Journal of King Saud University – Science 31 (2019) 21–23

Contents lists available at ScienceDirect

Journal of King Saud University – Science

journal homepage: www.sciencedirect.com

An extended differential form of Hilbert's inequality

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ARTICLE INFO

Article history:
Received 15 February 2017
Accepted 11 April 2017
Available online 18 April 2017

2000 Mathematics Subject Classification:
26D15

Keywords and phrases:
Hilbert's integral inequality
Hölder inequality
Hardy inequality

ABSTRACT

In this paper we introduce the extention of the differential form of Helbert's integral inequality, and the reverse form of it.

1. Introduction

The classical Hardy-Hilbert inequality for positive functions f, g and two conjugate parameters p and q such that $p > 1, \frac{1}{p} + \frac{1}{q} = 1$ is given as Hardy et al. (1952)

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{\int_0^\infty f^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty g^q(x)dx\right\}^{\frac{1}{q}} \quad (1.1)$$

provided that the integrals on the right-hand side are convergent. The constant $\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}$ is the best possible, in the sense that it cannot be decreased any more. Inequality (1.1) has several applications in mathematical analysis. There are many extensions of inequality (1.1), see for example the results in Hardy et al. (1952), Azar (2011), Yang et al. (2003), Yang (1991) and Mitrinović et al. (1991). A best extension of (1.1) was given in Krnic et al. (2005) as

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dxdy < B(1 - pA_2, \lambda + pA_2 - 1)$$
$$\left\{\int_0^\infty x^{pqA_1 - 1} f^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty y^{pqA_2 - 1} g^q(x)dx\right\}^{\frac{1}{q}}, \quad (1.2)$$

where $B(1 - pA_2, \lambda + pA_2 - 1)$ is the best possible constants ($B(u, v)$ is the beta function), $\lambda > 0, A_1 \in \left(\frac{1-\lambda}{q}, \frac{1}{q}\right)$, $A_2 \in \left(\frac{1-\lambda}{p}, \frac{1}{p}\right)$ and $pA_2 + qA_1 = 2 - \lambda$. For $0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1, \lambda > 0, A_1 \in \left(\frac{1}{q}, \frac{1-\lambda}{q}\right)$, $A_2 \in \left(\frac{1-\lambda}{p}, \frac{1}{p}\right)$ and $pA_2 + qA_1 = 2 - \lambda$ the reverse form of (1.2) is also valid. Recently, in Azar et al. (2014) a differential form of the Hilbert's inequality was obtained, namely

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dxdy \leqslant C\left(\int_0^\infty x^{p(n+1)-\lambda-1}\left(f^{(n)}(x)\right)^p dx\right)^{\frac{1}{p}}$$
$$\times \left(\int_0^\infty y^{q(n+1)-\lambda-1}\left(g^{(n)}(y)\right)^q dy\right)^{\frac{1}{q}} \quad (1.3)$$

where the constant $C = \frac{\Gamma\left(\frac{\lambda}{p}-n\right)\Gamma\left(\frac{\lambda}{q}-n\right)}{\Gamma(\lambda)}$ is the best possible. Here, $\Gamma(u)$ is the gamma function, $\lambda > 0, n = 0, 1, \ldots$. Note that if we let $n = 0, \lambda = 1$, and $p = q = 2$ we obtain the famous Hilbert's inequality. Therefore, one may consider inequality (1.3) as an extension of the Hilbert's inequality.

In this paper by introducing some parameters we obtain an extension to inequality (1.3) and the reverse form of it. The given inequalities as we will see are extensions of (1.2).

2. Preliminaries and Lemmas

We will frequently use the gamma and beta functions which are defined respectively as

$$\Gamma(\theta) = \int_0^\infty t^{\theta-1} e^{-t} dt, \quad \theta > 0, \quad (2.1)$$

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Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.jksus.2017.04.005
1018-3647/

$$B(\mu,\nu)=\int_0^\infty \frac{t^{\mu-1}}{(t+1)^{\mu+\nu}}dt,\quad \mu,\nu>0 \tag{2.2}$$

The technique of proving the main results is based on the following relation

$$\frac{1}{(x+y)^{\lambda}}=\frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}e^{-(x+y)t}dt. \tag{2.3}$$

The next two lemmas are also important tools for the proofs of the main results:

Lemma 2.1 *(see Azar et al., 2014, Lemma 2.2). Let $r>1, \frac{1}{r}+\frac{1}{s}=1, \varphi(x)>0$, the derivatives $\varphi',\varphi'',\cdots,\varphi^{(n)}$ exists and positive and $\varphi^{(n)}\in L(0,\infty), (n=0,1,2,\cdots), (\varphi^{(0)}:=\varphi)$, where L^p denotes the space of all Lebesgue integrable functions (If $p=1$ we obtain L), Moreover, suppose that $\varphi(0)=\varphi''(0)=\varphi''(0)=\cdots=\varphi^{(n-1)}(0)=0$, then for $t,\alpha>0$ we have*

$$\int_0^\infty e^{-tx}\varphi(x)dx\leqslant t^{-n-\frac{1}{s}-\alpha}\Gamma(\alpha s+1)^{\frac{1}{s}}\left(\int_0^\infty x^{-\alpha r}e^{-tx}\left(\varphi^{(n)}(x)\right)^r dx\right)^{\frac{1}{r}}. \tag{2.4}$$

Lemma 2.2. *Let $0<r<1, \frac{1}{r}+\frac{1}{s}=1$, Let φ be as in Lemma 2.2, then for $t>0$ and $\beta\in\mathbb{R}$ $(\beta s+1>0)$, we get*

$$\int_0^\infty e^{-tx}\varphi(x)dx\geqslant t^{-n-\frac{1}{s}-\beta}\Gamma(\beta s+1)^{\frac{1}{s}}\left(\int_0^\infty x^{-\beta r}e^{-tx}\left(\varphi^{(n)}(x)\right)^r dx\right)^{\frac{1}{r}}. \tag{2.5}$$

Proof. Using integration by parts n times, we get

$$\int_0^\infty e^{-tx}\varphi(x)dx=\frac{1}{t^n}\int_0^\infty e^{-tx}\varphi^{(n)}(x))dx.$$

Applying reverse Hölder inequality, then use (2.1), we obtain

$$\begin{aligned}\int_0^\infty e^{-tx}\varphi(x)dx&=\frac{1}{t^n}\int_0^\infty (x^{\beta}e^{-\frac{tx}{s}})(x^{-\beta}e^{-\frac{tx}{r}}\varphi^{(n)}(x))dx.\\ &\geqslant\frac{1}{t^n}\left(\int_0^\infty x^{\beta s}e^{-xt}dx\right)^{\frac{1}{s}}\left(\int_0^\infty x^{-\beta r}e^{-xt}((\varphi^{(n)})(x))^r dx\right)^{\frac{1}{r}}\\ &=\frac{1}{t^n}\left(\frac{1}{t^{\beta s+1}}\Gamma(\beta s+1)\right)^{\frac{1}{s}}\left(\int_0^\infty x^{-\beta r}e^{-xt}((\varphi^{(n)})(x))^r dx\right)^{\frac{1}{r}}\end{aligned}$$

this leads to (2.5). □

3. Main results

In this section, we give the main two inequalities of this paper, the first one is an extension of (1.3), and the second one is the reverse form.

Theorem 3.1. *Let $p>1, \frac{1}{p}+\frac{1}{q}=1, f(x), g(y)>0, \lambda>2n, \gamma\in(n-\frac{\lambda}{p}, \frac{\lambda}{q}-n), n=0,1,\ldots$ and $f(x), g(y)$ satisfies the conditions of Lemma 2.1 such that*

$\int_0^\infty x^{p(n+1)-p\gamma-\lambda-1}\left(f^{(n)}(x)\right)^p dx<\infty$,
$\int_0^\infty y^{q(n+1)+q\gamma-\lambda-1}\left(g^{(n)}(y)\right)^q dy<\infty$, then:

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^{\lambda}}dxdy\leqslant C\left(\int_0^\infty x^{p(n+1)-p\gamma-\lambda-1}\left(f^{(n)}(x)\right)^p dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{q(n+1)+q\gamma-\lambda-1}\left(g^{(n)}(y)\right)^q dx\right)^{\frac{1}{q}}, \tag{3.1}$$

where $C=\frac{\Gamma(\frac{\lambda}{p}+\gamma-n)\Gamma(\frac{\lambda}{q}-\gamma-n)}{\Gamma(\lambda)}$ is the best possible constant.

Proof. If we use the relation (2.3) and apply the Hölder's inequality, we obtain

$$\begin{aligned}I&:=\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{(x+y)^{\lambda}}dxdy\\ &=\frac{1}{\Gamma(\lambda)}\int_0^\infty\int_0^\infty f(x)g(y)\left(\int_0^\infty t^{\lambda-1}e^{-(x+y)t}dt\right)dxdy=\frac{1}{\Gamma(\lambda)}\\ &\quad\times\int_0^\infty\left(t^{\frac{\lambda-1}{p}+\gamma}\int_0^\infty e^{-tx}f(x)dx\right)\left(t^{\frac{\lambda-1}{q}-\gamma}\int_0^\infty e^{-ty}g(y))dy\right)dt\\ &\leqslant\frac{1}{\Gamma(\lambda)}\left(\int_0^\infty t^{\lambda-1+p\gamma}\left(\int_0^\infty e^{-tx}f(x)dx\right)^p dt\right)^{\frac{1}{p}}\\ &\quad\times\left(\int_0^\infty t^{\lambda-1-q\gamma}\left(\int_0^\infty e^{-ty}g(y)dy\right)^q dt\right)^{\frac{1}{q}}\end{aligned} \tag{3.2}$$

Using Lemma 2.2 for $r=p, s=q, \alpha=\alpha_1$, and then for $r=q, s=p, \alpha=\alpha_2$ we obtain respectively,

$$\left(\int_0^\infty e^{-tx}f(x)dx\right)^p\leqslant t^{-np-\alpha_1 p-p+1}\Gamma(\alpha_1 q+1)^{\frac{p}{q}}\times\int_0^\infty x^{-\alpha_1 p}e^{-tx}\left(f^{(n)}(x)\right)^p dx$$

and

$$\left(\int_0^\infty e^{-ty}g(y)dy\right)^q\leqslant t^{-nq-\alpha_2 q-q+1}\Gamma(\alpha_2 p+1)^{\frac{q}{p}}\times\int_0^\infty y^{-\alpha_2 q}e^{-ty}\left(g^{(n)}(y)\right)^q dy$$

If we substitute these two inequalities in (3.2) we get

$$\begin{aligned}I&\leqslant\frac{\Gamma(\alpha_1 q+1)^{\frac{1}{q}}\Gamma(\alpha_2 p+1)^{\frac{1}{p}}}{\Gamma(\lambda)}\left(\int_0^\infty x^{-\alpha_1 p}\left(f^{(n)}(x)\right)^p\left(\int_0^\infty t^{\lambda+p(\gamma-n-\alpha_1-1)}e^{-tx}dt\right)dx\right)^{\frac{1}{p}}\\ &\quad\times\left(\int_0^\infty t^{-\alpha_2 q}\left(g^{(n)}(y)\right)^q\left(\int_0^\infty t^{\lambda-q(\gamma+n+\alpha_2+1)}e^{-ty}dt\right)dy\right)^{\frac{1}{q}}\\ &=C_1\left(\int_0^\infty x^{p(n+1)-p\gamma-\lambda-1}\left(f^{(n)}(x)\right)^p dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{q(n+1)+q\gamma-\lambda-1}\left(g^{(n)}(y)\right)^q dy\right)^{\frac{1}{q}},\end{aligned}$$

where $C_1=\frac{\Gamma(\alpha_1 q+1)^{\frac{1}{q}}\Gamma(\alpha_2 p+1)^{\frac{1}{p}}\Gamma(\lambda+p(\gamma-n-\alpha_1-1)+1)^{\frac{1}{p}}\Gamma(\lambda-q(\gamma+n+\alpha_2+1)+1)^{\frac{1}{q}}}{\Gamma(\lambda)}$. Now, let $\alpha_1=\frac{\lambda+p(\gamma-n-1)}{pq}$ and $\alpha_2=\frac{\lambda-q(\gamma+n+1)}{pq}$, we obtain

$C_1=C=\frac{\Gamma(\frac{\lambda}{p}+\gamma-n)\Gamma(\frac{\lambda}{q}-\gamma-n)}{\Gamma(\lambda)}$. Hence, Inequality (3.1) is proved. It remains to show that the constant C is the best possible. To do that we define two functions:

$$f_{\varepsilon}(x)=\begin{cases}0, & 0<x<1\\ \frac{\Gamma\left(\frac{\lambda+q\gamma-\varepsilon}{p}-n\right)}{\Gamma\left(\frac{\lambda+p\gamma-\varepsilon}{p}\right)}x^{\frac{\lambda+p\gamma-\varepsilon}{p}-1}, & x\geqslant 1\end{cases}$$

and

$$g_{\varepsilon}(y)=\begin{cases}0, & 0<y<1\\ \frac{\Gamma\left(\frac{\lambda-q\gamma-\varepsilon}{q}-n\right)}{\Gamma\left(\frac{\lambda-q\gamma-\varepsilon}{q}\right)}y^{\frac{\lambda-q\gamma-\varepsilon}{q}-1}, & y\geqslant 1\end{cases},$$

where $0<\varepsilon<\min\{\lambda-q(\gamma+n), \lambda+p(\gamma-n)\}$. Thus, we find $f_{\varepsilon}^{(n)}(x)=x^{\frac{\lambda+p\gamma-\varepsilon}{p}-n-1}, x>1$, and $g_{\varepsilon}^{(n)}(y)=y^{\frac{\lambda-q\gamma-\varepsilon}{q}-n-1}, y>1$.

Suppose that $C=\frac{\Gamma(\frac{\lambda}{p}+\gamma-n)\Gamma(\frac{\lambda}{q}-\gamma-n)}{\Gamma(\lambda)}$ is not the best possible, then there exist $0<k<C$ such that

$$\int_0^\infty \int_0^\infty \frac{f_\varepsilon(x)g_\varepsilon(y)}{(x+y)^\lambda}dxdy < k\left(\int_0^\infty x^{p(n+1)-p\gamma-\lambda-1}\left(f_\varepsilon^{(n)}(x)\right)^p dx\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^\infty y^{q(n+1)+q\gamma-\lambda-1}\left(g_\varepsilon^{(n)}(y)\right)^q dx\right)^{\frac{1}{q}} = \frac{k}{\varepsilon} \quad (3.3)$$

On the other hand, we obtain (the constant $D = \frac{\Gamma\left(\frac{\lambda+p\gamma-\varepsilon}{p}-n\right)\Gamma\left(\frac{\lambda-q\gamma-\varepsilon}{q}-n\right)}{\Gamma\left(\frac{\lambda+p\gamma-\varepsilon}{p}\right)\Gamma\left(\frac{\lambda-q\gamma-\varepsilon}{q}\right)}$)

$$\begin{aligned} I &= D\int_1^\infty\int_1^\infty \frac{x^{\frac{\lambda+p\gamma-\varepsilon}{p}-1}y^{\frac{\lambda-q\gamma-\varepsilon}{q}-1}}{(x+y)^\lambda}dxdy \\ &= D\int_1^\infty x^{-\varepsilon-1}\int_{\frac{1}{x}}^\infty \frac{u^{\frac{\lambda-q\gamma-\varepsilon}{q}-1}}{(u+1)^\lambda}dudx \\ &= D\left[\int_1^\infty x^{-\varepsilon-1}\left\{\int_0^\infty \frac{u^{\frac{\lambda-q\gamma-\varepsilon}{q}-1}}{(u+1)^\lambda}du - \int_0^{\frac{1}{x}}\frac{u^{\frac{\lambda-q\gamma-\varepsilon}{q}-1}}{(u+1)^\lambda}du\right\}dx\right] \\ &= D\left[\frac{\beta\left(\frac{\lambda-q\gamma-\varepsilon}{q},\frac{\lambda}{p}+\gamma+\frac{\varepsilon}{q}\right)}{\varepsilon} - \int_1^\infty x^{-\varepsilon-1}\int_0^{\frac{1}{x}}\frac{u^{\frac{\lambda-q\gamma-\varepsilon}{q}-1}}{(u+1)^\lambda}dudx\right] \\ &> D\left[\frac{\beta\left(\frac{\lambda-q\gamma-\varepsilon}{q},\frac{\lambda}{p}+\gamma+\frac{\varepsilon}{q}\right)}{\varepsilon} - \int_1^\infty x^{-\varepsilon-1}\int_0^{\frac{1}{x}}u^{\frac{\lambda-q\gamma-\varepsilon}{q}-1}dudx\right] \\ &= D\left[\beta\left(\frac{\lambda}{q}-\gamma-\frac{\varepsilon}{q},\frac{\lambda}{p}+\gamma+\frac{\varepsilon}{q}\right)-O(1)\right] \end{aligned} \quad (3.4)$$

If we consider (3.3), (3.4) and the relation between the Beta and Gamma functions $B(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}$ and let $\varepsilon \to 0^+$, we get

$$k \geqslant \frac{\Gamma\left(\frac{\lambda+p\gamma}{p}-n\right)\Gamma\left(\frac{\lambda-q\gamma}{q}-n\right)}{\Gamma\left(\frac{\lambda}{p}+\gamma\right)\Gamma\left(\frac{\lambda}{q}-\gamma\right)}\beta\left(\frac{\lambda}{q}-\gamma,\frac{\lambda}{p}+\gamma\right)$$
$$= \frac{\Gamma\left(\frac{\lambda+p\gamma}{p}-n\right)\Gamma\left(\frac{\lambda-q\gamma}{q}-n\right)}{\Gamma(\lambda)},$$

which is in contradiction with our assumption. The theorem is proved. □

Theorem 3.2. *If $f,g>0$, $0<p<1$, $\frac{1}{p}+\frac{1}{q}=1$, $\lambda>2n$, $\gamma\in\left(n-\frac{\lambda}{p},\frac{\lambda}{q}-n\right)$, $n=0,1,\ldots$ and f,g satisfies the conditions of Lemma 2.1 such that $\int_0^\infty x^{p(n+1)-p\gamma-\lambda-1}\left(f^{(n)}(x)\right)^p dx<\infty$, and $\int_0^\infty y^{q(n+1)+q\gamma-\lambda-1}\left(g^{(n)}(y)\right)^q dy<\infty$, then we have the reverse form of (3.1) as*

$$I \geqslant C\left(\int_0^\infty x^{p(n+1)-p\gamma-\lambda-1}\left(f^{(n)}(x)\right)^p dx\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^\infty y^{q(n+1)+q\gamma-\lambda-1}\left(g^{(n)}(y)\right)^q dy\right)^{\frac{1}{q}}, \quad (3.5)$$

where C as in Theorem 3.1 is the best possible constant.

Proof. Using (2.3) and the reverse Hölder inequality, we have

$$I = \frac{1}{\Gamma(\lambda)}\int_0^\infty\left(t^{\frac{\lambda-1}{p}+\gamma}\int_0^\infty e^{-tx}f(x)dx\right)\left(t^{\frac{\lambda-1}{q}-\gamma}\int_0^\infty e^{-ty}g(y))dy\right)dt$$
$$\geqslant \frac{1}{\Gamma(\lambda)}\left(\int_0^\infty t^{\lambda-1+p\gamma}\left(\int_0^\infty e^{-tx}f(x)dx\right)^p dt\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^\infty t^{\lambda-1-q\gamma}\left(\int_0^\infty e^{-ty}g(y))dy\right)^q dt\right)^{\frac{1}{q}} \quad (3.6)$$

If we use Lemma 2.2 for $r=p, s=q$, $\beta=\beta_1$ and then for $r=q, s=p, \beta=\beta_2$ respectively, we find

$$\left(\int_0^\infty e^{-tx}f(x)dx\right)^p \geqslant t^{-np-\frac{p}{q}-\beta_1 p}\Gamma(\beta_1 q+1)^{\frac{p}{q}}\int_0^\infty x^{-\beta_1 p}e^{-tx}\left(f^{(n)}(x)\right)^p dx,$$

and

$$\left(\int_0^\infty e^{-ty}f(y)dy\right)^q \geqslant t^{-nq-\frac{q}{p}-\beta_2 q}\Gamma(\beta_1 q+1)^{\frac{q}{p}}\int_0^\infty y^{-\beta_2 q}e^{-ty}\left(f^{(n)}(y)\right)^q dy.$$

If we substitute the last two inequalities in (3.6) and make some computations and then letting: $\beta_1 = \frac{\lambda+p(\gamma-n-1)}{pq}$ and $\beta_2 = \frac{\lambda-q(\gamma+n+1)}{pq}$ we arrive at inequality (3.5). To prove that the constant is the best possible, we define $f_\varepsilon(x)$ and $g_\varepsilon(y)$ as in the proof of Theorem 3.1. If C is not the best possible one, then there is a positive number $k \geqslant C$ such that (3.6) is still valid as we replace C by k, then

$$\int_0^\infty\int_0^\infty \frac{f_\varepsilon(x)g_\varepsilon(y)}{(x+y)^\lambda}dxdy > \frac{k}{\varepsilon}. \quad (3.7)$$

On the other hand, we find

$$\begin{aligned} \int_0^\infty\int_0^\infty \frac{f_\varepsilon(x)g_\varepsilon(y)}{(x+y)^\lambda}dxdy &= D\int_1^\infty\int_1^\infty \frac{x^{\frac{\lambda+p\gamma-\varepsilon}{p}-1}y^{\frac{\lambda-q\gamma-\varepsilon}{q}-1}}{(x+y)^\lambda}dxdy \\ &= D\int_1^\infty x^{-1-\varepsilon}\int_{\frac{1}{x}}^\infty \frac{u^{\frac{\lambda-q\gamma-\varepsilon}{q}-1}}{(u+1)^\lambda}dudx \\ &< D\int_1^\infty x^{-1-\varepsilon}dx\int_0^\infty \frac{u^{\frac{\lambda-q\gamma-\varepsilon}{q}-1}}{(u+1)^\lambda}du \\ &= D\beta\left(\frac{\lambda}{p}+\gamma+\frac{\varepsilon}{q},\frac{\lambda}{q}-\gamma-\frac{\varepsilon}{q}\right). \end{aligned} \quad (3.8)$$

If we let $\varepsilon \to 0^+$, from (3.7) to (3.8) we find $k \geqslant C$. the theorem is proved. □

Remark 3.3. If we put $n=0$ in (3.1) we get

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy \leqslant C\left(\int_0^\infty x^{p-p\gamma-\lambda-1}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{q+q\gamma-\lambda-1}g^q(y)dx\right)^{\frac{1}{q}}. \quad (3.9)$$

Note that if we set in (3.9) $\gamma = \frac{p-\lambda-pqA_1}{p}$ $\left(-\frac{\lambda}{p}<\gamma<\frac{\lambda}{q}\right)$ under the condition $pA_2+qA_1 = 2-\lambda$ we obtain inequality (1.2) from the introduction. Similarly, we may obtain the reverse form of (1.2) from (3.5).

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