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$QTAG$-modules isomorphic to their fully invariant submodules

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ABSTRACT

Some algebraic structures are isomorphic to their substructures but it is not always true. Some times they are isomorphic to their substructures with certain properties. Grinshpon et. al. investigated abelian groups which are isomorphic to their subgroups with certain properties.This interesting fact motivates us to investigate *QTAG*-modules which are isomorphic to their proper submodules with special conditions. Here we study *If*-modules which are isomorphic to their fully invariant submodules. We define admissible sequence of the *Ulm-Kaplansky* invariants to define *If*-module and investigate their properties.

1. Introduction

Some basic definitions used in this paper have also been used in the works of one of the co-authors and these are presented as quotations and referred appropriately here.

"All the rings R considered here are commutative with unity and the modules M are unital $QTAG$-modules. An element $x \in M$ is uniform, if xR is a nonzero uniform (hence uniserial) module and for any R-module M with a unique decomposition series $d(M)$ denotes the decomposition length. For a uniform element x in $M, e(x) = d(xR)$ and $H_M(x) = \sup\{d(yR/xR)|\ y \in M, x \in yR$ and y uniform$\}$ are the exponent and height of x in M respectively. $H_k(M)$ denotes the submodule of M generated by the uniform elements of height at least k and M is h-divisible if M= $M^1 = \bigcap_{k=0}^{\infty} H_k(M) = H_\omega(M)$ and *h-reduced* if it does not contain any h-divisible submodule. In other words, if it is free from the elements of infinite height." (Mehdi et al., 2014)

"A submodule $N \subset M$ is said to be high if it is a complement of M^1 *i.e* $M = N \oplus M^1$. A submodule N of M is h-pure in M if $H_k(N) = N \cup H_k(M)$ for every $k = 0, 1, 2, \ldots, \infty$. The sum of all simple submodule of M is called the *socle* of M and is denoted by $Soc(M)$.

A $QTAG$-module M is said to be separable if every finite set $\{x_1, x_2, \ldots, x_n\} \subset M$, can be embedded in a direct summand K of M, which is a direct sum of uniserial modules.

The set of modules $\{H_k(M)\}_{k=0,1,\ldots,\infty}$ forms a base for the neighbourhood system of zero. This gives rise to a topology known as h-topology. The closure of a submodule $N \subset M$ is defined as $\overline{N} = \bigcap_{k=0}^{\infty}(N + H_k(M))$ and it is complete with respect to h-topology if $N = \overline{N}$ and N is h-dense in M if $\overline{N} = M$.

The cardinality of the minimal generating set of M is denoted by $g(M)$ and fin$g(M)$ is defined as the infimum of $g(H_k(M))$ for $k = 0, 1, 2, \ldots, \infty$. For all ordinal σ, the *σ-th Ulm-Kaplansky invariant* of $M, f_M(\sigma)$ is the cardinality of $g(Soc(H_\sigma(M))/Soc(H_{\sigma+1}(M)))$.

A submodule B of M is called a *basic submodule* of M, if B is a h-pure submodule of M, B is a direct sum of uniserial modules (we will abbreviated it as DSUM) and M/B is a direct sum of uniform modules of infinite length *i.e.* M/B is h-divisible." (Mehdi et al., 2016).

By closed QTAG-module M, we mean those modules which do not have any element of infinite height and has a limit in M for every Cauchy sequence.[5].

"A submodule N of a $QTAG$ module is fully invariant(characteristic) submodule if every endomorphism(automorphism)f of M maps N into N.

M is a HT-module if every homomorphism from M to N is small whenever N is DSUM. Equivalently, M is a HT-module if and only if

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$N \supset Soc(H_k(M))$ for some $k < \omega$ whenever M/N is a DSUM (Mehdi et al., 2015). A *QTAG*- module M is $(\omega + n)$- *projective*, if there exists a submodule $N \subset H^n(M)$ such that M/N is a DSUM" (Mehdi et al., 2006). The terminology is followed by Fuchs (1970) and Fuchs (1973).

2. Main results

We investigate *QTAG*-modules that are isomorphic to their direct summands/ *h*-pure submodules/fully invariant submodules. Among the fully invariant submodules of *QTAG*-modules, large submodules are very significant. In fact the fully invariant submodules of a *QTAG*-module Mwhich are not bounded are said to be the large submodules of M. Moreover if M is a *QTAG*-module which is the direct sum of the modules of length i and let K be a submodule of M, then $K = H_{n_i}(M)$ where $n_i \leqslant i$. If $K = 0$ then $n_i = i$ and if $K \neq 0$ then $n_i = min\{H(x)|x \in K\}$.

We begin this section with some definitions as follows:

Definition 1. A *QTAG*-module is *Ihp-module* provided it is isomorphic to a proper *h-pure* submodule and it is an *Id-module* in case it is isomorphic to its proper direct summand.

Remark 1. If a *QTAG*-module,M is *h-reduced* and also $\frac{M}{H_1(M)}$ is finitely generated then M is not an *Id-module*.

Definition 2. A *QTAG*-module is said to be an *If-module* provided it is isomorphic to a proper submodule which is fully invariant.

It is necessary to mention that if M_i and M_{i+j} are *QTAG*-modules of lengths i and $i+j$ respectively and $x \in M_i$, $y \in M_{i+j}$ then there exists a homomorphism $f : M_i \longrightarrow M_{i+j}$ such that $f(x) = y$ if and only if $e(x) \geqslant e(y)$ and there exists a homomorphism $f\prime : M_{i+j} \longrightarrow M_i$ such that $f\prime(y) = x$ if and only if $H(x) \geqslant H(y)$.

To study *If*-modules we generalize a result of Benabdallah et al. (1970) for *QTAG*-modules.

Theorem 1. *Let $M = \bigoplus_i M_i$, where each M_i is the DSUM of length i. Then N is fully invariant submodule of M if and only if $N = \bigoplus_i H_{n_i}(M_i)$ where $n_i \leqslant i$, for every $i \in \mathbb{Z}^+$ and $n_i \leqslant n_{i+j} \leqslant n_i + j$ for $i,j \in \mathbb{Z}^+$. A fully invariant submodule N is large in M if and only if $N = \bigoplus_i H_{n_i}(M_i)$, the above conditions hold and the sequence $< 1 - n_1, 2 - n_2, 3 - n_3, \ldots. >$ is unbounded if M itself is not bounded.*

Proof. Let N be a fully invariant submodule of M. Then $N = N \cap M = \bigoplus(M_i \cap N) = \bigoplus H_{n_i}(M_i)$. Now $n_i \leqslant i$ for all $i \in \mathbb{Z}^+$ and the first condition holds.

If $N = 0$, then $H_{n_i}(M_i) = 0$ for every i, therefore $n_i = i$ for every i and the second condition holds.

If $N \neq 0$, then a least positive integer k exists such that $H_{n_k}(M_k) \neq 0$. We claim that $H_{n_i}(M_i) \neq 0$ for all $i \geqslant k$ where $M_i \neq 0$. Since $Soc(M_k) = Soc(H_{k-1}(M_k)) \subseteq N$ implies that $Soc(H_{k-1}(M)) \subseteq Soc(N)$. Also $Soc(M_i) = Soc(H_{k-1}(M_i))$ for all $i \geqslant k$, we have $Soc_{(M_i) \subseteq N \cap M_i = H_{n_i}(M_i)}$ and the assertion follows.

Now suppose $N \neq 0$ and $M_i \neq 0 \neq M_{i+j}$. If $H_{n_{i+j}}(M_{i+j}) = 0$ then $H_{n_i}(M_i) = 0$ and $n_i = i, n_{i+j} = i+j = n_i + j$ and the second condition holds.

Therefore we assume that $H_{n_{i+j}}(M_{i+j}) \neq 0$. Consider $x \in M_i$ such that $H(x) \geqslant n_{i+j}$. $y \in H_{n_{i+j}}(M_{i+j})$ such that $H(y) = n_{i+j}$. Now there exists an endomorphism f of M which maps y onto x. Hence $x \in N$ and $H_{n_{i+j}}(M_i) \subseteq N \cap M_i = H_{n_i}(M_i)$. Thus, $n_i \leqslant n_{i+j}$.

Now suppose $H_{n_i}(M_i) = 0$. Then $n_i = i$ so $n_{i+j} \leqslant i + j = n_i + j$.

If $H_{n_i}(M_i) \neq 0$ and $y \in M_{i+j}$ such that $H(y) \geqslant n_i + j$. We may choose $x \in M_i$ such that $H(x) = n_i$. Then $e(x) = i - n_i$ and $e(y) \leqslant i + j - (n_i + j) = i - n_i$.

Again there exists an endomorphism f of M with $f(x) = y$. Thus $y \in N$ and we have $H_{n_{i+j}}(M_{i+j}) \subseteq N \cap M_{i+j} = H_{n_{i+j}}(M_{i+j})$, therefore $n_{i+j} \leqslant n_i + j$.

If $M_i \neq 0 \neq M_{i+j}$, then $n_i \leqslant n_{i+j} \neq n_i + j$ but if $M_i = 0$, we may define n_i so that this inequality holds for all i. Thus all fully invariant submodules of M are the direct sums of $H_{n_i}(M_i)$. If N is a large submodule of M and M is unbounded, N is also unbounded. Therefore $< 1 - n_1, 2 - n_2, 3 - n_3, \ldots.. >$ must be unbounded.

For the converse, suppose $N = \bigoplus H_{n_i}(M_i)$ where $n_i \leqslant i$ for all $i \in \mathbb{Z}^+$ and $n_i \leqslant n_{i+j} \leqslant n_i + j$ for all $i,j \in \mathbb{Z}^+$. To establish the fully invariance of N, we consider any $i \in \mathbb{Z}^+$ and $x \in H_{n_i}(M_i)$. We have to show that for any endomorphism f of $M, f(x) \in N$.

Consider $x \neq 0$, such that $f(x) = x_1 + x_2 + \ldots.. + x_3$ where $x_r \in M_r$ and $H(x) \leqslant H(f(x)) = min(H(x_k)), 1 \leqslant k \leqslant l, e(x) \geqslant e(f(x))$=max $(e(x_k)|1 \leqslant k \leqslant l)$. If $k \leqslant i$, then $H(x_k) \geqslant H(x) \geqslant n_i$ so $x_k \in H_{n_i}(M_k) \subseteq H_{n_k}(M_k)$, because $n_k \leqslant n_i$, hence $x_k \in N$. If $k = i + j$ then $e(x_k) \leqslant e(x) \leqslant i - n_i = i + j - (n_i + j) \leqslant i + j - n_{i+j}$ because $n_{i+k} \leqslant n_i + k$. Thus $x_k \in H^{(i+j-n_{i+j})}(M_{i+j}) = H_{n_{i+j}}(M_{i+j}) \subseteq N$.

This implies that N is a fully invariant submodule of M. If M is unbounded and $< 1 - n_1, 2 - n_2, 3 - n_3, \ldots. >$ is also unbounded, then N is unbounded and is therefore a large submodule of M.

We are now able to generalize the result of Grinshpon and Nikolskya (2014) for *QTAG*-modules along with some more results.

Theorem 2. *A bounded QTAG-module can not be an If*-module.

Proof. Let M be a bounded *QTAG*-module with $H(x) < k$ for every element x in M. Then M can be expressed as $\bigoplus_{i=1}^{k} M_i$ where each M_i is a DSUM of length i. If N is a fully invariant submodule of M, then by Theorem 1, $N = H_{n_1}(M_1) \oplus H_{n_2}(M_2) \oplus \ldots \oplus H_{n_k}(M_k)$ such that $n_j \leqslant j, n_j \leqslant n_{j+l} \leqslant n_j + l$, for all $j, l \in N$.

If $n_k = 0$ then $n_1 \leqslant n_2 \leqslant \ldots. \leqslant n_k = 0$ implies $N = M_1 \oplus M_2 \oplus \ldots.. \oplus M_k = M$, thus N is not a proper submodule of M. With no loss of generality, we assume that $n_k \geqslant 1$. Now $H_{n_k}(M_k)$ is a direct sum of uniserial modules of length $k - n_k$ which implies that no direct summand of N is a uniserial module of length k. Therefore N is not isomorphic to M.

Lemma 1. *Let $M = \bigoplus_{i \in I} M_i$ be a QTAG-module. If $N = \bigoplus_{i \in I} N_i, K = \bigoplus_{i \in I} K_i$ such that N_i and K_i are the submodules of $M_i, \forall i$ then $N_i = K_i$ for all $i \in I$.*

Proof. Let $x_i \in N_i$. Then $x_i \in N = \bigoplus K_i$ and we may write $x_i = y_{i1} + y_{i2} + \ldots + y_{ik}$ where $y_{ij} \in K_j, j = 1, 2, \ldots, k$. Since N_i and K_i are the submodules of $M_i, x_i \in M_i, y_{ij} \in M_{ij}$ and M is a direct sum of M_i's, $x_i = y_{ij}$ for some j. In fact $y_{ij} = x_i$ and $y_{ik} = 0$ if $j \neq k$. Therefore $x_i \in K_i$ or $N_i \subseteq K_i$. Similarly $K_i \subseteq N_i$ and $N_i = K_i$ for all i.

Theorem 3. *Consider a QTAG-module $M = \bigoplus M_i$ with each M_i fully invariant. M is an If*-module *if and only if there exists at least one M_i which is an If*-module.

Proof. Let us consider N as a proper submodule of M which is fully invariant such that $M \simeq N$. Since N is fully invariant, $N = \bigoplus (N \cap M_i) = \bigoplus N_i$ where $N_i = N \cap M_i$. Let $f : M \longrightarrow N$ be an isomorphism. Then f_i, the restriction of f on M_i is an endomorphism of M_i. For $x \in M, x = x_{i1} + \ldots + x_{ik}$ where $x_{ij} \in M_{ij}$ and we have $f(x) = f_{i1}(x_{i1}) + \ldots + f_{ik}(x_{ik})$. Thus $N = f(M) = \bigoplus_{i \in I} f_i(M_i) = \bigoplus N_i$ and by Lemma 1, $f_i(M_i) = N_i$. Since $Ker(f) = 0, Ker(f_i) = 0 \forall i \in I$ and each f_i is also an isomorphism from M_i onto N_i. As $M \neq N, M_{il} \simeq N_{il}$ and $M_{il} \neq N_{il}$ for some il implying that M_{il} is an *If-module*.

Conversely, consider $M = \bigoplus M_i$ where each M_i is a fully invariant submodule of M. Let M_{i_0} be an *If-module* for some $i_0 \in I$. Therefore there exists a submodule N_{i_0} of M_{i_0} which is proper and invariant such that $N_{i_0} \simeq M_{i_0}$. Consider $N = N_{i_0} \oplus \left(\bigoplus_{j \neq i_0} M_j \right)$. Since $N_{i_0} \neq M_{i_0}, N$ is a proper submodule of M. As $N_{i_0} \simeq M_{i_0}$, we have $N = N_{i_0} \oplus \left(\bigoplus_{j \neq i_0} M_j \right) \simeq M_{i_0} \oplus \left(\bigoplus_{j \neq i_0} M_j \right) = \bigoplus M_i = M$ *i.e* $N \simeq M$. Now consider an arbitrary endomorphism f of M and $x \in N$. Now $x = x_{i_0} + y_{i_1} + \ldots + y_{i_k}$ where $y_{ij} \in M_{ij}, x_{i_0} \in N_{i_0}$. We have $f(x) = f\left(x_{i_0} + y_{i_1} + \ldots + y_{i_k}\right) = f(x_{i_0}) + f(y_{i_1}) + \ldots + f\left(y_{i_k}\right)$. Since M_{ij}'s are fully invariant $f\left(y_{ij}\right) \in M_{ij}$. Also N_{i_0} is fully invariant in M_{i_0} which is fully invariant in M, thus $f(x_{i_0}) \in N_{i_0}$. Now we have $f(x) \in N \forall x \in N$, therefore N is fully invariant in M. Since N is a proper submodule of M with $N \simeq M, M$ is an *If-module* and we are done.

For a *QTAG*-module M, the k^{th} *Ulm-Kaplansky* invariant $f_M(k)$ is defined as the cardinality of the minimal generating set of $\frac{Soc(H_k(M))}{Soc(H_{k+1}(M))}$. This can be expressed as $f_M(k) = g\left(\frac{Soc(H_k(M))}{Soc(H_{k+1}(M))}\right)$. We study *If-modules* in the light of *Ulm-Kaplansky invariants*. We start with the following definition.

Definition 3. Let M be a separable *QTAG*-module. A strictly increasing sequence of positive integers $i_0, i_1, i_2, \ldots, i_n, \ldots$ is said to be *admissible* for M if $f_M(k) = \sum_{i=l_k}^{k+1} f_M(i), k < \omega$.

Theorem 4. *Let M be an unbounded QTAG-module with M is a direct sum of uniserial modules and all Ulm-Kaplansky* invariants *of M are finite. Then M is not an If*-module *if and only if there exists only one admissible sequence for M, consisting of all non negative numbers.*

Proof. We may express $M = \bigoplus_{k \in N} M_k$ where each M_k is a DSUM of length k. Thus $f_M(i) = g(M_{i+1})$ for each $i \in \mathbb{Z}^+$. Suppose M is not an *If-module* which has an admissible sequence $l_0, l_1, \ldots$ different from $0, 1, 2, \ldots$. Since the *Ulm-Kaplansky* invariants of M are finite we have two cases. Either $l_0 \neq 0$ or $l_0 = 0$. If $l_0 \neq 0$ consider the submodule N of M such that

$$\begin{aligned} N = {} & H_1(M_1) \oplus H_2(M_2) \oplus \ldots \oplus H_{l_0}(M_{l_0+1}) \oplus H_{l_0+1}(M_{l_0+2}) \oplus \ldots \\ & \oplus H_{l_1-1}(M_{l_1}) \oplus H_{l_1-1}(M_{l_1+1}) \oplus H_{l_1}(M_{l_1+2}) \oplus \ldots \oplus H_{l_2-2}(M_{l_2}) \\ & \oplus H_{l_2-2}(M_{l_2+1}) \oplus \ldots \oplus H_{i_2}(M_{l_2+3}) \oplus \ldots \oplus H_{i_3-3}(M_{i_3}) \oplus \ldots \end{aligned}$$

.

In short we may say that $N = \bigoplus H_{n_k}(M_k)$ where $n_{l_j} = n_{l_{j+1}} = l_j - j, n_{l_{j+m}} = l_j - j + m - 1$. Now N is a proper submodule of M. By Theorem 1, N is fully invariant in M. Since *Ulm-Kaplansky* invariants of M and N are equal $M \simeq N$ because $f_N(m) = f_M(l_m) + f_M(l_{m+1}) + \ldots + f_M(l_{m+1} - 1) = f_M(m)$ for each $m \in \mathbb{Z}^+$. Therefore $N \simeq M$ but $N \neq M$ and thus M is an *If-module*. This contradiction proves our assertion. If $l_0 = 0$, we have a least natural number j such that $l_{j+1} > j + 1$. Now $l_0 = 0, l_1 = 1, \ldots l_j = j$ and the admissible sequence has the form $0, 1 \ldots j, l_{j+1}, l_{j+2} \ldots$.For these sequences we may write

$$\begin{aligned} f_M(0) = {} & f_M(0) \\ f_M(j-1) = {} & f_M(j-1) \\ f_M(j) = {} & f_M(j) + f_M(j+1) + \ldots + f_M(j_m - 1) \\ f_M(k) = {} & f_M(l_k) + f_M(l_{k+1}) + \ldots + f_M(l_{k+1} - 1) \end{aligned}$$

Now the sum of the right hand side of the last inequality consists of more than one summand.

We consider

$$\begin{aligned} N = {} & M_1 \oplus M_2 \oplus \ldots \oplus M_{k+1} \oplus H_1(M_{k+2}) \oplus \ldots \\ & H_{l_{j+1}-j-1}\left(M_{l_{j+1}}\right) \oplus H_{l_{j+1}-j-1}\left(M_{l_{j+1}} + 1\right) \oplus \ldots \end{aligned}$$

By Theorem 1, N is fully invariant submodule of M. Now

$$\begin{aligned} f_N(0) = {} & f_M(0) \\ f_N(1) = {} & f_M(1) \\ f_N(j-1) = {} & f_M(j-1) \\ f_N(j) = {} & f_M(j) + f_M(j+1) + \ldots + f_M(l_{j+1} - 1) \\ f_N(r) = {} & f_M(l_r) + f_M(l_{r+1}) + \ldots + f_M(l_{r+1} - 1), r > j \end{aligned}$$

These two set of equalities ensure that $N \simeq M$. Since $N \neq M, M$ is an *If-module*, again a contradiction.

To prove it conversely, consider a fully invariant submodule N of M. Now by Theorem 1, $N = \bigoplus_{k \in N} H_{n_k}(M_k)$ such that $n_k \leqslant k, n_k \leqslant n_{k+l} \leqslant n_k + l$ for all $k, l \in N$.

Now $f_N(n) = g\left(\bigoplus_{n \in N} (H_{n_k}(M_k)) \right)$, where $H_{n_k}(M_k)$ is a DSUM of length $(n+1)$.

Therefore

$$\begin{aligned} f_N(n) & = g(\oplus H_{n_k}(M_k)) \text{ where } k - n_k = n + 1 \\ & = \sum_{k \in N} f_M(k-1) \text{ suchthat } k - n_k - 1 = n \end{aligned}$$

Again by Theorem 1,

$(k+1) - n_{k+1} - 1 \geqslant (k+1) - (n_k + 1) - 1 = k - n_k - 1$

$(k+1) - n_{k+1} - 1 \leqslant (k+1) - n_k - 1 = (k - n_k - 1) + 1$

On putting $j_n = \min_{k \in N} \{k - 1 | k - n_k - 1 = n\}$, we have.

$f_N(n) = \sum_{j=j_n}^{j_{n+1}-1} f_M(j)$

If $N \simeq M$, then $f_M(n) = f_N(n) = \sum_{j=j_n}^{j_{n+1}-1} f_M(j)$ for any $n \in \mathbb{Z}^+$.

Now the sequence $i_0, i_1, \ldots, i_n, ..$ is admissible for M, therefore $j_n = n$ for any n. Since $j_n = \min_{k \in N} \{k - 1 | k - n_k - 1 = n\}$ we have $n_k = 0$ for any k or $M = N$ implying that M is not an *If-module*. Hence proved.

The basic submodules of *QTAG*-modules are significant and we prove the following:

Theorem 5. *A separable QTAG-module M is not an If*-module *if its basic submodule is not an If-module.*

Proof. Let B be the basic submodule of the separable *QTAG*-module such that B is not an *If-module*. Without loss of generality we assume that M is *h-reduced*. If M is bounded then by Theorem 2, M is not an *If-module*.

Suppose M is unbounded *If-module*. Then there exists a proper fully invariant submodule N of M such that $N \simeq M$. Since M is *h-reduced*, so there is no element whose height is infinite. N is unbounded fully invariant submodule of M, therefore $N \cap B$ is a basic submodule of N. If $N \cap B = 0$, then N is *h-divisible* because $\frac{N+B}{B}$ is a submodule of a *h-divisible* module. Since M is *h-reduced*, it is not possible.

If $N \cap B = B$ then N is a large submodule of M and $B \subset N$ implies $N + B = N$ which is not possible as N is a proper submodule of M. Now we may conclude that $N \subset B$ is a proper submodule of M. Since $N \simeq M$, the basic submodule of M and N are isomorphic (Mehdi and Khan, 1984) and $N \cap B \simeq B$. As N is a large submodule of $M, N \cap B$ is a large submodule of B (Mehdi et al., 2014). Thus we obtain that the basic submodule B of M has a proper fully invariant submodule $N \cap B$, is isomorphic to B which is a contradiction proving the result.

Theorem 6. *Let M be a separable unbounded QTAG-module with finite Ulm – Kaplansky invariants. Then it is not an If*-module *if there exists only one admissible sequence for it consisting of all non negative integers.*

Proof. Let B be the basic submodule of a separable *QTAG*-module with finite Ulm-Kaplansky invariants. Let $0, 1, 2\ldots$ be the only admissible sequence for M. We have to show that B is not an *If-module*. Since B is a DSUM and $f_M(k) = f_B(k)$, by Theorem 4, B is not an *If-module* and the result follows.

Following is an immediate consequence of this result.

Corollary 1. *An unbounded separable QTAG- module is not an If-*module *if its Ulm – Kaplansky invariants are finite and form an increasing sequence.*

Proof. Let M be an unbounded separable *QTAG*-module. Suppose the sequence of Ulm-Kaplansky invariants of M is increasing and all the invariants are finite. Consider an admissible sequence $i_o, i_1, \ldots, i_n$ for M. Now

$f_M(k) = \sum_{i=i_k}^{i_{k+1}-1} f_M(i) = f_M(i_k) + f_M(i_k + 1) + \ldots + f_M(i_{k+1} - 1)$,

where k is an arbitrary non negative integer. Since the sequence $\{f_M(k)\}$ is increasing, $f_M(k) = f_M(i_k)$ and $i_k = k$ for every $k \in \mathbb{Z}^+$. Therefore the admissible sequence $i_o, i_1, \ldots, i_n$ coincides with $0, 1, 2, \ldots$ and by Theorem 6, M is not an *If-module*.

Mehdi and Khan (1984) studied closed modules which are significant in the study of *QTAG*-modules. The following results highlight the relation between *If-modules* and their basic submodules.

Theorem 7. *A closed QTAG-module M is an If*-module *if and only if its basic submodule B is an If*-module.

Proof. Let M be a closed *QTAG*-module and B its basic submodule which is an *If-module*. Now there exists a proper fully invariant submodule N of B such that $B \simeq N$. Now N is a proper large submodule of B. This N can be extended to $N\prime$, a proper large submodule of M such that $N\prime \cap B = N$. Now N is a basic submodule of $N\prime$. Since $N\prime$ is a large submodule of a closed module it is also closed. Now M has a proper fully invariant submodule $N\prime$ such that the basic submodule B of M is isomorphic to the basic submodule N of $N\prime$.

Now M and $N\prime$, both are closed therefore $M \simeq N\prime$ and M is an *If-module*. The converse is trivial.

Theorem 8. *Let M be a closed QTAG-module with finite Ulm-Kaplansky invariants. Then the following conditions are equivalent.*

(i) *M is not an If*-module.
(ii) *No basic submodule of M is an If*-module.
(iii) *There exists only one admissible sequence for M consisting of all nonnegative integers.*

Proof. Theorem 7 ensures that $(i) \Rightarrow (ii)$.

$(ii) \Rightarrow (iii)$. Let B be a basic submodule of M, and B is not an *If-module*. If M is bounded then $M = B$. Otherwise M is unbounded and B is also unbounded. Since $f_M(k) = f_B(k)$, by Theorem 4, we have that M has only one admissible sequence of the form $0, 1, 2, \ldots$

$(iii) \Rightarrow (i)$. If M is bounded then it is not an *If-module* by Theorem 2. If M is unbounded for which there exists only one admissible sequence of the form $0, 1, 2, \ldots$ then its basic submodule has the same property. Therefore by Theorem 4, B is not an *If-module*. Since (ii) and (i) are equivalent, M is not an *If-module*.

Following are the consequences of the above theorem.

Corollary 2. *Let M be a closed QTAG-module with finite Ulm-Kaplansky* invariants. *If for all* $n \in \mathbb{Z}^+, \exists k \in \mathbb{N}$ *such that* $f_M(n) = f_M(n + k), M$ *is an If*-module.

Proof. Let M be a closed *QTAG*-module with finite *Ulm-Kaplansky invariants*. Suppose there exists $k < \omega$ such that for all $n \in \mathbb{Z}^+, f_M(n) = f_M(n + k)$. Then the sequence $k, k + 1, k + 2, \ldots$ is admissible for this module and by Theorem 8, M is an *If-module*.

Corollary 3. *Let M be a closed QTAG-module such that there exists a* $m \in \mathbb{N}$ *such that* $f_M(n) = m \; \forall \; n \in \mathbb{Z}^+$, *then M is an If*-module.

Proof. Let M be a closed *QTAG*-module and $f_M(n) = m$ for every $n \in \mathbb{Z}^+$ where $m < \omega$. Now $f_M(n) = f_M(n + 1) \; \forall n \in \mathbb{Z}^+$ and the sequence $\{f_M(n)\}_{n<\omega}$ is *admissible* for M. Therefore by Corollary 2., M is an *If-module*.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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