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ORIGINAL ARTICLE

Exp-function method for some nonlinear PDE's and a nonlinear ODE's

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Abstract In this paper, we apply the Exp-function method to find some exact solutions for two nonlinear partial differential equations (NPDE) and a nonlinear ordinary differential equation (NODE), namely, Cahn-Hilliard equation, Allen-Cahn equation and Steady-State equation, respectively. It has been shown that the Exp-function method, with the help of symbolic computation, provides a very effective and powerful mathematical tool for solving NPDE's and NODE's. Mainly we try to present an application of Exp-function method taking to consideration rectifying a commonly occurring errors during some of recent works. The results of the other methods clearly indicate the reliability and efficiency of the used method.

1. Introduction

The study of exact solutions of nonlinear partial differential equations (NPDE) plays an important role in mathematical physics, engineering and the other sciences. In the past several decades, various methods for obtaining solutions of NPDEs and ODE's have been presented, such as, tanh-function method (Wazwaz, 2005, 2006a,b), Adomian decomposition method (Hashim et al., 2006; Tatari et al., 2007), Homotopy perturbation method (Rashidi et al., 2009; Biazar et al., 2009; Berberler and Yildirim, 2009), variational iteration method (Shakeri and Dehghan, 2008; Soliman and Abdou, 2007; Yusufoglu and Bekir, 2007), spectral method (Parand and Taghavi, 2009; Parand et al., 2009, 2010), sine–cosine method (Tascan and Bekir, 2009; Wazwaz, 2007), radial basis method (Tatari and Dehghan, 2010; Dehghan and Shokri, 2009) and so on. Recently, He and Wu (2006) proposed a novel method, so called Exp-function method, which is easy, succinct and powerful to implement to nonlinear partial differential equations arising in mathematical physics. The Exp-function method has been successfully applied to many kinds of NPDEs, such as, KdV equation with variable coefficients (Zhang, 2007), Maccari's system (Zhang, 2007), Boussinesq equations (Abdou et al., 2007), Burger's equations (Ebaid, 2007; Biazar and Ayati, 2009; Ebaid, 2009), Double Sine-Gordon equation (Domairry et al., 2010; He and Abdou, 2007), Fisher equation (Ozis and

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Koroglu, 2008), Jaulent–Miodek equations (He and Zhang, 2008) and the other important nonlinear partial differential equations (Koroglu and zis, 2009; Shin et al., 2009; Zhang, 2008). Recently, in some of papers applying the Exp-function method (He and Wu, 2006) have been occurred with common errors. Seven common errors are formulated and classified by (Kudryashov, 2009). In this paper we try to apply this method taking to rectifying these common errors to look exact solutions of three nonlinear differential equations, namely, Cahn-Hilliard equation, Allen-Cahn equation and Steady-State equation given by

$$u_t = \gamma u_x + 6u(u_x)^2 + (3u^2 - 1)u_{xx} - u_{xxxx},$$
$$u_t = u_{xx} - u^3 + u,$$
$$\alpha u''(x) = \beta u(x)(u(x) - m)(u(x) + m),$$

respectively, which α, β, m and γ are the constants.

The Cahn–Hilliard equation was proposed to describe phase separation phenomena in binary systems (Cahn et al., 1958). This equation is related with a number of interesting physical phenomena like the spinodal decomposition, phase separation and phase ordering dynamics. It is also very crucial in material sciences (Chan, 1961; Choo et al., 2004; Gurtin, 1996). On the other hand, this equation is very hard and difficult to solve. The Cahn-Hilliard equation has been extensively studied by Wang and Shi (1993), Jabbari and Peppas (1995), Puri and Binder (1991) for the study of interfaces. Global existence and uniqueness of the solution have been shown by Elliott and Zheng (1986). Jingxue (1992) has shown the existence of continuous solution for the problem with degenerate mobility. Recently, Dehghan and Mirzaei (2009) applied a numerical method based on the boundary integral equation and dual reciprocity methods for one-dimensional Cahn-Hilliard equation. Ugurlu and Kaya (2008) solved Cahn-Hilliard equation by tanh-function method. Furihata (2001) applied finite difference for Cahn-Hilliard equation. Many articles have investigated this equation mathematically and numerically this equation (Mello et al., 2005; Kim, 2007; Wells et al., 2006). Also, Allen-Cahn equation arise in many scientific applications such as mathematical biology, quantum mechanics and plasma physics. It is well known that wave phenomena of plasma media and fluid dynamics are modelled by kink shaped and tanh solution or bell shaped sech solutions (Wazwaz, 2007; Tascana and Bekir, 2009).

The rest of the paper is organized as follows: Section 2 describes Exp-function method for finding exact solutions to the NPDEs. The applications of the proposed analytical scheme presented in Section 3. The conclusions are discussed in the Section 4. exp-function calculations are provided in the end.

2. Basic idea of Exp-function method

We consider a general nonlinear PDE in the following form

$$N(u, u_x, u_t, u_{xx}, u_{tt}, u_{xt}, \ldots) = 0, \tag{1}$$

where N is a polynomial function with respect to the indicated variables or some functions which can be reduced to a polynomial function by using some transformation. We introduce a complex variation as

$$u(x,t) = U(\eta), \quad \eta = kx + \omega t, \tag{2}$$

where k and ω are constants. We can rewrite Eq. (1) in the following nonlinear ordinary differential equations

$$N(U, kU', \omega U', k^2U'', \ldots) = 0,$$

where the prime denotes the derivation with respect to η. According to the Exp-function method (He and Wu, 2006), we assume that the solution can be expressed in the form

$$U(\eta) = \frac{\sum_{i=-d}^{c} a_i \exp(i\eta)}{\sum_{j=-q}^{p} b_j \exp(j\eta)}, \tag{3}$$

where c, d, p and q are positive integers which can be freely chosen, a_i and b_j are unknown constants to be determined. To determine the values of c and p, we balance the highest order linear term with the highest order nonlinear term in Eq. (3). Similarly to determine the values of d and q. So by means of the exp-function method, we obtain the generalized solitary solution and periodic solution for nonlinear evolution equations arising in mathematical physics.

3. Applications of the Exp-function method

Example 1. Let us consider the Cahn-Hilliard equation (Ugurlu and Kaya, 2008; Dahmani and Benbachir, 2009) in the form

$$u_t = \gamma u_x + 6u(u_x)^2 + (3u^2 - 1)u_{xx} - u_{xxxx}, \tag{4}$$

that, by using the complex variation

$$u(x,t) = U(\eta), \quad \eta = kx + \omega t, \tag{5}$$

and integrating with respect to η, Eq. (4) can be converted to the ODE (for $\gamma = 1$)

$$(\omega - k)U + k^4U''' - 3k^2U^2U' + k^2U' = 0, \tag{6}$$

where the prime denotes the derivative with respect to η and also where the integration constant is chosen as zero. In other words, we are solved this problem for the case when integration constant is zero. In view of the Exp-function method, we assume that the solution of Eq. (6) can be expressed in the form

$$U(\eta) = \frac{a_c \exp(c\eta) + \cdots + a_{-d} \exp(-d\eta)}{b_p \exp(p\eta) + \cdots + b_{-q} \exp(-q\eta)},$$

where c, d, p and q are positive integers which are unknown to be determined later. In order to determine the values of c and p, we balance the linear term of the highest order with the highest order nonlinear terms in Eq. (6) i.e. U''' and U^2U'. By simple calculation, we have

$$U''' = \frac{c_1 \exp[(c + 7p)\eta] + \cdots}{c_2 \exp[8p\eta] + \cdots}, \tag{7}$$

and

$$U^2U' = \frac{c_3 \exp[(3c + p)\eta] + \cdots}{c_4 \exp[4p\eta] + \cdots} = \frac{c_3 \exp[(3c + 5p)\eta] + \cdots}{c_4 \exp[8p\eta] + \cdots}, \tag{8}$$

where c_i are determined coefficients only for simplicity. By balancing the highest order of Exp-function in Eqs. (8) and (7), we have

$$c + 7p = 3c + 5p,$$

which leads to the result

$$p = c.$$

Similarly, to determine the values of d and q, we balance the linear term of the lowest order in Eq. (6)

$$U''' = \frac{\cdots + d_1 \exp[-(7q + d)\eta]}{\cdots + d_2 \exp[-8q\eta]}, \tag{9}$$

and

$$U^2U' = \frac{\cdots + d_3 \exp[-(q+3d)\eta]}{\cdots + d_4 \exp[-4q\eta]} = \frac{\cdots + d_3 \exp[-(5q+3d)\eta]}{\cdots + d_4 \exp[-8q\eta]}, \tag{10}$$

where d_i are determined coefficients only for simplicity, we have

$$-(7q+d) = -(5q+3d),$$

which leads to results

$$q = d.$$

For simplicity, we set $p = c = 1$ and $q = d = 1$, so Eq. (3) reduces to

$$U(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1}\exp(-\eta)}{\exp(\eta) + b_0 + b_{-1}\exp(-\eta)}. \tag{11}$$

Substituting Eq. (11) into Eq. (6), equating to zero the coefficients of all powers of $\exp(n\eta)$ yields a set of algebraic equations for $a_0, b_0, a_{-1}, a_1, b_1, k$ and ω (see Appendix A). By solving the system of algebraic equations with a professional mathematical software, we obtain

Case 1.

$$\begin{cases} a_{-1} = b_{-1}, & b_{-1} = b_{-1}, \\ a_0 = 0, & b_0 = 0, \\ a_1 = -1, & k = \omega = \frac{\sqrt{2}}{2}. \end{cases} \tag{12}$$

where b_{-1} is free parameter which can be determined by initial or boundary conditions. Substituting these results into (11), we obtain

$$u(x,t) = \frac{-\exp\left[\frac{\sqrt{2}}{2}(x+t)\right] + b_{-1}\exp\left[-\frac{\sqrt{2}}{2}(x+t)\right]}{\exp\left[\frac{\sqrt{2}}{2}(x+t)\right] + b_{-1}\exp\left[-\frac{\sqrt{2}}{2}(x+t)\right]}. \tag{13}$$

These results cover some of special solutions Eq. (4) regarding to initial value conditions. For example, if initial conditions $u(x,t)$ becomes $u(x,0) = 0$, then by substituting $t = 0$ in the Eq. (13) we could obtain b_{-1} and exact solution of Eq. (4). If Eq. (4) were in the following form:

$$\begin{cases} u_t = u_x + 6u(u_x)^2 + (3u^2 - 1)u_{xx} - u_{xxxx}, \\ u(x,0) = -\tanh\left[\frac{\sqrt{2}}{2}x\right]. \end{cases} \tag{14}$$

From Eqs. (14) and (13), we obtain

$$b_{-1} = 1. \tag{15}$$

Thus, from substituting Eq. (15) into Eq. (13), we obtain

$$u(x,t) = \frac{-\exp\left[\frac{\sqrt{2}}{2}(x+t)\right] + \exp\left[-\frac{\sqrt{2}}{2}(x+t)\right]}{\exp\left[\frac{\sqrt{2}}{2}(x+t)\right] + \exp\left[-\frac{\sqrt{2}}{2}(x+t)\right]} = -\tanh\left[\frac{\sqrt{2}}{2}(x+t)\right],$$

which is the solution obtained by tanh method in Ugurlu and Kaya, 2008. Also, if Eq. (4) were in the following form:

$$\begin{cases} u_t = u_x + 6u(u_x)^2 + (3u^2 - 1)u_{xx} - u_{xxxx}, \\ u(x,0) = -\coth\left[\frac{\sqrt{2}}{2}x\right]. \end{cases} \tag{16}$$

From Eqs. (16) and (13), we derive

$$b_{-1} = -1. \tag{17}$$

Thus, from substituting Eq. (17) into Eq. (13), we obtain

$$u(x,t) = \frac{-\exp\left[\frac{\sqrt{2}}{2}(x+t)\right] - \exp\left[-\frac{\sqrt{2}}{2}(x+t)\right]}{\exp\left[\frac{\sqrt{2}}{2}(x+t)\right] - \exp\left[-\frac{\sqrt{2}}{2}(x+t)\right]} = -\coth\left[\frac{\sqrt{2}}{2}(x+t)\right],$$

which is the solution obtained by tanh method in Ugurlu and Kaya (2008).

Case 2.

$$\begin{cases} a_{-1} = -b_{-1}, & b_{-1} = b_{-1}, \\ a_0 = 0, & b_0 = 0, \\ a_1 = 1, & k = \omega = \frac{\sqrt{2}}{2}. \end{cases} \tag{18}$$

Substituting these result into Eq. (11), we obtain the following solution:

$$u(x,t) = \frac{\exp\left[\frac{\sqrt{2}}{2}(x+t)\right] - b_{-1}\exp\left[-\frac{\sqrt{2}}{2}(x+t)\right]}{\exp\left[\frac{\sqrt{2}}{2}(x+t)\right] + b_{-1}\exp\left[-\frac{\sqrt{2}}{2}(x+t)\right]}. \tag{19}$$

If Eq. (4) were in the following form:

$$\begin{cases} u_t = u_x + 6u(u_x)^2 + (3u^2 - 1)u_{xx} - u_{xxxx}, \\ u(x,0) = \coth\left[\frac{\sqrt{2}}{2}x\right]. \end{cases} \tag{20}$$

From Eq. (20) and Eq. (19), we obtain

$$b_{-1} = 1. \tag{21}$$

Thus, from substituting Eq. (21) into Eq. (19), we obtain

$$u(x,t) = \frac{\exp\left[\frac{\sqrt{2}}{2}(x+t)\right] - \exp\left[-\frac{\sqrt{2}}{2}(x+t)\right]}{\exp\left[\frac{\sqrt{2}}{2}(x+t)\right] + \exp\left[-\frac{\sqrt{2}}{2}(x+t)\right]} = \coth\left[\frac{\sqrt{2}}{2}(x+t)\right],$$

which is the same as Ugurlu's solution (Ugurlu and Kaya, 2008). Also, if Eq. (4) were in the following form:

$$\begin{cases} u_t = u_x + 6u(u_x)^2 + (3u^2 - 1)u_{xx} - u_{xxxx}, \\ u(x,0) = \tanh\left[\frac{\sqrt{2}}{2}x\right]. \end{cases} \tag{22}$$

From Eq. (22) and Eq. (19), we obtain

$$b_{-1} = -1. \tag{23}$$

Thus, from substituting Eq. (23) into Eq. (19), we obtain

$$u(x,t) = \frac{\exp\left[\frac{\sqrt{2}}{2}(x+t)\right] + \exp\left[-\frac{\sqrt{2}}{2}(x+t)\right]}{\exp\left[\frac{\sqrt{2}}{2}(x+t)\right] - \exp\left[-\frac{\sqrt{2}}{2}(x+t)\right]} = \tanh\left[\frac{\sqrt{2}}{2}(x+t)\right], \tag{24}$$

which is the exact solutions given by Ugurlu and Kaya (2008), Dahmani and Benbachir (2009). Fig. 1 depict the solution Eq. (24).

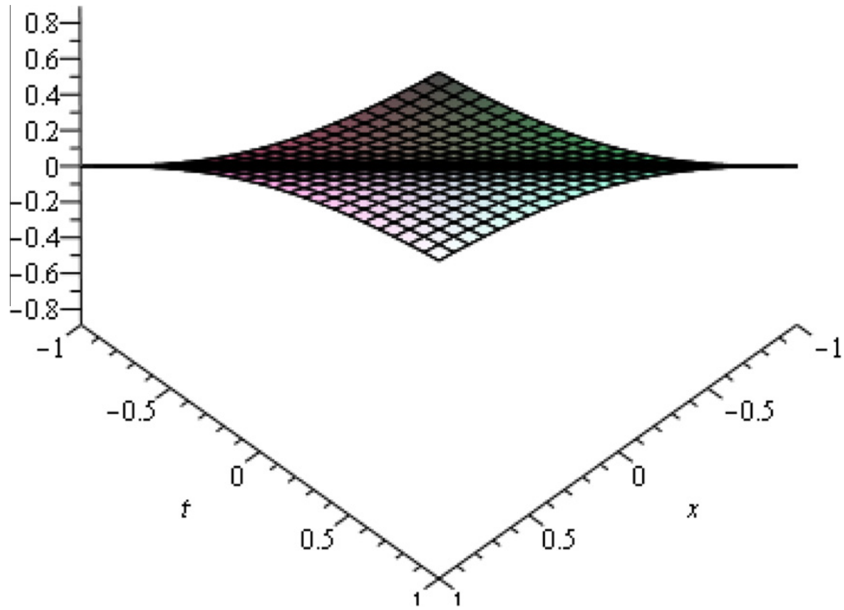

Figure 1 The solutions of Eq. (24).

Case 3.

$$\begin{cases} a_{-1} = 0, & b_{-1} = \frac{1}{8}a_0^2, \\ a_0 = a_0, & b_0 = 0, \\ a_1 = 0, & k = \omega = i. \end{cases} \tag{25}$$

In this case, k and ω are imaginary numbers. Substituting these results into (11), we obtain

$$u(x,t) = \frac{a_0}{\exp[i(x+t)] + \frac{1}{8}a_0^2 \exp[-i(x+t)]}. \tag{26}$$

If Eq. (4) were in the following form:

$$\begin{cases} u_t = u_x + 6u(u_x)^2 + (3u^2 - 1)u_{xx} - u_{xxxx}, \\ u(x,0) = \frac{\sqrt{2}}{2}\tan\left[\frac{1}{2}x\right] + \frac{\sqrt{2}}{2}\cot\left[l\frac{1}{2}x\right]. \end{cases} \tag{27}$$

From Eqs. (27) and (26), we obtain

$$a_0 = 2\sqrt{2}i. \tag{28}$$

Thus, from substituting Eq. (28) into Eq. (26), we have

$$u(x,t) = \frac{2\sqrt{2}i}{\exp[i(x+t)] - \exp[-i(x+t)]}, \tag{29}$$

we know

$$\exp[i(x+t)] - \exp[-i(x+t)] = 2i\sin(x+t), \tag{30}$$

and

$$\frac{1}{\sin(x+t)} = \frac{1}{2}\tan\left[\frac{1}{2}(x+t)\right] + \frac{1}{2}\cot\left[\frac{1}{2}(x+t)\right]. \tag{31}$$

Substituting Eqs. (30) and (31) into Eq. (29), we obtain

$$u(x,t) = \frac{\sqrt{2}}{2}\tan\left[\frac{1}{2}(x+t)\right] + \frac{\sqrt{2}}{2}\cot\left[\frac{1}{2}(x+t)\right], \tag{32}$$

which is the solution obtained by tanh method in Ugurlu and Kaya (2008). Fig. 2 depict the solution Eq. (32). Also, if Eq. (4) were in the following form:

$$\begin{cases} u_t = u_x + 6u(u_x)^2 + (3u^2 - 1)u_{xx} - u_{xxxx}, \\ u(x,0) = -\frac{\sqrt{2}}{2}\tan\left[\frac{1}{2}x\right] - \frac{\sqrt{2}}{2}\cot\left[\frac{1}{2}x\right]. \end{cases} \tag{33}$$

From Eqs. (33) and (26), we obtain

$$a_0 = -2\sqrt{2}i. \tag{34}$$

Thus, from substituting Eq. (34) into Eq. (26), we have

$$u(x,t) = \frac{-2\sqrt{2}i}{\exp[i(x+t)] - \exp[-i(x+t)]}. \tag{35}$$

Substituting Eqs. (30) and (31) into Eq. (35), we obtain

$$u(x,t) = -\frac{\sqrt{2}}{2}\tan\left[\frac{1}{2}(x+t)\right] - \frac{\sqrt{2}}{2}\cot\left[\frac{1}{2}(x+t)\right],$$

which is the solution obtained by tanh method in Ugurlu and Kaya (2008).

Case 4.

$$\begin{cases} a_{-1} = -\frac{1}{8}\frac{b_0^2(a_1^2-1)}{a_1}, & b_{-1} = -\frac{1}{8}\frac{b_0^2(a_1^2-1)}{a_1^2}, \\ a_0 = -\frac{(2a_1^2-1)b_0}{a_1}, & b_0 = b_0, \\ a_1 = a_1, & k = \omega = \sqrt{3a_1^2 - 1}, \end{cases} \tag{36}$$

where b_0 and a_1 are free parameters. Inserting these result into Eq. (11), we obtain

$$u(x,t) = \frac{a_1\exp(\eta) - \frac{(2a_1^2-1)b_0}{a_1} - \frac{1}{8}\frac{b_0^2(a_1^2-1)}{a_1}\exp(-\eta)}{\exp(\eta) + b_0 - \frac{1}{8}\frac{b_0^2(a_1^2-1)}{a_1^2}\exp(-\eta)}$$
$$= a_1 - \frac{3a_1b_0 - \frac{b_0}{a_1}}{\exp(\eta) + b_0 - \frac{1}{8}\frac{b_0^2(a_1^2-1)}{a_1^2}\exp(-\eta)},$$

with $\eta = \sqrt{3a_1^2 - 1}(x+t)$.

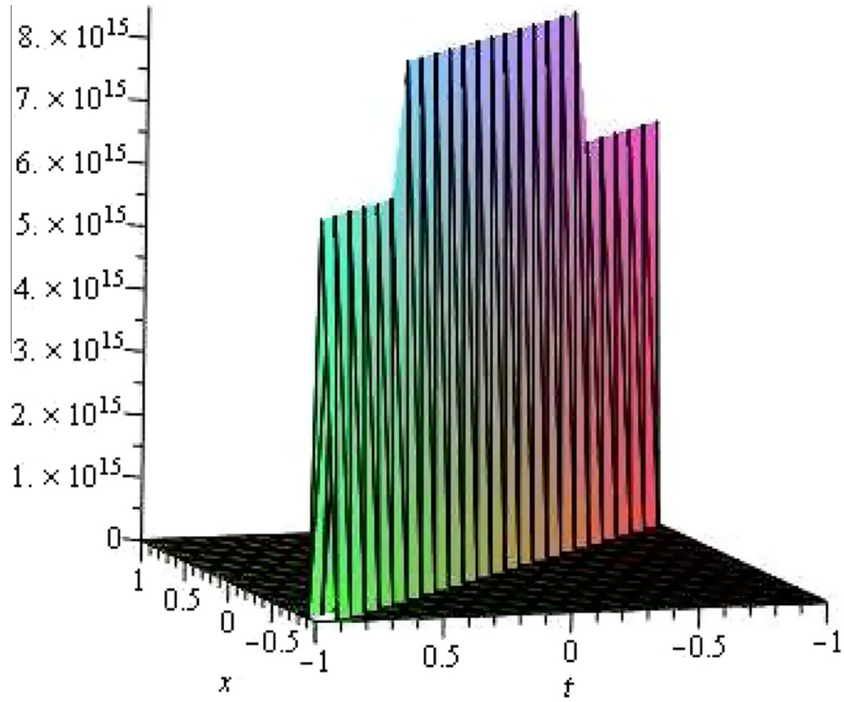

Figure 2 The solutions of Eq. (32).

Case 5.

$$\begin{cases} a_{-1} = \frac{1}{4}(a_0^2 - b_0^2), & b_{-1} = -\frac{1}{4}(a_0^2 - b_0^2), \\ a_0 = a_0, & b_0 = b_0, \\ a_1 = 1, & k = \omega = \sqrt{2}, \end{cases} \tag{37}$$

where b_0 and a_0 are free parameters. Inserting these result into Eq. (11), we obtain exact solution

$$u(x,t) = \frac{\exp(\eta) + a_0 + \frac{1}{4}(a_0^2 - b_0^2)\exp(-\eta)}{\exp(\eta) + b_0 - \frac{1}{4}(a_0^2 - b_0^2)\exp(-\eta)},$$

with $\eta = \sqrt{2}(x+t)$.

Example 2. Now we consider the Allen-Cahn equation (Wazwaz, 2007; Tascana and Bekir, 2009)

$$u_t = u_{xx} - u^3 + u. \tag{38}$$

Using the transformation (2), Eq. (38) becomes

$$k^2U'' - \omega U' - U^3 + U = 0, \tag{39}$$

where prime denotes the differential with respect to η. The highest nonlinear term U^3 is now given by

$$U^3 = \frac{c_1\exp[(3c+p)\eta] + \cdots}{c_2\exp[4p\eta] + \cdots}, \tag{40}$$

and the highest linear term U'' is given by

$$U'' = \frac{c_3\exp[(c+3p)\eta] + \cdots}{c_4\exp[4p\eta] + \cdots}. \tag{41}$$

Balancing the highest order of exp-function in Eqs. (41) and (40), we have $3c + p = c + 3p$, so $p = c$. As since illustrated in the previous example, we can also obtain that $q = d$. Here, we only consider the simplest case $p = c = 1$ and $q = d = 1$, so Eq. (3) reduces to

$$U(\eta) = \frac{a_1\exp(\eta) + a_0 + a_{-1}\exp(-\eta)}{\exp(\eta) + b_0 + b_{-1}\exp(-\eta)}. \tag{42}$$

Substituting Eq. (42) into Eq. (39), equating to zero the coefficients of all powers of $\exp(n\eta)$ yields a set of algebraic equations for $a_0, b_0, a_{-1}, a_1, b_1, k$ and ω (see Appendix B). By solving the system of algebraic equations with a professional mathematical software, we obtain

Case 1.

$$\begin{cases} a_1 = 1, & b_{-1} = 0, \\ a_0 = 0, & b_0 = b_0, \\ a_{-1} = 0, & k = \pm\frac{\sqrt{2}}{2}, \quad \omega = \frac{3}{2}. \end{cases} \tag{43}$$

Substituting these result into Eq. (42), we obtain

$$u(x,t) = \frac{1}{1 + b_0\exp\left[\frac{\sqrt{2}}{2}(\mp x - \frac{3\sqrt{2}}{2}t)\right]}, \tag{44}$$

where b_0 is free parameter. Replacing $b_0 = \pm 1$ in Eq. (44) gives the solutions obtained in Wazwaz (2007), Tascana and Bekir (2009). Fig. 3 depict the solution Eq. (44) with $b_0 = 1$.

Case 2.

$$\begin{cases} a_1 = -1, & b_{-1} = 0, \\ a_0 = 0, & b_0 = b_0, \\ a_{-1} = 0, & k = \pm\frac{\sqrt{2}}{2}, \quad \omega = \frac{3}{2}. \end{cases} \tag{45}$$

Substituting these result into Eq. (42), we obtain

$$u(x,t) = -\frac{1}{1 + b_0\exp\left[\frac{\sqrt{2}}{2}\left(\mp x - \frac{3\sqrt{2}}{2}t\right)\right]}, \tag{46}$$

where b_0 is free parameter. Replacing $b_0 = \pm 1$ in Eq. (46) gives the solutions obtained in Wazwaz (2007), Tascana and Bekir (2009).

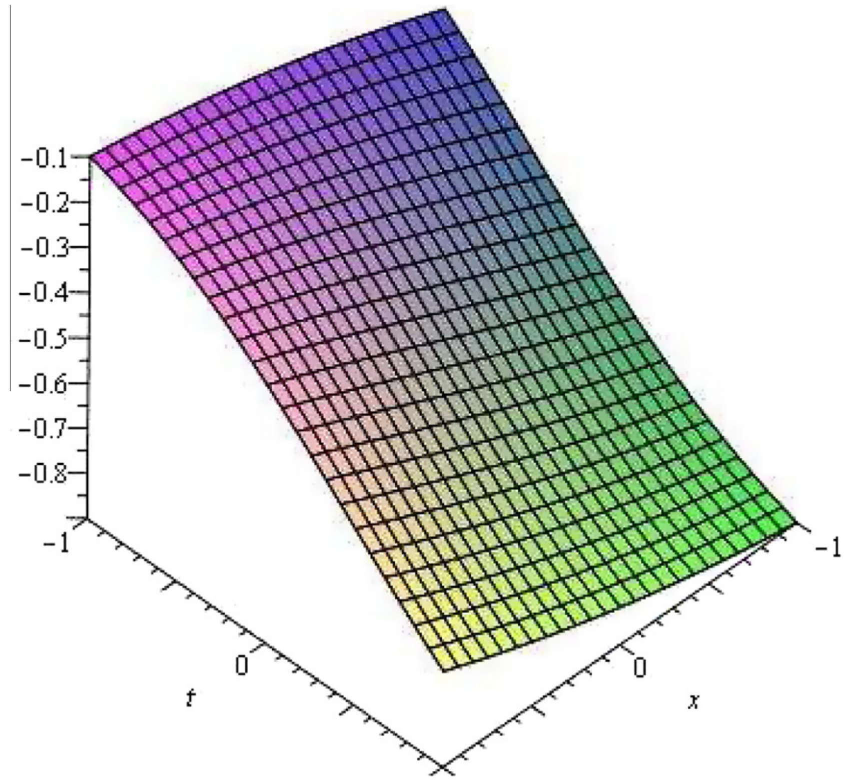

Figure 3 The solutions of Eq. (44) for the parameter value $b_0 = 1$.

Example 3. Finally, we consider the solution of steady-state equation that is presented by a ordinary differential equation in the following form (Elliott and French, 1987):

$$\alpha u''(x) = \beta u(x)(u(x) - m)(u(x) + m), \tag{47}$$

where α, β, m are the constants. By making the transformation

$$v(x) = m^{-1}(u(\varepsilon x) + m),$$

where $\varepsilon = \left(\frac{\alpha}{\beta m^2}\right)^{1/2}$, Eq. (47) becomes

$$v''(x) = v(x)(v(x) - 1)(v(x) - 2). \tag{48}$$

According to the Exp-function method (He and Wu, 2006), we assume that the solution of Eq. (48) can be expressed in the following from:

$$v(x) = \frac{a_1 \exp[kx + \omega] + a_0 + a_{-1} \exp[-kx - \omega]}{\exp[kx + \omega] + b_0 + b_{-1} \exp[-kx - \omega]}, \tag{49}$$

where $a_1, a_0, a_{-1}, b_0, b_{-1}, k$ and ω are constants which are unknown to be further determined. Substituting Eq. (49) into Eq. (48) and equating the coefficients of all powers of $\exp[i(kx + \omega)] (i = 0, \pm 1, \pm 2, \ldots)$ to zero yields a set of algebraic equations for $a_1, a_0, a_{-1}, b_0, b_{-1}, k$ and ω (see Appendix C). By solving the system of algebraic equations with a professional mathematical software, we obtain

Case 1.

$$\begin{cases} a_{-1} = 0, & b_{-1} = b_{-1}, \\ a_0 = 0, & b_0 = 0, \\ a_1 = 2, & k = \frac{\sqrt{2}}{2}, \quad \omega = \omega. \end{cases} \tag{50}$$

Substituting these results into (49), we obtain

$$v(x) = \frac{2 \exp\left[\frac{\sqrt{2}}{2} x + \omega\right]}{\exp\left[\frac{\sqrt{2}}{2} x + \omega\right] + b_{-1} \exp\left[-\frac{\sqrt{2}}{2} x - \omega\right]}. \tag{51}$$

By the some manipulation in Eq. (51), we obtain

$$v(x) = \frac{\exp\left[\frac{\sqrt{2}}{2} x + \omega\right] + b_{-1} \exp\left[-\frac{\sqrt{2}}{2} x - \omega\right] + \exp\left[\frac{\sqrt{2}}{2} x + \omega\right] - b_{-1} \exp\left[-\frac{\sqrt{2}}{2} x - \omega\right]}{\exp\left[\frac{\sqrt{2}}{2} x + \omega\right] + b_{-1} \exp\left[-\frac{\sqrt{2}}{2} x - \omega\right]},$$

or equivalently

$$v(x) = 1 + \frac{\exp\left[\frac{\sqrt{2}}{2} x + \omega\right] - b_{-1} \exp\left[-\frac{\sqrt{2}}{2} x - \omega\right]}{\exp\left[\frac{\sqrt{2}}{2} x + \omega\right] + b_{-1} \exp\left[-\frac{\sqrt{2}}{2} x - \omega\right]}. \tag{52}$$

If we choose $b_{-1} = 1, \omega = -\frac{\sqrt{2}}{2}\varepsilon$ in our solution Eq. (52) gives

$$v(x) = 1 + \tanh\left[\frac{\sqrt{2}}{2}(x - \varepsilon)\right].$$

So

$$u(x) = m \tanh\left[\frac{\sqrt{2}}{2}\left(m\sqrt{\frac{\beta}{\alpha}} x - \varepsilon\right)\right], \tag{53}$$

which this the solution gives the solution obtained in Elliott and French (1987). Fig. 4 depict the solution of Eq. (47), when $m = \alpha = \beta = 1$.

Case 2.

$$\begin{cases} a_{-1} = 0, & b_{-1} = 0, \\ a_0 = 2b_0, & b_0 = b_0, \\ a_1 = 0, & k = \sqrt{2}, \quad \omega = \omega. \end{cases} \tag{54}$$

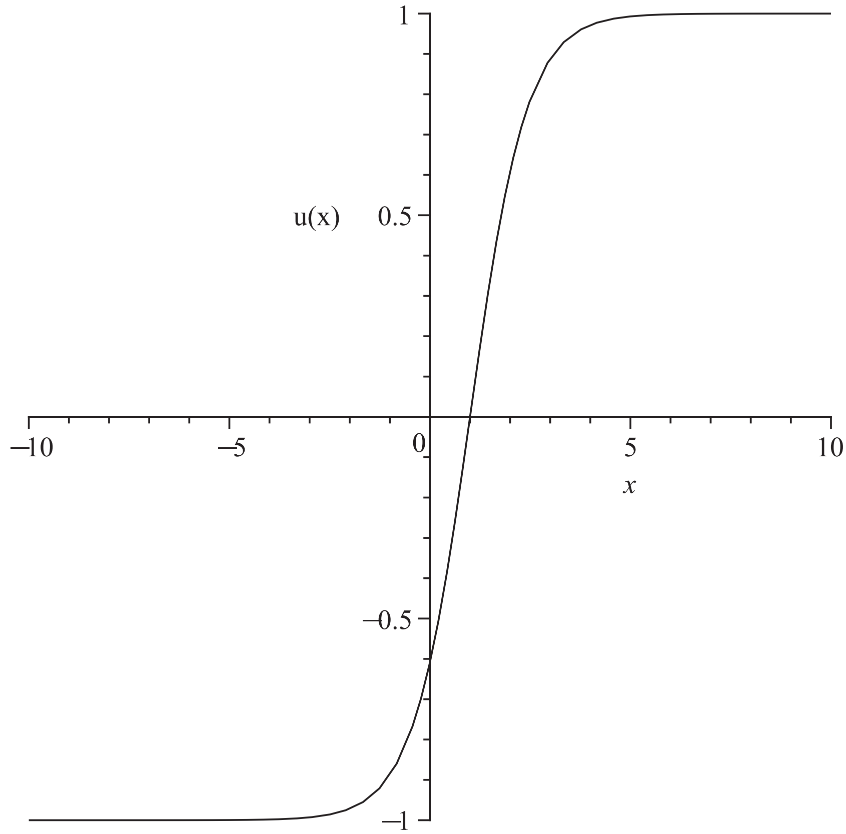

Figure 4 The solutions of Eq. (53) for the parameter value $m = \alpha = \beta = 1$.

Substituting these results into (49), we obtain

$$v(x) = \frac{2b_0}{\exp\left[\sqrt{2}x + \omega\right] + b_0}. \tag{55}$$

By the some manipulation in Eq. (55), we obtain

$$v(x) = \frac{b_0 + \exp\left[\sqrt{2}x + \omega\right] + b_0 - \exp\left[\sqrt{2}x + \omega\right]}{\exp\left[\sqrt{2}x + \omega\right] + b_0}.$$

Or equivalently

$$v(x) = 1 - \frac{\exp\left[\sqrt{2}x + \omega\right] - b_0}{\exp\left[\sqrt{2}x + \omega\right] + b_0}.$$

Or equivalently

$$v(x) = 1 - \frac{\exp\left[\frac{\sqrt{2}}{2}x + \frac{\omega}{2}\right] - b_0 \exp\left[-\frac{\sqrt{2}}{2}x - \frac{\omega}{2}\right]}{\exp\left[\frac{\sqrt{2}}{2}x + \frac{\omega}{2}\right] + b_0 \exp\left[-\frac{\sqrt{2}}{2}x - \frac{\omega}{2}\right]}. \tag{56}$$

If we choose $b_0 = 1, \omega = -\sqrt{2}\varepsilon$ in our solution Eq. (56) gives

$$v(x) = 1 - \tanh\left[\frac{\sqrt{2}}{2}(x - \varepsilon)\right].$$

So

$$u(x) = -m \tanh\left[\frac{\sqrt{2}}{2}(m\sqrt{\frac{\beta}{\alpha}}x - \varepsilon)\right],$$

which this the solution gives the solution obtained in Elliott and French (1987).

4. Conclusion

In this paper, the Exp-function method has been tested by applying it successfully to the Cahn–Hilliard equations, Allen–Cahn equation and Steady-State equation. In the Exp-function method, the free parameters may imply some physical meaningful results for the problem considered. The free parameters, of course, might be related to initial conditions as well. The performance of the Exp-function method is reliable and effective. Many methods (such as Adomian decomposition method (ADM), Homotopy analysis method (HAM) or variational iteration method (VIM)) can only obtain a special equation with special boundary (initial) conditions; some obtained solutions are physically meaningless. This paper obtains solutions with free parameters that can be determined via boundary (initial) conditions. In applications of Exp-function method in past decade common errors in finding exact solutions of nonlinear problems have been omitted (53). In this paper we present an application of this method with tackling these common errors. The solving procedure reveals that the exp-function method is a straightforward, succinct and promising tool for solving nonlinear partial differential equation and nonlinear ordinary differential equation.

Appendix A

$$\begin{aligned}&-\omega a_{-1}b_0b_{-1}^3 - k^4a_{-1}b_0b_{-1}^3 - 3k^2a_{-1}^2a_0b_{-1}^2 + 3k^2a_{-1}^3b_0b_{-1}\\&-k^2a_{-1}b_0b_{-1}^3 + ka_{-1}b_0b_{-1}^3 + \omega a_0b_{-1}^4 + k^4a_0b_{-1}^4 + k^2a_0b_{-1}^4\\&-ka_0b_{-1}^4 = 0,\end{aligned}$$

$$
\begin{aligned}
&-3ka_0b_{-1}^3b_0 - 3\omega a_{-1}b_0^2b_{-1}^2 + 3\omega a_0b_{-1}^3b_0 + 11k^4a_{-1}b_0^2b_{-1}^2\\
&\quad - 11k^4a_0b_{-1}^3b_0 - 12k^2a_{-1}a_0^2b_{-1}^2 - k^2a_{-1}b_0^2b_{-1}^2 + k^2a_0b_{-1}^3b_0\\
&\quad + 3ka_{-1}b_0^2b_{-1}^2 - 16k^4a_{-1}b_{-1}^3 + 12k^2a_{-1}^3b_{-1} + 4k^2a_1b_{-1}^4\\
&\quad - 4k^2a_{-1}b_{-1}^3 - 2ka_1b_{-1}^4 - 9k^2a_{-1}^3b_0^2 - 2\omega a_{-1}b_{-1}^3 + 16k^4a_1b_{-1}^4\\
&\quad + 2\omega a_1b_{-1}^4 + 2ka_{-1}b_{-1}^3 - 12k^2a_{-1}^2a_1b_{-1}^2 + 21k^2a_{-1}^2b_0a_0b_{-1}\\
&\quad = 0,
\end{aligned}
$$

$$
\begin{aligned}
&2\omega a_0b_{-1}^3 - 2ka_0b_{-1}^3 - 76k^4a_0b_{-1}^3 - 4k^2a_0b_{-1}^3 + 77k^4a_{-1}b_0b_{-1}^2\\
&\quad + 11k^2a_1b_0b_{-1}^3 - 7k^2a_{-1}b_0b_{-1}^2 - 7ka_1b_0b_{-1}^3 + 9ka_{-1}b_0b_{-1}^2\\
&\quad - 9\omega a_{-1}b_0b_{-1}^2 + 7\omega a_1b_0b_{-1}^3 - 9k^2a_0^3b_{-1}^2 - 33k^2a_{-1}^3b_0\\
&\quad + 21k^2a_{-1}b_0a_0^2b_{-1} - 54k^2a_{-1}a_1b_{-1}^2a_0 + 21k^2a_{-1}^2a_1b_{-1}b_0\\
&\quad - k^4a_1b_0b_{-1}^3 - 3ka_0b_{-1}^2b_0^2 - 3\omega a_{-1}b_0^3b_{-1} + 3\omega a_0b_{-1}^2b_0^2\\
&\quad - 11k^4a_{-1}b_0^3b_{-1} + 11k^4a_0b_{-1}^2b_0^2 - 12k^2a_0a_{-1}^2b_0^2\\
&\quad + 66k^2a_{-1}^2a_0b_{-1} + k^2a_{-1}b_0^3b_{-1} - k^2a_0b_{-1}^2b_0^2 + 3ka_{-1}b_0^3b_{-1}\\
&\quad = 0,
\end{aligned}
$$

$$
\begin{aligned}
&-36k^2a_{-1}^3 + 18k^2a_0a_1b_{-1}a_{-1}b_0 - 51k^2a_{-1}^2b_0a_0 + 84k^2a_{-1}^2a_1b_{-1}\\
&\quad - 3k^2a_1a_{-1}^2b_0^2 + 11k^2a_1b_0^2b_{-1}^2 - ka_0b_{-1}b_0^3 - 58k^4a_{-1}b_0^2b_{-1}\\
&\quad + 47k^4a_0b_{-1}^2b_0 - 6ka_1b_{-1}^3 - 4k^2a_{-1}b_{-1}^2 + 6ka_{-1}b_{-1}^2\\
&\quad + 4k^2a_1b_{-1}^3 - 176k^4a_1b_{-1}^3 - 6\omega a_{-1}b_{-1}^2 + 176k^4a_{-1}b_{-1}^2\\
&\quad + 6\omega a_1b_{-1}^3 - \omega a_{-1}b_0^4 + k^4a_{-1}b_0^4 + 3k^2a_0^3b_{-1}b_0 + 2k^2a_{-1}b_0^2b_{-1}\\
&\quad - 13k^2a_0b_{-1}^2b_0 - 9ka_1b_0^2b_{-1}^2 + 12ka_{-1}b_0^2b_{-1} - 3ka_0b_{-1}^2b_0\\
&\quad + 9\omega a_1b_0^2b_{-1}^2 - 12\omega a_{-1}b_0^2b_{-1} + 3\omega a_0b_{-1}^2b_0 + 11k^4a_1b_0^2b_{-1}^2\\
&\quad + \omega a_0b_{-1}b_0^3 - k^4a_0b_{-1}b_0^3 - 48k^2a_1b_{-1}^2a_0^2 + 84k^2a_{-1}a_0^2b_{-1}\\
&\quad - 48k^2a_{-1}a_1^2b_{-1}^2 - 3k^2a_0^2a_{-1}b_0^2 - k^2a_0b_{-1}b_0^3 + k^2a_{-1}b_0^4\\
&\quad + ka_{-1}b_0^4 = 0,
\end{aligned}
$$

$$
\begin{aligned}
&15ka_{-1}b_0b_{-1} + 5\omega a_1b_0^3b_{-1} + 15\omega a_1b_0b_{-1}^2 - 15ka_1b_0b_{-1}^2\\
&\quad - 10k^4a_0b_{-1}b_0^2 + 5k^2a_1b_0^3b_{-1} + 5k^2a_1b_0b_{-1}^2 + 5k^2a_{-1}b_0b_{-1}\\
&\quad - 10k^2a_0b_{-1}b_0^2 - 5ka_1b_0^3b_{-1} - 115k^4a_{-1}b_0b_{-1} - 75k^2a_0a_1^2b_{-1}^2\\
&\quad - 5\omega a_{-1}b_0^3 + 5k^4a_{-1}b_0^3 + 230k^4a_0b_{-1}^2 + 30k^2a_0^3b_{-1}\\
&\quad - 75k^2a_0a_{-1}^2 - 15k^2a_1a_{-1}^2b_0 - 15k^2a_{-1}a_0^2b_0\\
&\quad + 180k^2a_0a_1b_{-1}a_{-1} - 15k^2a_1^2b_{-1}a_{-1}b_0 - 15k^2a_1b_{-1}a_0^2b_0\\
&\quad + 5k^2a_{-1}b_0^3 + 5ka_{-1}b_0^3 - 10k^2a_0b_{-1}^2 - 15\omega a_{-1}b_0b_{-1}\\
&\quad + 5k^4a_1b_0^3b_{-1} - 115k^4a_1b_0b_{-1}^2 = 0,
\end{aligned}
$$

$$
\begin{aligned}
&47k^4a_0b_{-1}b_0 - 3k^2a_1^2b_0^2a_{-1} + 2k^2a_1b_0^2b_{-1} - 13k^2a_0b_{-1}b_0\\
&\quad - 12ka_1b_0^2b_{-1} + 3ka_0b_{-1}b_0 + 12\omega a_1b_0^2b_{-1} - 3\omega a_0b_{-1}b_0\\
&\quad - 58k^4a_1b_0^2b_{-1} + 84k^2a_1^2b_{-1}a_{-1} + 84k^2a_1b_{-1}a_0^2 + 9ka_{-1}b_0^2\\
&\quad + \omega a_1b_0^4 - 176k^4a_{-1}b_{-1} - 9\omega a_{-1}b_0^2 - 6ka_1b_{-1}^2\\
&\quad + 11k^2a_{-1}b_0^2 - 4k^2a_1b_{-1}^2 - 51k^2a_1^2b_0a_0b_{-1}\\
&\quad + 18k^2a_0a_1b_0a_{-1} - 3k^2a_0^2a_1b_0^2 + 4k^2a_{-1}b_{-1} + 6ka_{-1}b_{-1}\\
&\quad + 176k^4a_1b_{-1}^2 + 11k^4a_{-1}b_0^2 - \omega a_0b_0^3 - k^4a_0b_0^3 - 6\omega a_{-1}b_{-1}\\
&\quad + 6\omega a_1b_{-1}^2 + k^4a_1b_0^4 - 36k^2a_1^3b_{-1}^2 + 3k^2a_0^3b_0 - 48k^2a_1a_{-1}^2\\
&\quad - 48k^2a_{-1}a_0^2 - k^2a_0b_0^3 - ka_1b_0^4 + ka_0b_0^3 + k^2a_1b_0^4 = 0,
\end{aligned}
$$

$$
\begin{aligned}
&-76k^4a_0b_{-1} + 3ka_0b_0^2 - 3\omega a_0b_0^2 - 7\omega a_{-1}b_0 + 11k^2a_{-1}b_0\\
&\quad - k^4a_{-1}b_0 - 11k^4a_1b_0^3 + 3\omega a_1b_0^3 + 7ka_{-1}b_0 + 11k^4a_0b_0^2\\
&\quad + k^2a_1b_0^3 - 9ka_1b_0b_{-1} - 7k^2a_1b_0b_{-1} + 9\omega a_1b_0b_{-1}\\
&\quad + 77k^4a_1b_0b_{-1} - 33k^2a_1^3b_0b_{-1} + 21k^2a_1^2b_0a_{-1} - 54k^2a_1a_{-1}a_0\\
&\quad + 21k^2a_1b_0a_0^2 - 12k^2a_0a_1^2b_0^2 + 66k^2a_1^2b_{-1}a_0 - 2\omega a_0b_{-1}\\
&\quad - 3ka_1b_0^3 + 2ka_0b_{-1} - k^2a_0b_0^2 - 4k^2a_0b_{-1} - 9k^2a_0^3 = 0,
\end{aligned}
$$

$$
\begin{aligned}
&21k^2a_1^2b_0a_0 - 12k^2a_1^2a_{-1} + 12k^2a_1^3b_{-1} - 4k^2a_1b_{-1} - 2ka_1b_{-1}\\
&\quad + 3\omega a_1b_0^2 - 3\omega a_0b_0 + 11k^4a_1b_0^2 - 11k^4a_0b_0 - 12k^2a_0^2a_1\\
&\quad - 9k^2a_1^3b_0^2 - k^2a_1b_0^2 + k^2a_0b_0 - 3ka_1b_0^2 + 3ka_0b_0 - 16k^4a_1b_{-1}\\
&\quad + 2\omega a_1b_{-1} - 2\omega a_{-1} + 2ka_{-1} + 16k^4a_{-1} + 4k^2a_{-1}\\
&\quad = 0,
\end{aligned}
$$

$$
\begin{aligned}
&k^2a_0 - \omega a_0 + k^4a_0 + ka_0 + \omega a_1b_0 - k^4a_1b_0 + 3k^2a_1^3b_0\\
&\quad - 3k^2a_0a_1^2 - k^2a_1b_0 - ka_1b_0 = 0.
\end{aligned}
$$

Appendix B

$$
\begin{aligned}
&a_{-1}b_{-1}^2 - a_{-1}^3 = 0,\\
&\quad 2a_{-1}b_0b_{-1} + k^2a_0b_{-1}^2 - \omega a_0b_{-1}^2 - 3a_0a_{-1}^2 + a_0b_{-1}^2 - k^2a_{-1}b_0b_{-1}\\
&\qquad + \omega a_{-1}b_0b_{-1} = 0,\\
&\quad 2a_0b_{-1}b_0 + k^2a_{-1}b_0^2 + \omega a_{-1}b_0^2 + 4k^2a_1b_{-1}^2 - 4k^2a_{-1}b_{-1}\\
&\qquad - 2\omega a_1b_{-1}^2 + 2\omega a_{-1}b_{-1} - 3a_0^2a_{-1} + a_{-1}b_0^2 - 3a_1a_{-1}^2\\
&\qquad + 2a_{-1}b_{-1} + a_1b_{-1}^2 - k^2a_0b_{-1}b_0 - \omega a_0b_{-1}b_0 = 0, 2a_1b_0b_{-1}\\
&\qquad - 6k^2a_0b_{-1} + 3k^2a_{-1}b_0 + 3\omega a_{-1}b_0 - 6a_1a_0a_{-1}\\
&\qquad + 3k^2a_1b_0b_{-1} - 3\omega a_1b_0b_{-1} + a_0b_0^2 + 2a_0b_{-1} + 2a_{-1}b_0 - a_0^3 = 0,\\
&\quad k^2a_1b_0^2 - k^2a_0b_0 - \omega a_1b_0^2 + \omega a_0b_0 - 2\omega a_1b_{-1} - 4k^2a_1b_{-1}\\
&\qquad + a_1b_0^2 + 2a_1b_{-1} + 2\omega a_{-1} - 3a_1a_0^2 + 2a_0b_0\\
&\qquad - 3a_1^2a_{-1} + a_{-1} + 4k^2a_{-1} = 0,\\
&\qquad - k^2a_1b_0 - \omega a_1b_0 + \omega a_0 + k^2a_0 - 3a_1^2a_0 + 2a_1b_0 + a_0 = 0,\\
&\qquad - a_1^3 + a_1 = 0.
\end{aligned}
$$

Appendix C

$$
\begin{aligned}
&-a_{-1}^3 - 2a_{-1}b_{-1}^2 + 3a_{-1}^2b_{-1} = 0,\\
&\quad - k^2a_{-1}b_0b_{-1} + k^2a_0b_{-1}^2 + 6a_0a_{-1}b_{-1} - 4a_{-1}b_0b_{-1}\\
&\quad + 3a_{-1}^2b_0 - 3a_0a_{-1}^2 - 2a_0b_{-1}^2 = 0,\\
&\quad - 4a_0b_{-1}b_0 + 6a_0a_{-1}b_0 + 3a_{-1}^2 + 4k^2a_1b_{-1}^2 - 4k^2a_{-1}b_{-1}\\
&\quad + 6a_1a_{-1}b_{-1} + k^2a_{-1}b_0^2 - 2a_{-1}b_0^2\\
&\quad - 3a_0^2a_{-1} + 3a_0^2b_{-1} - k^2a_0b_{-1}b_0\\
&\quad - 3a_1a_{-1}^2 - 2a_1b_{-1}^2 - 4a_{-1}b_{-1} = 0,\\
&\quad - a_0^3 - 4a_1b_0b_{-1} + 3k^2a_{-1}b_0\\
&\quad - 6k^2a_0b_{-1} - 6a_1a_0a_{-1}\\
&\quad + 6a_1a_0b_{-1} + 6a_1a_{-1}b_0 + 6a_0a_{-1}\\
&\quad - 4a_{-1}b_0 + 3a_0^2b_0 - 2a_0b_0^2\\
&\quad - 4a_0b_{-1} + 3k^2a_1b_0b_{-1} = 0,
\end{aligned}
$$

$$3a_0^2 - k^2a_0b_0 + 6a_1a_0b_0 - 2a_{-1} - 4k^2a_1b_{-1} + k^2a_1b_0^2 - 2a_1b_0^2 - 4a_0b_0 - 3a_1a_0^2 + 3a_1^2b_{-1} + 6a_1a_{-1} - 4a_1b_{-1} + 4a_{-1}k^2 - 3a_1^2a_{-1} = 0,$$
$$-2a_0 - k^2a_1b_0 - 4a_1b_0 + k^2a_0 - 3a_1^2a_0 + 3a_1^2b_0 + 6a_1a_0 = 0, 3a_1^2 - 2a_1 - a_1^3 = 0.$$

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