleft(r, s)) - \phi_p(t-2)d_s - 1)((d_r + 1)^2 + d_s^2) + (p^{t-2}(d_s - \triangleleft(r, s)) - \phi_p(t-2)d_r - 1)(d_r^2 + (d_s + 1)^2) + (p^{t-2}(\triangleleft(r, s) + 1) + \phi_p(t-2)(d_r + d_s + 1) + 2)((d_r + 1)^2 + (d_s + 1)^2)] + p(d_w^2 + d_x^2) = \sum_{rs \in E(\mathcal{H})} [(\phi_p(t) + 2\phi_p(t-1))(d_r^2 + d_s^2) + (3\phi_p(t-1) + 2)(d_r + d_s) + 2(\phi_p(t-1) + 1)] + p(d_w^2 + d_x^2) = (\phi_p(t) + 2\phi_p(t-1))F(\mathcal{H}) + (3\phi_p(t-1) + 2)M_1(\mathcal{H}) + 2q(\phi_p(t-1) + 1) + p(d_w^2 + d_x^2).$$

\square

Corollary 2.14. Let P_p, S_p, C_p and K_p be path, star, cycle and complete graphs of order p . Then forgotten topological index for extended Sierpiński graph with dimension $t \geq 1$ of these graphs is given as

$$F(ES(P_p, t)) = \phi_p(t)(8p - 14) + \phi_p(t-1)(13p - 48) + p^3 + 19p - 24$$

$$F(ES(S_p, t)) = \phi_p(t)(p^3 - 3p^2 + 4p - 2) + \phi_p(t-1)(2p^3 - 3p^2 + 7p - 6) + p^3 + 3p^2 + 4p - 6$$

$$F(ES(C_p, t)) = \phi_p(t)8p + \phi_p(t-1)(30p) + p^3 + 19p$$

$$F(ES(K_p, t)) = \phi_p(t)p(p-1)^3 + \phi_p(t-1)(2p^4 - 3p^3 + p^2) + 4p^3 - 3p^2 + p.$$

Proof. From Theorem 2.13, we have

$$F(ES(\mathcal{H}, t)) = (3\phi_p(t-1) + 2)M_1(\mathcal{H}) + (\phi_p(t) + 2\phi_p(t-1))F(\mathcal{H}) + 2q(\phi_p(t-1) + 1) + p(d_w^2 + d_x^2).$$

Now, by replacing the value of $M_1(\mathcal{H}), q, F(\mathcal{H})$ and $p(d_w^2 + d_x^2)$ by taking path, Star, Cycle and complete graph as a base graph in above equation, then we will obtain

$$F(ES(P_p, t)) = (3\phi_p(t-1) + 2)(4p - 6) + (\phi_p(t) + 2\phi_p(t-1))(8p - 14) + 2(p-1)(\phi_p(t-1) + 1) + 2(2+p)^2 + (p-2)^2(p+3)^2 = \phi_p(t)(8p - 14) + \phi_p(t-1)(13p - 48) + p^3 + 19p - 24$$

$$F(ES(S_p, t)) = (3\phi_p(t-1) + 2)p(p-1) + (\phi_p(t) + 2\phi_p(t-1))(p^3 - 3p^2 + 4p - 2) + 2(p-1)(\phi_p(t-1) + 1) + 2p^2 + (p-1)(2^2 + p^2) = \phi_p(t)(p^3 - 3p^2 + 4p - 2) + \phi_p(t-1)(2p^3 - 3p^2 + 7p - 6) + p^3 + 3p^2 + 4p - 6$$

$$F(ES(C_p, t)) = (3\phi_p(t-1) + 2)(4p) + (\phi_p(t) + 2\phi_p(t-1))(8p) + 2p(\phi_p(t-1) + 1) + p(9 + p^2) = \phi_p(t)8p + \phi_p(t-1)(30p) + p^3 + 19p$$

$$F(ES(K_p, t)) = (3\phi_p(t-1) + 2)p(p-1)^2 + (\phi_p(t) + 2\phi_p(t-1))p(p-1)^3 + p(p-1)(\phi_p(t-1) + 1) + p(2p^2) = \phi_p(t)p(p-1)^3 + \phi_p(t-1)(2p^4 - 3p^3 + p^2) + 4p^3 - 3p^2 + p$$

\square

Theorem 2.15. If \mathcal{H} is a base graph, where δ and Δ are minimum and maximum degrees respectively. Then for extended Sierpiński graph, we have $2q\delta^2(\phi_p(t) + 2\phi_p(t-1)) + 2q\delta(3\phi_p(t-1) + 2) + 2q(\phi_p(t-1) + 1) + p(d_w^2 + d_x^2) \leq F(ES(\mathcal{H}, t)) \leq 2q\Delta^2(\phi_p(t) + 2\phi_p(t-1)) + 2q\Delta(3\phi_p(t-1) + 2) + 2q(\phi_p(t-1) + 1) + p(d_w^2 + d_x^2)$ left equality holds if $\mathcal{H} \cong \delta$ -regular graph and right equality holds if $\mathcal{H} \cong \Delta$ -regular graph.

Proof. Let p and q be order and size respectively of a graph \mathcal{H} . Then forgotten topological index of $ES(\mathcal{H}, t)$ can be stated as

$$F(ES(\mathcal{H}, t)) = \sum_{rs \in E(G)} \sum_{i,j=0}^1 |d_r + i, d_s + j| ((d_r + i)^2 + (d_s + j)^2) + |d_w, d_x|_{ES(\mathcal{H}, t)} (d_w^2 + d_x^2)$$

Now, by using Lemma 2.1 we have

$$F(ES(\mathcal{H}, t)) = \sum_{rs \in E(G)} [p^{t-2}(p - d_r - d_s + \triangleleft(r, s))(d_r^2 + d_s^2) + (p^{t-2}(d_r - \triangleleft(r, s)) - \phi_p(t-2)d_s - 1)((d_r + 1)^2 + d_s^2) + (p^{t-2}(d_s - \triangleleft(r, s)) - \phi_p(t-2)d_r - 1)(d_r^2 + (d_s + 1)^2) + (p^{t-2}(\triangleleft(r, s) + 1) + \phi_p(t-2)(d_r + d_s + 1) + 2)((d_r + 1)^2 + (d_s + 1)^2)] + p(d_w^2 + d_x^2)$$

As $\delta(\mathcal{H}) = \delta$ is the minimum degree of graph \mathcal{H} . Then we have

$$\begin{aligned}
 F(ES(\mathcal{H}, t)) &\geq \sum_{rs \in E(\mathcal{H})} \left[p^{t-2} (p - 2\delta + \triangleleft(r, s)) 2\delta^2 \right. \\
 &\quad + 2(p^{t-2} (\delta - \triangleleft(r, s)) - 2\phi_p(t-2)\delta - 1) (2\delta^2 + 2\delta + 1) \\
 &\quad \left. + (p^{t-2} (\triangleleft(r, s) + 1) + \phi_p(t-2)(2\delta + 1) + 2) (2\delta^2 + 4\delta + 2) \right] \\
 &\quad + p(d_w^2 + d_x^2) \\
 &= 2q\delta^2 (\phi_p(t) + 2\phi_p(t-1)) + 2q\delta(3\phi_p(t-1) + 2) + 2q(\phi_p(t-1) + 1) \\
 &\quad + p(d_w^2 + d_x^2)
 \end{aligned}$$

and equality holds if $\mathcal{H} \cong \delta$ -regular graph

As $\Delta(\mathcal{H}) = \Delta$ is the maximum degree of graph \mathcal{H} . Then we obtained

$$\begin{aligned}
 F(ES(\mathcal{H}, t)) &\leq \sum_{rs \in E(\mathcal{H})} \left[p^{t-2} (p - 2\Delta + \triangleleft(r, s)) 2\Delta^2 \right. \\
 &\quad + 2(p^{t-2} (\Delta - \triangleleft(r, s)) - 2\phi_p(t-2)\Delta - 1) (2\Delta^2 + 2\Delta + 1) \\
 &\quad \left. + (p^{t-2} (\triangleleft(r, s) + 1) + \phi_p(t-2)(2\Delta + 1) + 2) (2\Delta^2 + 4\Delta + 2) \right] \\
 &\quad + p(d_w^2 + d_x^2) \\
 &= 2q\Delta^2 (\phi_p(t) + 2\phi_p(t-1)) + 2q\Delta(3\phi_p(t-1) + 2) + 2q(\phi_p(t-1) + 1) \\
 &\quad + p(d_w^2 + d_x^2)
 \end{aligned}$$

and equality holds if $\mathcal{H} \cong \Delta$ -regular graph. \square

Corollary 2.16. *Let $p \geq 3$ be the order of a base graph \mathcal{H} . Then $\phi_p(t)8p + \phi_p(t-1)30p + p^3 + 19p \leq F(ES(\mathcal{H}, t)) \leq \phi_p(t)p(p-1)^3 + \phi_p(t-1)(2p^4 - 3p^3 + p^2) + 4p^3 - 3p^2 + p$ left equality holds if $\mathcal{H} \cong C_p$ and right equality holds if $\mathcal{H} \cong K_p$.*

3. Conclusion

The extended Sierpiński graphs are obtained by introducing a new vertex in generalized Sierpiński graph and attached this vertex with extreme vertices. In this paper, we have compute the Zagreb and forgotten invariants for extended Sierpiński graphs using any base graph \mathcal{H} . Moreover, for these topological indices of extended Sierpiński graph, we attained some sharp bounds by applying numerous parameters. In future, we want to extend this work by applying other topological indices on extended Sierpiński graphs and attained the fruitful results.

Data availability statements

All the data used to finding the results is included in the manuscript.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

This research is supported by the University program of Advanced Research (UPAR) and UAEU-AUA grants of United Arab Emirates University (UAEU) via Grant No.G00003271 and Grant No. G00003461.

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