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On boundedness and compactness of a generalized Srivastava–Owa fractional derivative operator

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Abstract The purpose of this present effort is to define a new fractional differential operator $\mathfrak{T}_z^{\beta,\tau,\gamma}$, involving Srivastava–Owa fractional derivative operator. Further, we investigate some geometric properties such as univalency, starlikeness, convexity for their normalization, we also study boundedness and compactness of analytic and univalent functions on weighted μ-Bloch space for this operator. The method in this study is based on the generalized hypergeometric function.

1. Introduction

The study of fractional operators (integral and differential) plays a vital and essential role in mathematical analysis. Recently, there is a flurry of activity to define generalized differential operators and study their basic properties in a loosely defined area of holomorphic analytic functions in open unit disk. Many authors generalized fractional differential operators on well known classes of analytic and univalent functions to discover and modify new classes and to investigate multi various interesting properties of new classes, for example (see Kiryakova et al., 1998; Dziok and Srivastava, 1999; Srivastava, 2007; Kiryakova, 2010).

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{\kappa=2}^{\infty} a_\kappa z^\kappa \tag{1.1}$$

which are analytic functions in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = 1 - f'(0) = 0$, and let $\mathcal{S}$ be the subclass of the $\mathcal{A}$ of the univalent functions in $\mathbb{U}$. Further, a function $f(z) \in \mathcal{S}$ is said to be starlike and convex of order $\lambda (0 \leqslant \lambda < 1)$, if and only if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \lambda \quad \text{and} \quad \Re\left\{\frac{zf''(z)}{f'(z)} + 1\right\} > \lambda,$$

respectively, these subclasses of $\mathcal{S}$ are denoted by $\mathcal{S}^*$ and $\mathcal{K}$.

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Theorem 1.1 (Bieberbach's Conjecture). *If the function $f(z)$ defined by (1.1) is in the class $\mathcal{S}^*$ then $|a_\kappa| \leqslant \kappa$ for all $\kappa \geqslant 2$ and if it is in the class $\mathcal{K}$ then $|a_\kappa| \leqslant 1$ for all $\kappa \geqslant 2$ (Duren, 1983).*

For $f(z)$ given by (1.1) and $g(z) = z + \sum_{\kappa=2}^{\infty} b_\kappa z^\kappa$, the convolution (or Hadamard product) $f * g$ is defined by

$$f * g(z) = z + \sum_{\kappa=2}^{\infty} a_\kappa b_\kappa z^\kappa. \tag{1.2}$$

The operator $\mathcal{O}_z^{\beta,\tau}$ is defined in terms of Riemann–Liouville fractional differential operator $\mathcal{D}_z^{\beta-\tau}$ as

$$\mathcal{O}_z^{\beta,\tau} f(z) = \frac{\Gamma(\tau)}{\Gamma(\beta)} z^{1-\tau} \mathcal{D}_z^{\beta-\tau} z^{\beta-1} f(z) \quad z \in \mathbb{U}. \tag{1.3}$$

This operator is given by Tremblay (1974). Recently, Ibrahim and Jahangiri (2014) extended Tremblay's operator in terms of Srivastava–Owa fractional derivative of $f(z)$ of order $(\beta-\tau)$ and is defined as follows

$$\mathfrak{T}_z^{\beta,\tau} f(z) = \frac{\Gamma(\tau)}{\Gamma(\beta)} z^{1-\tau} D_z^{\beta-\tau} z^{\beta-1} f(z) \tag{1.4}$$

Often, the generalized fractional differential operators and their applications associated with special functions, Dziok and Srivastava (1999), defined a linear operator as a Hadamard product with an arbitrary ${}_pF_q$-function $p \leqslant q+1$, several authors interested Dziok–Srivastava operator as well as Srivastava–Wright operator, which is defined and investigated by Srivastava (2007). Recently Kiryakova (2011), considered those operators and studied their criteria univalence properties in the class $\mathcal{A}$.

Definition 1.1. The Fox–Wright ${}_p\Psi_q$ generalization of the hypergeometric ${}_pF_q$ function is defined as:

$${}_p\Psi_q[z] = {}_p\Psi_q\left[\begin{matrix}(a_j, A_j)_{1,p} \\ (b_j, B_j)_{1,q}\end{matrix}\middle| z\right] = \sum_{\kappa=0}^{\infty} \frac{\Pi_{j=1}^{p}\Gamma(a_j + \kappa A_j)}{\Pi_{j=1}^{q}\Gamma(b_j + \kappa B_j)(1)_\kappa} z^\kappa, \tag{1.5}$$

where a_j, b_j are parameters in complex plane $\mathbb{C}$. $A_j > 0,\ B_j > 0$ for all $j = 1, \ldots, q$ and $j = 1, \ldots, p$, such that $0 \leqslant 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_i$ for fitting values $|z| < 1$ and it is well know that

$${}_p\Psi_q\left[\begin{matrix}(a_j, 1)_{1,p} \\ (b_j, 1)_{1,q}\end{matrix}\middle| z\right] = \Delta^{-1}{}_pF_q(a_j, b_j; z), \quad \text{with} \quad \Delta = \frac{\Pi_{j=1}^{q}\Gamma(b_j)}{\Pi_{j=1}^{p}\Gamma(a_j)} \tag{1.6}$$

where ${}_pF_q$ is the generalized hypergeometric function and $(v)_\kappa = \Gamma(v+\kappa)/\Gamma(v)$ is the *Pochhammer* symbol (see Kilbas et al., 2006).

Remark 1.1. By usage the Hadamard product technique, Srivastava (2007) provided families of analytic and univalent functions associated with the Fox–Wright generalized hypergeometric functions ${}_p\Psi_q$ in the open unit disk $\mathbb{U}$.

In the present paper the new generalized fractional differential operator $\mathfrak{T}_z^{\beta,\tau,\gamma}$ of analytic function is defined. Also, the univalence properties of the normalization generalized operator are investigated and proved. Further, the boundedness and compactness of this operator are studied.

2. Background and results

In this section, we consider the generalized type fractional differential operator and then we determine the generalized fractional differential of some special functions. For this main purpose, we begin by recalling the Srivastava–Owa fractional derivative operators of $f(z)$ of order β defined by

$$D_z^{\beta} f(z) := \frac{1}{\Gamma(1-\beta)} \frac{d}{dz} \int_0^z f(\zeta)(z-\zeta)^{-\beta} d\zeta, \tag{2.1}$$

where $0 \leqslant \beta < 1$, and the function $f(z)$ is analytic in simply-connected region of the complex z-plane containing the origin and the multiplicity of $(z-\zeta)^{-\beta}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$ (see Owa, 1978; Owa and Srivastava, 1987). Then under the conditions of the above definition the Srivastava–Owa fractional derivative of $f(z) = z^\kappa$ is defined by

$$D_z^{\beta}\{z^\kappa\} = \frac{\Gamma(\kappa+1)}{\Gamma(\kappa-\rho+1)} z^{\kappa-\beta}.$$

The theory of fractional integral and differential operators has found significant importance applications in various areas, for example (see Dziok and Srivastava, 2003). Recently, many mathematicians have developed various generalized fractional derivatives of Srivastava–Owa type, for example, (Srivastava et al., 2010 and Kiryakova, 2011). Further, we consider a generalized Srivastava–Owa type fractional derivative formulas which recently appeared.

Definition 2.1 (Ibrahim, 2011). The generalized Srivastava–Owa fractional derivative of $f(z)$ of order β is defined by

$$\mathcal{D}_z^{\beta,\gamma} f(z) := \frac{(\gamma+1)^{\beta}}{\Gamma(1-\beta)} \frac{d}{dz} \int_0^z \left(z^{\gamma+1} - \zeta^{\gamma+1}\right)^{-\beta} \zeta^\gamma f(\zeta) d\zeta, \tag{2.2}$$

where $0 \leqslant \beta < 1$, $\gamma > 0$ and $f(z)$ is analytic in simply-connected region of the complex z-plane $\mathbb{C}$ containing the origin. In particular, the generalized Srivastava–Owa fractional derivative of function $f(z) = z^\kappa$ is defined by

$$\mathcal{D}_z^{\beta,\gamma}\{z^\kappa\} = \frac{(\gamma+1)^{\beta-1}\Gamma\left(\frac{\kappa}{\gamma+1}+1\right)}{\Gamma\left(\frac{\kappa}{\gamma+1}+1-\beta\right)} z^{(1-\beta)(\gamma+1)+\kappa-1}.$$

Now, we present a new generalized fractional differential operator $\mathfrak{T}_z^{\beta,\tau,\gamma}$ as follows:

Definition 2.2. The generalized fractional differential of $f(z)$ of two parameters β and τ is defined by

$$\mathfrak{T}_z^{\beta,\tau,\gamma} f(z) := \frac{(\gamma+1)^{\beta-\tau}\Gamma(\tau)}{\Gamma(\beta)\Gamma(1-\beta-\tau)} \left(z^{1-\tau}\frac{d}{dz}\right) \int_0^z \frac{\zeta^{\gamma+\beta-1} f(\zeta)}{\left(z^{\gamma+1} - \zeta^{\gamma+1}\right)^{\beta-\tau}} d\zeta, \tag{2.3}$$

$(\gamma \geqslant 0;\quad 0 < \beta \leqslant 1;\quad 0 < \tau \leqslant 1;\quad 0 \leqslant \beta - \tau < 1)$,

where the function $f(z)$ is analytic in simple-connected region of the complex z-plane $\mathbb{C}$ containing the origin.

Remark 2.1. For $f(z) \in \mathcal{A}$, we have

i. when $\gamma = 0$ in (2.3), is reduced to the classical known one (1.4) and

ii. when $\tau=\beta$ in (2.3), we have $\mathfrak{T}_z^{\beta,\beta,\gamma}f(z)=f(z)$.

Now, we investigate the generalized fractional differential of the function $f(z)=z^{v};\ v\geqslant 0$.

Theorem 2.1. *Let $0\leqslant\beta-\tau<1$ for some $0<\beta\leqslant 1;\ 0<\tau\leqslant 1$ and $\upsilon\in N$, then we have*

$$\mathfrak{T}_z^{\beta,\tau,\gamma}\{z^{\upsilon}\}=\frac{(\gamma+1)^{\beta-\tau}\Gamma(\frac{\upsilon+\beta-1}{\gamma+1}+1)\Gamma(\tau)}{\Gamma(\frac{\upsilon+\beta-1}{\gamma+1}+1-\beta+\tau)\Gamma(\beta)}z^{(1-\beta+\tau)\gamma+\upsilon}. \tag{2.4}$$

Proof.

Applying (2.3) in Definition 2.2 to the function z^{υ}, we obtain

$$\mathfrak{T}_z^{\beta,\tau,\gamma}\{z^{\upsilon}\}=\frac{(\gamma+1)^{\beta-\tau}\Gamma(\tau)}{\Gamma(\beta)\Gamma(1-\beta+\tau)}\left(z^{1-\tau}\frac{\mathrm{d}}{\mathrm{d}z}\right)\times\int_0^z z^{(\gamma+1)(\tau-\beta)}\zeta^{\gamma+\beta+\upsilon-1}\left(1-\frac{\zeta^{\gamma+1}}{z^{\gamma+1}}\right)^{\tau-\beta}d\zeta,$$

Let use the substitution $w:=(\frac{\zeta}{z})^{\gamma+1}$ in this expression, we have

$$\mathfrak{T}_z^{\beta,\tau,\gamma}\{z^{\upsilon}\}=\frac{[(1-\beta+\tau)\gamma+\upsilon+\tau](\gamma+1)^{\beta-\tau-1}\Gamma(\tau)}{\Gamma(\beta)\Gamma(1-\beta+\tau)}\times\left(z^{(1-\beta+\tau)\gamma+\upsilon}\right)B\left(\frac{\upsilon+\beta-1}{\gamma+1}+1,1-\beta+\tau\right), \tag{2.5}$$

where $B(.,.)$ in (2.5) is the Beta function. Thus, we obtain (2.4). □

In the following, we apply some special functions in Theorem 2.1 to obtain their modifications.

Theorem 2.2. *Let $f(z)=z(1-z)^{-\alpha},\alpha\geqslant 1$ and $z\in\mathbb{U}$, then*

$$\mathfrak{T}_z^{\beta,\tau,\gamma}f(z)=\frac{(\gamma+1)^{\beta-\tau}\Gamma(\tau)}{\Gamma(\beta)}z^{(1-\beta+\tau)\gamma+1}{}_3\Psi_2\times\left[\begin{matrix}(1,1),(\alpha+1,1),(1+\frac{\beta}{\gamma+1},\frac{1}{\gamma+1})\\(2,1),(1-\beta+\tau+\frac{\beta}{\gamma+1},\frac{1}{\gamma+1})\end{matrix}\middle|z\right],$$

when $\alpha=2$, the equality holds true for the Koebe function.

Theorem 2.3. *Let $f(z)=z\,{}_1F_1(\alpha,\lambda;z)$, $0<\beta\leqslant 1$ and $0<\tau\leqslant 1$ such that $0\leqslant\beta-\tau<1$. Then*

$$\mathfrak{T}_z^{\beta,\tau,\gamma}\{z\,{}_1F_1(\alpha,\lambda;z)\}=\frac{\alpha(\gamma+1)^{\beta-\tau}\Gamma(\tau)}{\lambda\Gamma(1+\tau)}z^{(1-\beta+\tau)\gamma+1}\sum_{\kappa=0}^{\infty}\frac{(1)_\kappa(\alpha+1)_\kappa}{(\lambda+1)_\kappa(2)_\kappa}\times\frac{B(\frac{\kappa+\beta}{\gamma+1}+1,1-\beta+\tau)}{B(\beta,1-\beta+\tau)}\frac{z^{\kappa}}{(1)_\kappa},$$

where ${}_1F_1(\alpha,\lambda;z)$ is the confluent hypergeometric function (see Kilbas et al., 2006).

Theorem 2.4. *Let $0<\beta\leqslant 1$ and $0<\tau\leqslant 1$ such that $0\leqslant\beta-\tau<1$. Then, we obtain*

$$\mathfrak{T}_z^{\beta,\tau,\gamma}\left\{z\,{}_p\Psi_q(z)\left[\begin{matrix}(a_j,A_j)_{1,p}\\(b_j,B_j)_{1,q}\end{matrix}\middle|z\right]\right\}=\frac{(\gamma+1)^{\beta-\tau}\Gamma(\tau)}{\Gamma(1+\tau)}z^{(1-\beta+\tau)\gamma+1}\times\sum_{\kappa=0}^{\infty}\left(\frac{\Pi_{j=1}^{p}\Gamma(a_j,(\kappa+1)A_j)}{\Pi_{j=1}^{q}\Gamma(b_j+(\kappa+1)B_j)\Gamma(\kappa+2)}\frac{B(\frac{\kappa+\beta}{\gamma+1}+1,1-\beta+\tau)}{B(\beta,1-\beta+\tau)}z^{\kappa}\right)$$

where ${}_p\Psi_q(z)$ is given by (1.5), for all $|z|<1$.

Theorem 2.5. *Let $0<\beta\leqslant 1$ and $0<\tau\leqslant 1$ such that $0\leqslant\beta-\tau<1$. Then*

$$\mathfrak{T}_z^{\beta,\tau,\gamma}\{z\Omega_{\alpha,\lambda,\rho}(z,s,a)\}=\frac{\alpha\lambda(\gamma+1)^{\beta-\tau}\Gamma(\tau)}{\rho\Gamma(\beta)\Gamma(1-\beta+\tau)}z^{(1-\beta+\tau)\gamma+1}\times\sum_{\kappa=0}^{\infty}\left(\frac{(\alpha+1)_\kappa(\lambda+1)_\kappa}{(\rho+1)_\kappa\Gamma(\kappa+2)}\times B\left(\frac{\kappa+\beta}{\gamma+1}+1,1-\beta+\tau\right)\cdot\frac{z^{\kappa}}{(\kappa+1+a)^{s}}\right)$$

where $\Omega_{\alpha,\lambda,\rho}(z,s,a)$ is the extended general Hurwitz-Lerch Zeta function (see Bateman and Erdélyi, 1953; Lin and Srivastava, 2004*).*

3. Generalized operator

In this section, we normalize the generalized operator $\mathfrak{T}_z^{\beta,\tau,\gamma}$ of type fractional differential of analytic univalent functions in $\mathbb{U}$ and define as follows:

Let the following conditions to be realized: $0\leqslant\beta-\tau<1,\gamma\geqslant 0$, the operator $\Theta^{\beta,\tau,\gamma}f(z):\mathcal{A}\to\mathcal{A}$ is defined by

$$\Theta^{\beta,\tau,\gamma}f(z)=z+\sum_{\kappa=2}^{\infty}\Phi_{\beta,\tau,\gamma}(\kappa)\,a_\kappa z^{\kappa}, \tag{3.1}$$

where

$$\Phi_{\beta,\tau,\gamma}(\kappa):=\frac{\Gamma(\frac{\beta}{\gamma+1}+1-\beta+\tau)\Gamma(\frac{\kappa+\beta-1}{\gamma+1}+1)}{\Gamma(\frac{\beta}{\gamma+1}+1)\Gamma(\frac{\kappa+\beta-1}{\gamma+1}+1-\beta+\tau)}>0. \tag{3.2}$$

In terms of product (1.2), we represent the operator $\Theta^{\beta,\tau,\gamma}$ in $\mathcal{A}$ as follows

$$\Theta^{\beta,\tau,\gamma}f(z):=\hbar(z)*f(z) \tag{3.3}$$

with

$$\hbar(z)=\frac{\Gamma\left(\frac{1+\beta}{\gamma+1}+1-\beta+\tau\right)}{\Gamma\left(\frac{1+\beta}{\gamma+1}+1\right)}z\,{}_2\Psi_1\left[\begin{matrix}(1,1),(1+\frac{\beta}{\gamma+1},\frac{1}{\gamma+1})\\(1-\beta+\tau+\frac{\beta}{\gamma+1},\frac{1}{\gamma+1})\end{matrix}\middle|z\right]. \tag{3.4}$$

Motivated by Theorem 2 and Theorem 3 in Kiryakova (2011), we proceed to study the univalence properties of operator $\Theta^{\beta,\tau,\gamma}$ in $\mathbb{U}$.

Theorem 3.1. *Let $f\in\mathcal{S}$. If*

(i) for $0<\beta\leqslant 1,\ 0<\tau\leq 1$ such that $0\leqslant\beta-\tau<1$ and
(ii) $\rho_i>0,i=1,\ldots,p$ and $\lambda_j>0,\ j=1,\ldots,q;\ p\leqslant q+1$,

then the operator $\Theta^{\beta,\tau,\gamma}f(z)\in\mathcal{S}$ in open unite disk $\mathbb{U}$.

$$ {}_2\Psi_1\left[\begin{matrix}(3,1),(1+\frac{\beta}{\gamma+1}+\frac{1}{\gamma+1},\frac{1}{\gamma+1})\\(1-\beta+\tau+\frac{\beta}{\gamma+1}+\frac{1}{\gamma+1},\frac{1}{\gamma+1})\end{matrix}\middle|1\right]+{}_2\Psi_1\left[\begin{matrix}(2,1),(1+\frac{\beta}{\gamma+1},\frac{1}{\gamma+1})\\(1-\beta+\tau+\frac{\beta}{\gamma+1},\frac{1}{\beta+1})\end{matrix}\middle|1\right] < 2\left(\frac{\Gamma(\frac{\beta}{\gamma+1}+1)}{\Gamma(\frac{\beta}{\gamma+1}+1-\beta+\tau)}\right). $$

Proof.

Suppose the function $f\in\mathcal{S}$ and let $\Theta^{\beta,\tau,\gamma}f(z)=z+\sum_{\kappa=2}^{\infty}w_\kappa z^\kappa$ be defined by equality (3.1), where $w_\kappa:=\Phi_{\beta,\tau,\gamma}(\kappa)\,a_\kappa$. In view of Theorem 1.1, we give the estimate for the coefficients of an univalent function belonging to $\mathcal{S}$ in $\mathbb{U}$ also, by using this estimate, we can get another estimate for ℓ_1 in $\mathcal{S}$ as follows:

$$ \ell_1=\sum_{\kappa=2}^{\infty}\kappa\Phi_{\beta,\tau,\gamma}|a_\kappa|\leqslant\sum_{\kappa=2}^{\infty}\frac{(\kappa)^2}{\kappa!}(\Phi_{\beta,\tau,\gamma}(\kappa)\kappa!) =\sum_{\kappa=2}^{\infty}\frac{(\kappa)^2}{\kappa!}\ell(\kappa)<1 \tag{3.5} $$

where $\ell(\kappa)=\frac{\Gamma(\frac{\beta}{\gamma+1}+1-\beta+\tau)}{\Gamma(\frac{\beta}{\gamma+1}+1)}\frac{\Gamma(\frac{\kappa+\beta-1}{\gamma+1}+1)(1)_\kappa}{\Gamma(\frac{\kappa+\beta-1}{\gamma+1}+1-\beta+\tau)}$. The series in (3.5) is transformed by employing the following links: $\frac{\kappa^2}{(1)_\kappa}=\frac{\kappa}{(1)_{\kappa-1}}=\frac{1}{(1)_{\kappa-1}}+\frac{1}{(1)_{\kappa-2}}$. Depending on $(1)_\kappa=\kappa!$ and $(1)_{\kappa-1}=(\kappa-1)!$, the estimate (3.5) satisfies

$$ \begin{aligned}\ell_1&\leqslant\sum_{\kappa=2}^{\infty}\frac{\kappa^2}{(1)_\kappa}\ell(\kappa)=\sum_{\kappa=2}^{\infty}\left(\frac{1}{(1)_{\kappa-1}}+\frac{1}{(1)_{\kappa-2}}\right)\ell(\kappa)\\&=\sum_{\kappa=2}^{\infty}\frac{\Gamma\left(\frac{\beta}{\gamma+1}+1-\beta+\tau\right)}{\Gamma\left(\frac{\beta}{\gamma+1}+1\right)}\frac{\Gamma\left(\frac{\kappa+\beta-1}{\gamma+1}+1\right)}{\Gamma\left(\frac{\kappa+\beta-1}{\gamma+1}+1-\beta+\tau\right)}\frac{(1)_\kappa}{(1)_{\kappa-1}}\\&\quad+\sum_{\kappa=2}^{\infty}\frac{\Gamma\left(\frac{\beta}{\gamma+1}+1-\beta+\tau\right)}{\Gamma\left(\frac{\beta}{\gamma+1}+1\right)}\frac{\Gamma\left(\frac{\kappa+\beta-1}{\gamma+1}+1\right)}{\Gamma\left(\frac{\kappa+\beta-1}{\gamma+1}+1-\beta+\tau\right)}\frac{(1)_\kappa}{(1)_{\kappa-2}}\\&=\frac{\Gamma\left(\frac{\beta}{\gamma+1}+1-\beta+\tau\right)}{\Gamma\left(\frac{\beta}{\gamma+1}+1\right)}\left(\sum_{\kappa=1}^{\infty}\frac{\Gamma(\kappa+2)\Gamma\left(\frac{\kappa+\beta}{\gamma}+1\right)}{\Gamma\left(\frac{\kappa+\beta}{\gamma+1}+1-\beta+\tau\right)}\frac{1}{(1)_\kappa}\right.\\&\quad\left.+\sum_{\kappa=0}^{\infty}\frac{\Gamma(\kappa+3)\Gamma\left(\frac{\kappa+\beta+1}{\gamma+1}+1\right)}{\Gamma\left(\frac{\kappa+\beta+1}{\gamma+1}+1-\beta+\tau\right)}\frac{1}{(1)_\kappa}\right),\end{aligned} $$

by employing (1.5), we can transform the estimate ℓ_1 at $z=1$, then we get (3.5). Hence, $\Theta^{\beta,\tau,\gamma}:\mathcal{S}\to\mathcal{S}$. □

Similarly, we may prove the convexity of $\Theta^{\beta,\tau,\gamma}$ in the next result.

Theorem 3.2. *Let the condition (i) as the Theorem 3.1 be satisfied. If $0\leqslant\beta-\tau<1$,*

$$ {}_2\Psi_1\left[\begin{matrix}(2,1),\left(1+\frac{\beta}{\gamma+1},\frac{1}{\gamma+1}\right)\\\left(1-\beta+\tau+\frac{\beta}{\gamma+1},\frac{1}{\gamma+1}\right)\end{matrix}\middle|1\right]<2\left(\frac{\Gamma(\frac{\beta}{\gamma+1}+1)}{\Gamma(\frac{\beta}{\gamma+1}+1-\beta+\tau)}\right) $$

then $\Theta^{\beta,\tau,\gamma}:\mathcal{K}\to\mathcal{K}$.

4. Generalized operator on Bloch space

In this section we study the boundedness and compactness of operator $\Theta^{\beta,\tau,\gamma}$ given by (3.3) on the weighted μ-Bloch space $\mathbb{B}_w^\mu$ (see Duren, 1983; Hedenmalm et al., 2012).

Definition 4.1. A holomorphic function $f\in\mathcal{H}(\mathbb{U})$ is said to be in Bloch space $\mathbb{B}$ whenever $\|f\|_{\mathbb{B}}=\sup_{z\in\mathbb{U}}\left(1-|z|^2\right)|f'(z)|<\infty$ and the little Bloch space $\mathbb{B}_0$ is given as follows

$$ \lim_{|z|\to1-}\left(1-|z|^2\right)|f'(z)|=0. $$

Definition 4.2. Let $w:[0.1)\to[0,\infty)$ and f be an analytic function on open unit disk $\mathbb{U}$ which is said to be in the weighted Bloch space $\mathbb{B}_w$ if

$$ (1-|z|)|f'(z)|<h\,w(1-|z|),\quad z\in\mathbb{U} $$

for some $h>0$. Note that, if $w=1$ then $\mathbb{B}_1\equiv\mathbb{B}$. Further, the weighted μ-Bloch space $\mathbb{B}_w^\mu$, covering of all $f\in\mathbb{B}_w^\mu$ defined by

$$ \lim_{|z|\to-1}\frac{(1-|z|)^\mu|f'(z)|}{w(1-|z|)}=0 $$

and

$$ \|f\|_{\mathbb{B}_w^\mu}=\sup_{z\in\mathbb{U}}\left\{|f'(z)|\frac{(1-|z|)^\mu}{w(1-|z|)}\right\}<\infty. \tag{4.1} $$

It is easy to note that if an analytic function $g(z)\in\mathbb{B}_w^\mu$, then

$$ \sup_{z\in\mathbb{U}}\{|\Bbbk g(z)|\}\frac{(1-|z|)^\mu}{w(1-|z|)}\leqslant c<\infty. \tag{4.2} $$

where $\Bbbk$ is a *positive* number.

Lemma 4.1. *(Ruscheweyh, 1982) Let f and g be two analytic functions. Then*

$$ z(g*f)'(z):=g(z)*zf'(z)\Longleftrightarrow(g*f)'(z)=\frac{g(z)}{z}*f'(z). $$

Theorem 4.1. *Let f be an analytic function on the open unit disk $\mathbb{U}$, and $\mathbb{B}_w^\mu$; $w:[0,1)\to[0,\infty)$. Then*

$$ f\in\mathbb{B}_w^\mu\Longleftrightarrow\Theta^{\beta,\tau,\gamma}f\in\mathbb{B}_w^\mu. $$

Proof.

By suppose $f\in\mathbb{B}_w^\mu$ and following Lemma 4.1, we obtain

$$ \begin{aligned}\|\Theta^{\beta,\tau,\gamma}f\|_{\mathbb{B}_w^\mu}&=\sup_{z\in\mathbb{U}}\left\{|[\hbar(z)*f(z)]'|\frac{(1-|z|)^\mu}{w(1-|z|)}\right\}\\&=\sup_{z\in\mathbb{U}}\left\{\left|\frac{\hbar(z)}{z}*f'(z)\right|\frac{(1-|z|)^\mu}{w(1-|z|)}\right\}\leqslant c\|f\|_{\mathbb{B}_w^\mu}<\infty\end{aligned} $$

where $\sup_{z\in\mathbb{U}}\left\{|\hbar(z)|\frac{(1-|z|)^\mu}{w(1-|z|)}\right\}\leqslant c|z|\leqslant c$, for $|z|<1$. Hence, $\Theta^{\beta,\tau,\gamma}f\in\mathbb{B}_w^\mu$, which prove the first part of Theorem 4.1. On the other hand, if $\Theta^{\beta,\tau,\gamma}f\in\mathbb{B}_w^\mu$, we then aim to show that

$$\|f\|_{\mathbb{B}^{\mu}_{w}} = \sup_{z\in\mathbb{U}}\left\{|f'(z)|\frac{(1-|z|)^{\mu}}{w(1-|z|)}\right\} < \infty \quad (z\in\mathbb{U}).$$

Let now define an analytic function $\mathcal{F}^{\beta,\tau,\gamma}(z)$ by

$$\mathcal{F}^{\beta,\tau,\gamma}(z) = \frac{\hbar(z)}{z} \quad (z\in\mathbb{U})$$

such that

$$\mathcal{F}^{\beta,\tau,\gamma}(z) * \left(\mathcal{F}^{\beta,\tau,\gamma}\right)^{-1}(z) = (1-z)^{-1}.$$

Hence, we get

$$\begin{aligned}\|f\|_{\mathbb{B}^{\mu}_{w}} &= \sup_{z\in\mathbb{U}}\left\{|f'(z)|\frac{(1-|z|)^{\mu}}{w(1-|z|)}\right\}\\ &= \sup_{z\in\mathbb{U}}\left\{|\mathcal{F}^{\beta,\tau,\gamma}(z) * f'(z) * \left(\mathcal{F}^{\beta,\tau,\gamma}\right)^{-1}(z)|\frac{(1-|z|)^{\mu}}{w(1-|z|)}\right\}\\ &= \sup_{z\in\mathbb{U}}\left\{|\left(\Theta^{\beta,\tau,\gamma}(z)\right)' * \left(\mathcal{F}^{\beta,\tau,\gamma}\right)^{-1}(z)|\frac{(1-|z|)^{\mu}}{w(1-|z|)}\right\}\\ &\leqslant c_{\beta,\tau,\gamma}\|\Theta^{\beta,\tau,\gamma}\|_{\mathbb{B}^{\mu}_{w}},\end{aligned}$$

where $c_{\beta,\tau,\gamma} = \|(\mathcal{F}^{\beta,\tau,\gamma})^{-1}(z)\|$. This completes the proof of Theorem 4.1. □

Theorem 4.2. *Let f be an analytic function on open unit disk $\mathbb{U}$, and $\mathbb{B}^{\mu}_{w}$; $w:[0,1)\to[0,\infty)$, then the operator $\Theta^{\beta,\tau,\gamma}f:\mathbb{B}_{w}\to\mathbb{B}^{\mu}_{w}$ is compact.*

Proof.

If $\Theta^{\beta,\tau,\gamma}f$ is compact, then it is bounded and by Theorem 4.1 it satisfies that $f\in\mathbb{B}_{w}$ because $\mathbb{B}_{w}\subset\mathbb{B}^{\mu}_{w}$. Let us assume that $f\in\mathbb{B}_{w}$, that $(f_n)_{n\in\mathbb{N}}\subset\mathbb{B}^{\mu}_{w}$ be such that $f_n\to 0$ converges uniformly on $\overline{\mathbb{U}}$ as $n\to\infty$. Since $(f)_{n\in\mathbb{N}}$ convergence uniformly on each compact $\mathbb{U}$, we have that there in $\mathbb{N}>0$ such that for every $n>\mathbb{N}$ and every $z\in\mathbb{U}$, there is an $0<\delta<1$, such that for every $n\geqslant 1, |\hbar(z)/z|<\varepsilon$, where $\delta<|z|<1$, Since δ is arbitrary, then we can choose $|\frac{(1-|z|)}{(1-|z|)^{\mu}}|<1$, for all $\delta<|z|<1$ and

$$\begin{aligned}\|\Theta^{\beta,\tau,\gamma}f_n\|_{\mathbb{B}_{w}} &= \sup_{z\in\mathbb{U}\setminus\mathbb{U}_{\delta}}\left\{|(\hbar(z)*f_n(z))'|\frac{(1-|z|)}{w(1-|z|)}\right\},\\ &= \sup_{z\in\mathbb{U}\setminus\mathbb{U}_{\delta}}\left\{\left|\left(\frac{\hbar(z)}{z}*f_n(z)\right)'\right|\frac{(1-|z|)}{w(1-|z|)}\right\},\\ &\leqslant \varepsilon\|f_n\|_{\mathbb{B}^{\mu}_{w}},\end{aligned} \tag{4.3}$$

Since for $f_n\to 0$ on $\overline{\mathbb{U}}$ we get $\|f_n\|_{\mathbb{B}^{\mu}_{w}}\to 0$, and that $\varepsilon>0$, by letting $n\to\infty$ in (4.3), we have that $\lim_{n\to\infty}\|\Theta^{\beta,\tau,\gamma}f_n\|_{\mathbb{B}_{w}}=0$. Thus $\Theta^{\beta,\tau,\gamma}$ is compact. □

5. Conclusion

In open unit disk $\mathbb{U}$ we defined a new generalized differential operator of fractional formula and viewed some of their applications with several special functions. On the another hand, we gave the normalization for this generalized differential operator and discussed their univalence (starlikeness and convexity) characteristics. Further, compactness and boundedness for this operator in weight μ-Bloch space are introduced.

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