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Original article

Estimation methods for the discrete Poisson-Lindley and discrete Lindley distributions with actuarial measures and applications in medicine

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ABSTRACT

Discrete distributions have their important in modeling count data in several applied fields such as epidemiology, public health, sociology, medicine, and agriculture. This paper discusses the estimation of the parameters of two discrete models called discrete Poisson-Lindley and discrete Lindley distributions, using several frequentist estimation methods. Parameter estimation can provide a guideline for choosing the best method of estimation for the model parameters, which would be very important to reliability engineers and applied statisticians. The finite sample properties of the estimates are explored using extensive simulation results. The ordering performance of the proposed estimators is determined by the partial and overall ranks of different parametric values. We also derived two important actuarial measures of the two discrete models. A computational study for the two risk measures is conducted. Finally, applications of the two discrete distributions have been examined and compared with other discrete distributions via three data sets from the medicine field including two COVID-19 data sets.

1. Introduction

The discrete distributions are very useful in modeling count data in several applied fields such as medicine, public health, epidemiology, agriculture, sociology, and applied science. Many discrete distributions have been proposed for modeling count data. However, the traditional discrete distributions including geometric and Poisson have limited applicability as models for failure times, reliability, counts, etc. This is so, since many real count data show either under-dispersion, in which the variance is smaller than the mean or over-dispersion, in which the variance is greater than the mean. This has motivated many authors to develop some new discrete distributions based on classical continuous distributions for failure times, reliability, etc.

Sankaran (1970) introduced the discrete Poisson-Lindley (DPL) distribution which is specified by the probability mass function (PMF)

$$p(x) = \frac{\alpha^2(x+\alpha+2)}{(\alpha+1)^{x+3}}, \quad x = 0, 1, 2, \ldots, \quad \alpha > 0, \tag{1}$$

and the cumulative distribution function (CDF)

$$F(x) = P(X \leqslant x) = 1 - (\alpha+1)^{-x-3}[1 + \alpha(\alpha + x + 3)]. \tag{2}$$

The quantile function (QF) and expected value of the DPL distribution are given by

$$x_u = -3 - \alpha - \frac{1}{\alpha} - \frac{1}{\log(\alpha+1)} W\left[\frac{(\alpha+1)^{-\frac{\alpha^2+1}{\alpha}}(u-1)\log(\alpha+1)}{\alpha}\right] \tag{3}$$

and

$$\mu = \frac{\alpha+2}{\alpha(\alpha+1)}, \tag{4}$$

respectively, where $W[.]$ is the negative branch of the Lambert function.

Bakouch et al. (2014) proposed the discrete Lindley (DL) distribution as a discrete version of the continuous Lindley distribution. The PMF of the DL distribution takes the form

$$p(x) = \frac{\rho^x}{1+\beta}[\beta(1-2\rho) + (1-\rho)(1+\beta x)], \quad x = 0, 1, 2, \ldots, \tag{5}$$

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where $\rho = \exp(-\beta), \beta > 0$. Its CDF reduces to

$$F(x) = P(X \leqslant x) = 1 - \frac{\rho^{x+1}}{1+\beta}[1 + \beta(2+x)]. \quad (6)$$

The QF and mean of the DL distribution are, respectively, defined by

$$x_u = -2 - \frac{1}{\beta} - \frac{1}{\beta} W\left[e^{-\beta-1}(\beta+1)(u-1)\right] \quad (7)$$

and

$$\mu = \frac{\rho(2\beta+1) - \rho^2(\beta+1)}{(1-\rho)^2(\beta+1)}, \quad (8)$$

where $W[.]$ is the negative branch of the Lambert function.

Our aim in this paper is to explore the estimation of the DPL and DL parameters by five methods of estimation, such as the maximum likelihood estimator (MLE), Cramér-von Mises estimator (CVME), least-square estimator (OLSE), weighted least squares estimator (WLSE) and percentile estimator (PCE). We compare the proposed estimators using an extensive computational simulations to develop a guideline for choosing the best estimation method that provides better estimates for the parameters of the DPL and DL models.

In this regard, comparing several frequentist estimators for estimating the parameters of different continuous distributions are conducted by several authors. Notable among them are Dey et al. (2017), Nassar et al. (2018), Rodrigues et al. (2018), Afify and Mohamed (2020), Shakhatreh et al. (2020), Afify et al. (2020), Al-Mofleh et al. (2020), Aldahlan and Afify (2020) and Nassar et al. (2020) for exponentiated-Gumbel, transmuted exponentiated Pareto, Poisson-exponential, extended odd Weibull exponential, generalized extended exponential Weibull, generalized Ramos-Louzada, exponentiated half-logistic exponential and alpha power-exponential distributions.

As far as we know, there are no reports on estimation of parameters of the DPL and DL distribution based on several frequentist estimators. To the best of our knowledge, Sankaran (1970) proposed the moments and maximum likelihood methods for estimating the DPL parameter. However, he only used the moments estimator for the two analyzed data sets he studied. Ghitany and Al-Mutairi (2009) presented a simulation study to compare the moments and maximum likelihood estimators and showed that the two estimators are consistent and asymptotically normal.

Further, we derive two important risk measures for the DPL and DL distributions, called value at risk and tail value at risk which are useful in evaluating the exposure to market risk in a portfolio of instruments. The numerical simulations of the two risk measures are presented for the two discrete distributions using several parametric values.

The rest of the paper is organized as follows. In Section 2, we derive the value at risk and tail value at risk for the DPL and DL distributions along with detailed numerical simulations for these measures. In Section 3, different classical methods of parameter estimation are discussed. In Section 4, we present the simulation results to compare and assess the performance of the proposed estimators. Three COVID-19 data sets are analyzed to validate the use of DPL and DL distributions in fitting lifetime count data are explored in Section 5. Finally, concluding remarks are presented in Section 6.

2. Actuarial measures

In this section, we determine the value at risk (VaR) and tail value at risk (TVaR) for the DPL and DL distributions, which play a crucial role in portfolio optimization under uncertainty.

2.1. VaR measure

The VaR of any random variable X is the p^{th} quantile of its CDF as shown in Artzner (1999), and it is defined, for a probability level $p \in (0,1)$, by $Var_p = F_X^{-1}(p)$.

The VaR of the DPL distribution with PMF (1) is derived as

$$VaR_p = -3 - \alpha - \frac{1}{\alpha} - \frac{1}{\log(\alpha+1)} W\left[\frac{(\alpha+1)^{-\frac{\alpha^2+1}{\alpha}}(p-1)\log(\alpha+1)}{\alpha}\right].$$

The VaR of the DL model with PMF (5) takes the form

$$VaR_p = -2 - \frac{1}{\beta} - \frac{1}{\beta} W\left[e^{-\beta-1}(\beta+1)(p-1)\right].$$

2.2. TVaR Measure

The TVaR is an important actuarial measure, and it is used to determine the expected value of the loss given that an event outside a given probability level has occurred. The TVaR is defined in Klugman et al. (2012) by the following equation

$$\begin{aligned} TVaR_p = & \; E(X|X > x_p) = x_p + \frac{\sum_{x=x_p}^{\infty}(x-x_p)}{p(x)} \\ = & \; x_p + \frac{E(X) - x_p + \sum_{x=0}^{x_p-1}(x_p - x)p(x)}{1 - F(x_p)}. \end{aligned}$$

Using Eqs. (3) and (4), the TVaR of the DPL distribution is derived as

$$TVaR_p = x_p + \frac{(\alpha+1)\left[\frac{1}{\alpha(\alpha+3)+\alpha x_p+1} + 1\right]}{\alpha}.$$

Similarly, based on Eqs. (7) and (8), the TVaR of the DL distribution follows as

$$TVaR_p = x_p + \frac{e^{\beta}\left[-\beta + e^{\beta}(2\beta+1) + (e^{\beta}-1)\beta x_p - 1\right]}{(e^{\beta}-1)^2(2\beta + \beta x_p + 1)}.$$

2.3. Simulations for risk measures

In this sub-section, we present some numerical computations for the VaR and TVaR measures of the DPL and DL distributions for different parametric values. The results can be obtained as follows.

1. A random sample of size $n = 100$ is generated from the DPL and DL distributions and their parameters are estimated by the maximum likelihood method.
2. The VaR and TVaR of the two distributions are calculated from 2,000 repetitions.

Simulation results of the VaR and TVaR for the DPL and DL distributions are reported in Tables 1 and 2. For visual presentation purpose, we summarize the results in the two tables graphically in Fig. 1.

The results in Tables 1 and 2 and the plots in Fig. 1 reveal that the VaR and TVaR measures are increasing functions in the parameter α, for the DPL distribution, and β, for the DL distribution.

3. Estimation methods

In this section, we discuss the estimation of the parameters of the DPL and DL distributions using several classical methods of estimation including the maximum likelihood estimator (MLE), Cramér-von Mises estimator (CVE), least-squares estimator (LSE), weighted least squares estimator (WLSE) and percentile estimator (PCE).

3.1. Maximum likelihood estimation

Let $x_1, x_2, \ldots, x_n$be a random sample of size n from the DPL model with PMF (2), then the log-likelihood function reduces to

$$L_{DPL} = 2n\log\alpha - \log(\alpha+1)\sum_{i=1}^{n}(x_i+3) + \sum_{i=1}^{n}\log(x_i+\alpha+2), \qquad (9)$$

The MLE of α, as shown in Sankaran (1970), is the solution of the following nonlinear equation

$$\frac{2n}{\alpha} - \frac{n(\bar{x}+3)}{\alpha+1} + \sum_{i=1}^{n}(x_i+\alpha+2)^{-1} = 0. \qquad (10)$$

Ghitany and Al-Mutairi (2009) proved that Eq. (10) has a unique solution for all n.

Let $x_1, x_2, \ldots, x_n$be a random sample of size n from the DL model with PMF (5), then the log-likelihood function is given by

$$L_{DL} = -\beta\sum_{i=1}^{n}x_i - n\log(\beta+1) + \sum_{i=1}^{n}\log[1+(1+x_i)\beta - (1+2\beta+\beta x_i)\exp(\beta)], \qquad (11)$$

The MLE of β is follows as the root of the following equation (Bakouch et al., 2014)

$$-\frac{n}{\beta+1} - \sum_{i=1}^{n}x_i + \sum_{i=1}^{n}\frac{1+x_i-(1-2\beta+x_i-\beta x_i)\exp(-\beta)}{1+(1+x_i)\beta-(1+2\beta+\beta x_i)\exp(\beta)} = 0.$$

3.2. Least-squares and weighted least-squares estimation

Let $x_{1:n}, x_{2:n}, \ldots, x_{n:n}$be the order statistics from the DPL distribution. By minimizing the following equation, we have LSE (Swain et al., 1988) of the parameter α of the DPL distribution

Table 1
Numerical values of VaR of the DPL and DL distributions.

Significance Level	DPL($\alpha = 0.25$)	DL($\beta = 0.25$)
0.70	8.90197	8.26617
0.75	9.95805	9.2078
0.80	11.21761	10.32968
0.85	12.79907	11.73696
0.90	14.96585	13.66341
0.95	18.5491	16.84654
0.99	26.50909	23.91149
Significance Level	DPL($\alpha = 0.5$)	DL($\beta = 0.5$)
0.70	4.03788	3.50114
0.75	4.56447	3.9304
0.80	5.19558	4.44301
0.85	5.99151	5.08736
0.90	7.08663	5.97111
0.95	8.90529	7.43411
0.99	12.96347	10.68758
Significance Level	DPL($\alpha = 1.5$)	DL($\beta = 1.5$)
0.70	1.02010	0.51384
0.75	1.17239	0.58836
0.80	1.35813	0.67874
0.85	1.59661	0.79409
0.90	1.93101	0.95472
0.95	2.49861	1.22504
0.99	3.8006	1.83774

Table 2
Numerical values of TVaR of the DPL and DL distributions.

Significance Level	DPL($\alpha = 0.25$)	DL($\beta = 0.25$)
0.70	13.81718	12.58693
0.75	14.79489	13.45659
0.80	15.97354	14.50447
0.85	17.46878	15.83318
0.90	19.53854	17.67156
0.95	22.99892	20.74363
0.99	30.78289	27.65028
Significance Level	DPL($\alpha = 0.5$)	DL($\beta = 0.5$)
0.70	6.28894	5.23761
0.75	6.78275	5.63693
0.80	7.37937	6.11848
0.85	8.13786	6.72957
0.90	9.19001	7.57572
0.95	10.9531	8.99085
0.99	14.92949	12.17532
Significance Level	DPL($\alpha = 1.5$)	DL($\beta = 1.5$)
0.70	1.44267	0.59833
0.75	1.59315	0.67166
0.80	1.7768	0.76072
0.85	2.01277	0.87453
0.90	2.34394	1.03330
0.95	2.90669	1.30100
0.99	4.1999	1.90943

$$LS_{DPL} = \sum_{i=1}^{n}\left[F(x_{i:n}) - \frac{i}{n+1}\right]^2 = \sum_{i=1}^{n}\left\{1-(\alpha+1)^{-x_{i:n}-3}[1+\alpha(\alpha+x_{i:n}+3)] - \frac{i}{n+1}\right\}^2,$$

it can be also obtained by solving the following non-linear equation

$$\sum_{i=1}^{n}\left\{1-(\alpha+1)^{-x_{i:n}-3}[1+\alpha(\alpha+x_{i:n}+3)] - \frac{i}{n+1}\right\}\Delta(x_{i:n}) = 0,$$

where

$$\Delta(x_{i:n}) = \frac{\partial}{\partial\alpha}F(x_{i:n}) = \alpha(x_{i:n}+1)(\alpha+1)^{-x_{i:n}-4}(\alpha+x_{i:n}+4). \qquad (13)$$

The WLSE (Swain et al., 1988) of the DPL parameter can be determined by minimizing the following equation

$$W_{DPL} = \sum_{i=1}^{n}\tfrac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_{i:n}) - \tfrac{i}{n+1}\right]^2 = \sum_{i=1}^{n}\tfrac{(n+1)^2(n+2)}{i(n-i+1)}\left\{1-(\alpha+1)^{-x_{i:n}-3}[1+\alpha(\alpha+x_{i:n}+3)] - \tfrac{i}{n+1}\right\}^2,$$

or by solving the following non-linear equation

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_{i:n}) - \frac{i}{n+1}\right]\Delta(x_{i:n}) = 0,$$

where $\Delta(x_{i:n})$ is defined by Eq. (13).

Let $x_{1:n}, x_{2:n}, \ldots, x_{n:n}$be the order statistics of a random sample from the DL distribution. The LSE of the DL parameter β follows by minimizing the following equation with respect to β

$$LS_{DL} = \sum_{i=1}^{n}\left[F(x_{i:n}) - \frac{i}{n+1}\right]^2 = \sum_{i=1}^{n}\left\{1 - \frac{\exp[-\beta(x_{i:n}+1)]}{1+\beta}[1+\beta(2+x_{i:n})] - \frac{i}{n+1}\right\}^2.$$

The LSE of β can also be obtained by solving the following non-linear equation

$$\sum_{i=1}^{n}\left\{1 - \frac{\exp[-\beta(x_{i:n}+1)]}{1+\beta}[1+\beta(2+x_{i:n})] - \frac{i}{n+1}\right\}\Phi(x_{i:n}) = 0,$$

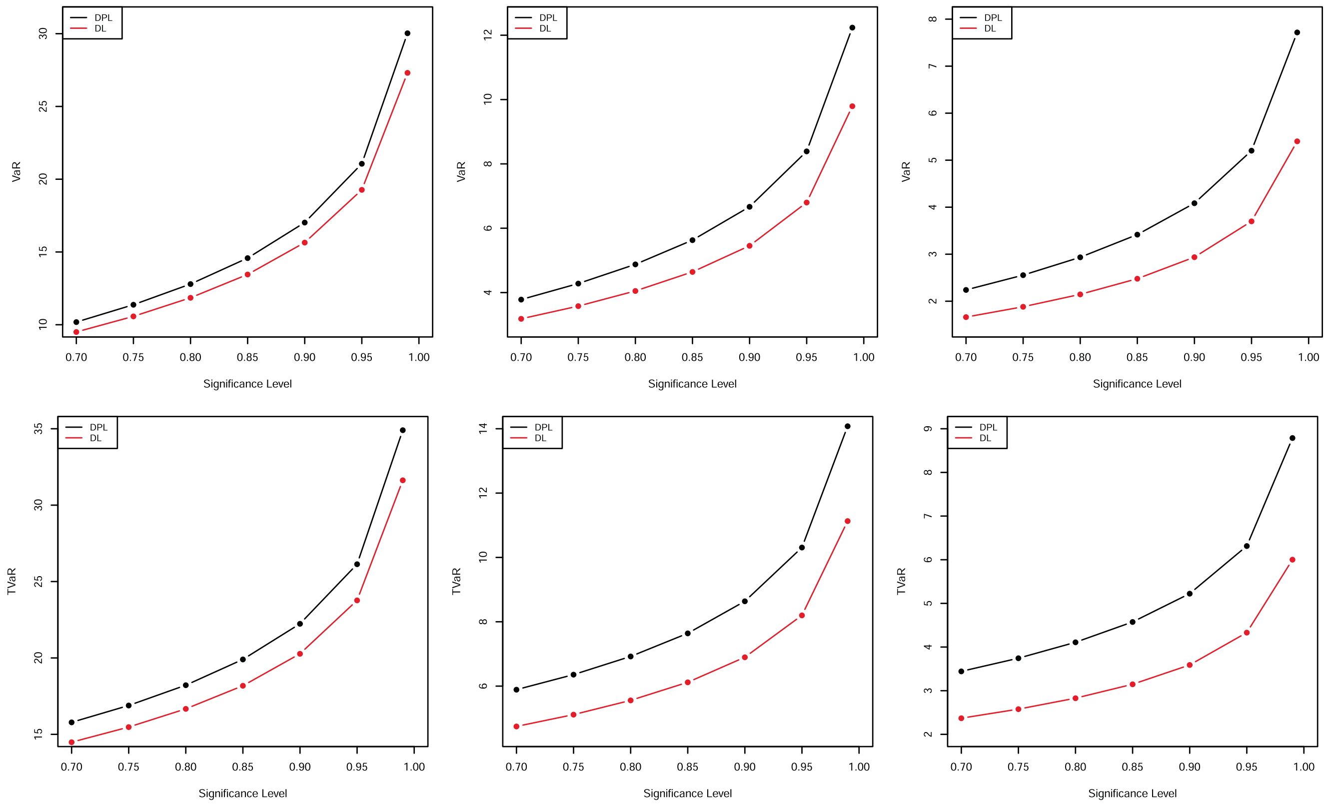

Fig. 1. Plots of the VaR (top panel) and TVaR (bottom panel) of the DPL and DL distributions using the values in Tables 1 and 2.

where

$$\Phi(x_{i:n}) = \frac{\partial}{\partial\beta}F(x_{i:n}) = \frac{\beta(x_{i:n}+1)\exp[-\beta(x_{i:n}+1)][\beta(x_{i:n}+2)+x_{i:n}+3]}{(\beta+1)^2}. \quad (14)$$

The WLSE of the DL parameter can be determined by minimizing the following equation

$$W_{DL} = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_{i:n}) - \frac{i}{n+1}\right]^2 = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left\{1-\frac{\exp[-\beta(x_{i:n}+1)]}{1+\beta}[1+\beta(2+x_{i:n})] - \frac{i}{n+1}\right\}^2,$$

or by solving the following non-linear equation

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left\{1-\frac{\exp[-\beta(x_{i:n}+1)]}{1+\beta}[1+\beta(2+x_{i:n})] - \frac{i}{n+1}\right\} = 0,$$

where $\Phi(x_{i:n})$ is given by Eq. (14).

3.3. Cramér-von Mises Estimation

The CVME of the parameter α of the DPL distribution is obtained by minimizing the following equation

$$CV_{DPL} = \frac{1}{12n} + \sum_{i=1}^{n}\left[F(x_{i:n}) - \frac{2i-1}{2n}\right]^2 = \frac{1}{12n} + \sum_{i=1}^{n}\left\{1-(\alpha+1)^{-x_{i:n}-3}[1+\alpha(\alpha+x_{i:n}+3)] - \frac{2i-1}{2n}\right\}^2,$$

or by solving the following non-linear equation

$$\sum_{i=1}^{n}\left\{1-(\alpha+1)^{-x_{i:n}-3}[1+\alpha(\alpha+x_{i:n}+3)] - \frac{2i-1}{2n}\right\}\Delta(x_{i:n}) = 0,$$

where $\Delta(x_i)$ is defined in Eq. (13). Further details about the CVME can be explored in Macdonald (1971) and Luceño (2006). The CVME of the DL parameter β can be calculated by minimizing the following equation

$$CV_{DL} = \frac{1}{12n} + \sum_{i=1}^{n}\left[F(x_{i:n}) - \frac{2i-1}{2n}\right]^2 = \frac{1}{12n} + \sum_{i=1}^{n}\left\{1-\frac{\exp[-\beta(x_{i:n}+1)]}{1+\beta}[1+\beta(2+x_{i:n})] - \frac{2i-1}{2n}\right\}^2,$$

or by solving the following non-linear equation

$$\sum_{i=1}^{n}\left\{1-\frac{\exp[-\beta(x_{i:n}+1)]}{1+\beta}[1+\beta(2+x_{i:n})] - \frac{2i-1}{2n}\right\} = 0,$$

where $\Phi(x_i)$ is given in Eq. (14).

3.4. Percentile Estimation

This estimation method was introduced by Kao (1958, 1959). Let $p_i = \frac{i}{n+1}$ be an estimate of $F(x_i)$, then the PCE of DPL parameter α follows by minimizing

$$PC_{DPL} = \sum_{i=1}^{n}[x_{i:n} - Q(p_i)]^2 = \sum_{i=1}^{n}\left\{x_{i:n} + 3 + \alpha + \frac{1}{\alpha} + \frac{W(z)}{\log(\alpha+1)}\right\}^2,$$

where

$$z = \frac{(\alpha+1)^{-\frac{\alpha^2+1}{\alpha}}(p_i-1)\log(\alpha+1)}{\alpha}.$$

Table 3
Simulation results of the AVEs, ABBs, MSEs and MREs of DPL distribution.

				$\alpha = 0.2$		
n	Est.	MLE	CVME	LSE	PCE	WLSE
20	AVEs	0.19606	0.20512	0.20529	0.19537	0.20363
	ABBs	0.02708$^{\{1\}}$	0.03063$^{\{4\}}$	0.03196$^{\{5\}}$	0.03037$^{\{3\}}$	0.03011$^{\{2\}}$
	MSEs	0.00114$^{\{1\}}$	0.00161$^{\{4\}}$	0.00180$^{\{5\}}$	0.00147$^{\{2\}}$	0.00154$^{\{3\}}$
	MREs	0.13540$^{\{1\}}$	0.15316$^{\{4\}}$	0.15982$^{\{5\}}$	0.15185$^{\{3\}}$	0.15055$^{\{2\}}$
	$\sum$ Ranks	3$^{\{1\}}$	12$^{\{4\}}$	15$^{\{5\}}$	8$^{\{3\}}$	7$^{\{2\}}$
40	AVEs	0.19276	0.20266	0.20228	0.19585	0.20133
	ABBs	0.01950$^{\{1\}}$	0.02193$^{\{5\}}$	0.02124$^{\{4\}}$	0.02109$^{\{2\}}$	0.02139$^{\{3\}}$
	MSEs	0.00057$^{\{1\}}$	0.00079$^{\{5\}}$	0.00075$^{\{3\}}$	0.00070$^{\{2\}}$	0.00076$^{\{4\}}$
	MREs	0.09751$^{\{1\}}$	0.10966$^{\{5\}}$	0.10622$^{\{3\}}$	0.10547$^{\{2\}}$	0.10693$^{\{4\}}$
	$\sum$ Ranks	3$^{\{1\}}$	15$^{\{5\}}$	10$^{\{4\}}$	6$^{\{2\}}$	11$^{\{3\}}$
60	AVEs	0.19162	0.20134	0.20073	0.19573	0.20103
	ABBs	0.01659$^{\{1\}}$	0.01744$^{\{5\}}$	0.01743$^{\{4\}}$	0.01738$^{\{3\}}$	0.01693$^{\{2\}}$
	MSEs	0.00041$^{\{1\}}$	0.00050$^{\{5\}}$	0.00049$^{\{4\}}$	0.00047$^{\{3\}}$	0.00046$^{\{2\}}$
	MREs	0.08294$^{\{1\}}$	0.08722$^{\{5\}}$	0.08713$^{\{4\}}$	0.08691$^{\{3\}}$	0.08463$^{\{2\}}$
	$\sum$ Ranks	3$^{\{1\}}$	15$^{\{5\}}$	12$^{\{4\}}$	9$^{\{3\}}$	6$^{\{2\}}$
80	AVEs	0.19137	0.20129	0.20019	0.19652	0.20056
	ABBs	0.01525$^{\{4\}}$	0.01527$^{\{5\}}$	0.01492$^{\{2\}}$	0.01504$^{\{3\}}$	0.01472$^{\{1\}}$
	MSEs	0.00035$^{\{2.5\}}$	0.00037$^{\{5\}}$	0.00036$^{\{4\}}$	0.00035$^{\{2.5\}}$	0.00034$^{\{1\}}$
	MREs	0.07625$^{\{1\}}$	0.07635$^{\{2\}}$	0.07458$^{\{4\}}$	0.07521$^{\{5\}}$	0.07362$^{\{3\}}$
	$\sum$ Ranks	7.5$^{\{2\}}$	12$^{\{5\}}$	10$^{\{3\}}$	10.5$^{\{4\}}$	5$^{\{1\}}$
100	AVEs	0.19106	0.20086	0.20045	0.19704	0.20071
	ABBs	0.01373$^{\{5\}}$	0.01355$^{\{3\}}$	0.01365$^{\{4\}}$	0.01307$^{\{1\}}$	0.01320$^{\{2\}}$
	MSEs	0.00028$^{\{3\}}$	0.00029$^{\{4.5\}}$	0.00029$^{\{4.5\}}$	0.00027$^{\{1.5\}}$	0.00027$^{\{1.5\}}$
	MREs	0.06863$^{\{5\}}$	0.06775$^{\{3\}}$	0.06826$^{\{4\}}$	0.06533$^{\{1\}}$	0.06600$^{\{2\}}$
	$\sum$ Ranks	13$^{\{5\}}$	10.5$^{\{3\}}$	12.5$^{\{4\}}$	3.5$^{\{1\}}$	5.5$^{\{2\}}$
				$\alpha = 0.5$		
20	AVEs	0.45460	0.51588	0.50524	0.48614	0.50106
	ABBs	0.07653$^{\{1\}}$	0.08695$^{\{4\}}$	0.50127$^{\{5\}}$	0.08315$^{\{3\}}$	0.08104$^{\{2\}}$
	MSEs	0.00876$^{\{1\}}$	0.01364$^{\{5\}}$	0.01230$^{\{4\}}$	0.01095$^{\{2\}}$	0.01121$^{\{3\}}$
	MREs	0.15306$^{\{1\}}$	0.17390$^{\{5\}}$	0.16956$^{\{4\}}$	0.16631$^{\{3\}}$	0.16209$^{\{2\}}$
	$\sum$Ranks	3$^{\{1\}}$	14$^{\{5\}}$	13$^{\{4\}}$	8$^{\{3\}}$	7$^{\{2\}}$
40	AVEs	0.44802	0.50510	0.50127	0.48845	0.50064
	ABBs	0.06435$^{\{5\}}$	0.06126$^{\{4\}}$	0.05984$^{\{2\}}$	0.06108$^{\{3\}}$	0.05762$^{\{1\}}$
	MSEs	0.00580$^{\{3\}}$	0.00595$^{\{5\}}$	0.00579$^{\{2\}}$	0.00586$^{\{4\}}$	0.00527$^{\{1\}}$
	MREs	0.12870$^{\{5\}}$	0.12252$^{\{4\}}$	0.11968$^{\{2\}}$	0.12216$^{\{3\}}$	0.11524$^{\{1\}}$
	$\sum$Ranks	13$^{\{4.5\}}$	13$^{\{4.5\}}$	6$^{\{2\}}$	10$^{\{3\}}$	3$^{\{1\}}$
60	AVEs	0.44914	0.49937	0.49984	0.48864	0.49723
	ABBs	0.05935$^{\{5\}}$	0.04785$^{\{2\}}$	0.04898$^{\{3\}}$	0.04990$^{\{4\}}$	0.04661$^{\{1\}}$
	MSEs	0.00479$^{\{5\}}$	0.00366$^{\{2\}}$	0.00380$^{\{3\}}$	0.00393$^{\{4\}}$	0.00354$^{\{1\}}$
	MREs	0.11870$^{\{5\}}$	0.09569$^{\{2\}}$	0.09796$^{\{3\}}$	0.09981$^{\{4\}}$	0.09322$^{\{1\}}$
	$\sum$Ranks	15$^{\{5\}}$	6$^{\{2\}}$	9$^{\{3\}}$	12$^{\{4\}}$	3$^{\{1\}}$
80	AVEs	0.44615	0.50067	0.50012	0.48867	0.49659
	ABBs	0.05782$^{\{5\}}$	0.04173$^{\{3\}}$	0.04291$^{\{4\}}$	0.04138$^{\{2\}}$	0.03986$^{\{1\}}$
	MSEs	0.00442$^{\{5\}}$	0.00283$^{\{3\}}$	0.00299$^{\{4\}}$	0.00266$^{\{2\}}$	0.00255$^{\{1\}}$
	MREs	0.11563$^{\{5\}}$	0.08347$^{\{3\}}$	0.08582$^{\{4\}}$	0.08275$^{\{2\}}$	0.07972$^{\{1\}}$
	$\sum$Ranks	15$^{\{5\}}$	9$^{\{3\}}$	12$^{\{4\}}$	6$^{\{2\}}$	3$^{\{1\}}$
100	AVEs	0.44547	0.49984	0.49761	0.49103	0.49653
	ABBs	0.05679$^{\{5\}}$	0.03785$^{\{2\}}$	0.03921$^{\{4\}}$	0.03798$^{\{3\}}$	0.03770$^{\{1\}}$
	MSEs	0.00415$^{\{5\}}$	0.00232$^{\{3\}}$	0.00238$^{\{4\}}$	0.00227$^{\{2\}}$	0.00220$^{\{1\}}$
	MREs	0.11359$^{\{5\}}$	0.07571$^{\{2\}}$	0.07841$^{\{4\}}$	0.07596$^{\{3\}}$	0.07540$^{\{1\}}$
	$\sum$Ranks	15$^{\{5\}}$	7$^{\{2\}}$	12$^{\{4\}}$	8$^{\{3\}}$	3$^{\{1\}}$

We can also obtain the PCE of the parameter α by solving the following non-linear equation

$$\sum_{i=1}^{n}\left\{x_{i:n}+3+\alpha+\frac{1}{\alpha}+\frac{W(z)}{\log(\alpha+1)}\right\}\Psi(x_{i:n})=0,$$

where

$$\Psi(x_{i:n})=\frac{\partial}{\partial\alpha}Q(p_i)$$
$$=\frac{\alpha W(z)\left[(\alpha^2+\alpha+2)\log(\alpha+1)+\alpha W(z)\right]-(\alpha-1)(\alpha+1)^2\log^2(\alpha+1)}{\alpha^2(\alpha+1)\log^2(\alpha+1)(W(z)+1)}.$$

Let $p_i=\frac{i}{n+1}$ is an estimate of $F(x_i)$, then the PCE of the DL parameter β follows by minimizing

$$PC_{DL}=\sum_{i=1}^{n}\left[x_{i:n}-Q(p_i)\right]^2$$
$$=\sum_{i=1}^{n}\left\{x_{i:n}+\frac{2\beta+W\left[e^{-\beta-1}(\beta+1)(p_i-1)\right]+1}{\beta}\right\}^2.$$

The PCE of the parameter β can also be determined by solving the non-linear equation

Table 4
Simulation results of the AVEs, ABBs, MSEs and MREs of DPL distribution.

n	Est.	MLE	CVME	LSE	PCE	WLSE
			$\alpha = 1$			
20	AVEs	0.82634	1.00301	0.98853	0.39414	0.98526
	ABBs	$0.20754^{\{5\}}$	$0.18880^{\{3\}}$	$0.18821^{\{2\}}$	$0.19711^{\{4\}}$	$0.18754^{\{1\}}$
	MSEs	$0.05792^{\{1\}}$	$1.13242^{\{5\}}$	$0.06146^{\{2\}}$	$0.06696^{\{3\}}$	$0.18977^{\{4\}}$
	MREs	$0.20754^{\{5\}}$	$0.18880^{\{3\}}$	$0.18821^{\{2\}}$	$0.19711^{\{4\}}$	$0.18754^{\{1\}}$
	$\sum$ Ranks	$11^{\{4\}}$	$11^{\{4\}}$	$6^{\{1.5\}}$	$11^{\{4\}}$	$6^{\{1.5\}}$
40	AVEs	0.81185	0.98221	0.97546	0.39857	0.97555
	ABBs	$0.19456^{\{5\}}$	$0.13690^{\{4\}}$	$0.13365^{\{1\}}$	$0.13547^{\{3\}}$	$0.13535^{\{2\}}$
	MSEs	$0.04710^{\{3\}}$	$1.10559^{\{5\}}$	$0.02827^{\{1\}}$	$0.02922^{\{2\}}$	$0.18057^{\{4\}}$
	MREs	$0.19456^{\{5\}}$	$0.13690^{\{4\}}$	$0.13365^{\{3\}}$	$0.13547^{\{2\}}$	$0.13535^{\{1\}}$
	$\sum$ Ranks	$13^{\{4.5\}}$	$13^{\{4.5\}}$	$5^{\{1\}}$	$7^{\{2.5\}}$	$7^{\{2.5\}}$
60	AVEs	0.80381	0.97510	0.97837	0.40102	0.97408
	ABBs	$0.04639^{\{1\}}$	$0.11094^{\{3\}}$	$0.11260^{\{4\}}$	$0.11619^{\{5\}}$	$0.10662^{\{2\}}$
	MSEs	$0.19875^{\{4\}}$	$1.09942^{\{5\}}$	$0.01983^{\{1\}}$	$0.02099^{\{2\}}$	$0.17540^{\{3\}}$
	MREs	$0.19875^{\{5\}}$	$0.11094^{\{2\}}$	$0.11260^{\{3\}}$	$0.11619^{\{4\}}$	$0.10662^{\{1\}}$
	$\sum$ Ranks	$10^{\{3.5\}}$	$10^{\{3.5\}}$	$8^{\{2\}}$	$11^{\{5\}}$	$6^{\{1\}}$
80	AVEs	0.80129	0.96981	0.97179	0.40182	0.97040
	ABBs	$0.19996^{\{5\}}$	$0.09579^{\{1\}}$	$0.09660^{\{4\}}$	$0.09630^{\{3\}}$	$0.09627^{\{2\}}$
	MSEs	$0.04532^{\{3\}}$	$1.09837^{\{5\}}$	$0.01423^{\{1\}}$	$0.01460^{\{2\}}$	$0.17315^{\{4\}}$
	MREs	$0.19996^{\{5\}}$	$0.09579^{\{1\}}$	$0.09660^{\{4\}}$	$0.09630^{\{3\}}$	$0.09627^{\{2\}}$
	$\sum$ Ranks	$13^{\{5\}}$	$7^{\{1\}}$	$9^{\{4\}}$	$8^{\{2.5\}}$	$8^{\{2.5\}}$
100	AVEs	0.80080	0.97775	0.96876	0.40777	0.97156
	ABBs	$0.19928^{\{5\}}$	$0.08687^{\{3\}}$	$0.08656^{\{1\}}$	$0.08786^{\{4\}}$	$0.08629^{\{2\}}$
	MSEs	$0.04415^{\{3\}}$	$1.09223^{\{5\}}$	$0.01158^{\{1\}}$	$0.01212^{\{2\}}$	$0.17547^{\{4\}}$
	MREs	$0.19928^{\{5\}}$	$0.08687^{\{3\}}$	$0.08656^{\{2\}}$	$0.08786^{\{4\}}$	$0.08629^{\{1\}}$
	$\sum$ Ranks	$13^{\{5\}}$	$11^{\{4\}}$	$4^{\{1\}}$	$10^{\{3\}}$	$7^{\{2\}}$
			$\alpha = 1.5$			
20	AVEs	1.13242	1.45882	1.42722	1.47038	1.43834
	ABBs	$0.39414^{\{5\}}$	$0.31306^{\{3\}}$	$0.30263^{\{1\}}$	$0.34243^{\{4\}}$	$0.30863^{\{2\}}$
	MSEs	$0.18977^{\{4\}}$	$0.16389^{\{3\}}$	$0.14361^{\{1\}}$	$0.20584^{\{5\}}$	$0.15506^{\{2\}}$
	MREs	$0.26276^{\{5\}}$	$0.20871^{\{3\}}$	$0.20175^{\{1\}}$	$0.22829^{\{4\}}$	$0.20575^{\{2\}}$
	$\sum$ Ranks	$14^{\{4.5\}}$	$9^{\{3\}}$	$3^{\{1\}}$	$14^{\{4.5\}}$	$6^{\{2\}}$
40	AVEs	1.10559	1.41598	1.40194	1.47368	1.43224
	ABBs	$0.39857^{\{5\}}$	$0.22208^{\{2\}}$	$0.21993^{\{1\}}$	$0.24205^{\{4\}}$	$0.22424^{\{3\}}$
	MSEs	$0.18057^{\{5\}}$	$0.07536^{\{2\}}$	$0.07308^{\{1\}}$	$0.09610^{\{4\}}$	$0.07802^{\{3\}}$
	MREs	$0.26571^{\{5\}}$	$0.14806^{\{2\}}$	$0.14662^{\{1\}}$	$0.16136^{\{4\}}$	$0.14949^{\{3\}}$
	$\sum$ Ranks	$15^{\{5\}}$	$6^{\{2\}}$	$3^{\{1\}}$	$12^{\{4\}}$	$9^{\{3\}}$
60	AVEs	1.09942	1.40902	1.40520	1.46148	1.43142
	ABBs	$0.40102^{\{5\}}$	$0.18479^{\{2\}}$	$0.18468^{\{1\}}$	$0.19474^{\{4\}}$	$0.18779^{\{3\}}$
	MSEs	$0.17540^{\{5\}}$	$0.05021^{\{1\}}$	$0.05158^{\{2\}}$	$0.06121^{\{4\}}$	$0.05393^{\{3\}}$
	MREs	$0.26735^{\{5\}}$	$0.12320^{\{2\}}$	$0.12312^{\{1\}}$	$0.12983^{\{4\}}$	$0.12519^{\{3\}}$
	$\sum$ Ranks	$15^{\{5\}}$	$5^{\{2\}}$	$4^{\{1\}}$	$12^{\{4\}}$	$9^{\{3\}}$
80	AVEs	1.09837	1.40565	1.40449	1.46497	1.41964
	ABBs	$0.40182^{\{5\}}$	$0.16934^{\{4\}}$	$0.16823^{\{3\}}$	$0.16752^{\{2\}}$	$0.15990^{\{1\}}$
	MSEs	$0.17515^{\{5\}}$	$0.04256^{\{3\}}$	$0.04121^{\{2\}}$	$0.04326^{\{4\}}$	$0.03823^{\{1\}}$
	MREs	$0.26788^{\{5\}}$	$0.11289^{\{4\}}$	$0.11215^{\{3\}}$	$0.11168^{\{2\}}$	$0.10660^{\{1\}}$
	$\sum$ Ranks	$15^{\{5\}}$	$11^{\{4\}}$	$8^{\{2.5\}}$	$8^{\{2.5\}}$	$3^{\{1\}}$
100	AVEs	1.09223	1.39648	1.39322	1.47211	1.42241
	ABBs	$0.40777^{\{5\}}$	$0.15884^{\{4\}}$	$0.15503^{\{3\}}$	$0.14919^{\{1\}}$	$0.15027^{\{2\}}$
	MSEs	$0.17347^{\{5\}}$	$0.03649^{\{4\}}$	$0.03476^{\{2\}}$	$0.03480^{\{3\}}$	$0.03379^{\{1\}}$
	MREs	$0.27185^{\{5\}}$	$0.10590^{\{4\}}$	$0.10336^{\{3\}}$	$0.09946^{\{1\}}$	$0.10018^{\{2\}}$
	$\sum$ Ranks	$15^{\{5\}}$	$12^{\{4\}}$	$8^{\{3\}}$	$5^{\{1.5\}}$	$5^{\{1.5\}}$

$$\sum_{i=1}^{n}\left\{x_{i:n}+\frac{2\beta+W\left[e^{-\beta-1}(\beta+1)(p_i-1)\right]+1}{\beta}\right\}\Omega(x_{i:n})=0,$$

where

$$\Omega(x_{i:n})=\frac{1+W\left[e^{-\beta-1}(\beta+1)(p-1)\right]}{\beta^2}+\left\{\beta+\frac{\beta+1}{W[e^{-\beta-1}(\beta+1)(p-1)]}+1\right\}^{-1}.$$

4. Simulation Study

In this section, the performance of different estimation methods for estimating the DPL(α) and DL(β) parameters is explored via numerical simulations. Different sample sizes, $n=\{20,40,60,80,100\}$, and several parametric values are considered for the parameters $\alpha=\{0.2,0.5,1,1.5\}$ and $\beta=\{0.2,0.5,1,1.5\}$. We generate $N=2,000$ random samples from the two distributions using Eqs. (3) and (7), respectively. The performance of these estimators are assessed by determining some indices such as the average values of the estimates (AVEs), average mean square errors (MSEs), aver-

Table 5
Simulation results of the AVEs, ABBs, MSEs and MREs of DL distribution.

n	Est.	MLE	CVME	LSE	PCE	WLSE
			$\beta = 0.2$			
20	AVEs	0.20428	0.21867	0.21764	0.19561	0.21647
	ABBs	0.02624$^{\{1\}}$	0.03455$^{\{4\}}$	0.03515$^{\{5\}}$	0.02759$^{\{2\}}$	0.03290$^{\{3\}}$
	MSEs	0.00113$^{\{1\}}$	0.45543$^{\{5\}}$	0.00224$^{\{3\}}$	0.00117$^{\{2\}}$	0.00662$^{\{4\}}$
	MREs	0.13122$^{\{1\}}$	0.17276$^{\{4\}}$	0.17576$^{\{5\}}$	0.13794$^{\{3\}}$	0.16450$^{\{2\}}$
	$\sum$ Ranks	3$^{\{1\}}$	13$^{\{4.5\}}$	13$^{\{4.5\}}$	7$^{\{2\}}$	9$^{\{3\}}$
40	AVEs	0.20179	0.21546	0.21517	0.19499	0.21510
	ABBs	0.01819$^{\{1\}}$	0.02512$^{\{5\}}$	0.02509$^{\{4\}}$	0.01912$^{\{2\}}$	0.02470$^{\{3\}}$
	MSEs	0.00054$^{\{1\}}$	0.44796$^{\{5\}}$	0.00111$^{\{3\}}$	0.00057$^{\{2\}}$	0.00489$^{\{4\}}$
	MREs	0.09097$^{\{1\}}$	0.12561$^{\{5\}}$	0.12543$^{\{4\}}$	0.09560$^{\{2\}}$	0.12348$^{\{3\}}$
	$\sum$ Ranks	3$^{\{1\}}$	15$^{\{5\}}$	11$^{\{4\}}$	6$^{\{2\}}$	10$^{\{3\}}$
60	AVEs	0.20131	0.21501	0.21413	0.19588	0.21386
	ABBs	0.01507$^{\{1\}}$	0.02128$^{\{4\}}$	0.02150$^{\{5\}}$	0.01566$^{\{2\}}$	0.02058$^{\{3\}}$
	MSEs	0.00036$^{\{1\}}$	0.44746$^{\{5\}}$	0.00076$^{\{3\}}$	0.00037$^{\{2\}}$	0.00416$^{\{4\}}$
	MREs	0.07537$^{\{1\}}$	0.10641$^{\{4\}}$	0.10748$^{\{5\}}$	0.07832$^{\{2\}}$	0.10290$^{\{3\}}$
	$\sum$ Ranks	3$^{\{1\}}$	13$^{\{4.5\}}$	13$^{\{4.5\}}$	6$^{\{2\}}$	10$^{\{3\}}$
80	AVEs	0.20195	0.21412	0.21352	0.19634	1.41964
	ABBs	0.01258$^{\{1\}}$	0.01948$^{\{4\}}$	0.01927$^{\{3\}}$	0.01344$^{\{2\}}$	0.21311$^{\{5\}}$
	MSEs	0.00026$^{\{1\}}$	0.44678$^{\{5\}}$	0.00060$^{\{3\}}$	0.00028$^{\{2\}}$	0.01814$^{\{4\}}$
	MREs	0.06291$^{\{2\}}$	0.09740$^{\{5\}}$	0.09633$^{\{4\}}$	0.06718$^{\{3\}}$	0.00393$^{\{1\}}$
	$\sum$ Ranks	4$^{\{1\}}$	14$^{\{5\}}$	10$^{\{3.5\}}$	7$^{\{2\}}$	10$^{\{3.5\}}$
100	AVEs	0.20061	0.21319	0.21407	0.19668	0.21336
	ABBs	0.01161$^{\{1\}}$	0.01757$^{\{4\}}$	0.01809$^{\{5\}}$	0.01189$^{\{2\}}$	0.01713$^{\{3\}}$
	MSEs	0.00021$^{\{1\}}$	0.44578$^{\{5\}}$	0.00051$^{\{3\}}$	0.00023$^{\{2\}}$	0.00376$^{\{4\}}$
	MREs	0.05804$^{\{1\}}$	0.08784$^{\{4\}}$	0.09044$^{\{5\}}$	0.05943$^{\{2\}}$	0.08563$^{\{3\}}$
	$\sum$ Ranks	3$^{\{1\}}$	13$^{\{4.5\}}$	13$^{\{4.5\}}$	6$^{\{2\}}$	10$^{\{3\}}$
			$\beta = 0.5$			
20	AVEs	0.45543	0.50757	0.50711	0.48569	0.50300
	ABBs	0.06791$^{\{2\}}$	0.07239$^{\{4\}}$	0.07257$^{\{5\}}$	0.06734$^{\{1\}}$	0.06946$^{\{3\}}$
	MSEs	0.00662$^{\{1\}}$	0.00900$^{\{5\}}$	0.00897$^{\{4\}}$	0.00723$^{\{2\}}$	0.00795$^{\{3\}}$
	MREs	0.13582$^{\{2\}}$	0.14478$^{\{4\}}$	0.14514$^{\{5\}}$	0.13468$^{\{1\}}$	0.13892$^{\{3\}}$
	$\sum$ Ranks	5$^{\{2\}}$	13$^{\{4\}}$	14$^{\{5\}}$	4$^{\{1\}}$	9$^{\{3\}}$
40	AVEs	0.44796	0.50448	0.50012	0.48775	0.49911
	ABBs	0.05983$^{\{5\}}$	0.05015$^{\{4\}}$	0.05004$^{\{3\}}$	0.04821$^{\{1\}}$	0.04852$^{\{2\}}$
	MSEs	0.00489$^{\{5\}}$	0.00414$^{\{4\}}$	0.00397$^{\{3\}}$	0.00360$^{\{1\}}$	0.00371$^{\{2\}}$
	MREs	0.11966$^{\{5\}}$	0.10030$^{\{4\}}$	0.10008$^{\{3\}}$	0.09643$^{\{1\}}$	0.09704$^{\{2\}}$
	$\sum$ Ranks	15$^{\{5\}}$	12$^{\{4\}}$	9$^{\{3\}}$	3$^{\{1\}}$	6$^{\{2\}}$
60	AVEs	0.44746	0.49718	0.50018	0.48812	0.49580
	ABBs	0.05631$^{\{5\}}$	0.03982$^{\{2\}}$	0.04011$^{\{3\}}$	0.04035$^{\{4\}}$	0.03790$^{\{1\}}$
	MSEs	0.00416$^{\{5\}}$	0.00250$^{\{2\}}$	0.00257$^{\{4\}}$	0.00253$^{\{3\}}$	0.00227$^{\{1\}}$
	MREs	0.11263$^{\{5\}}$	0.07963$^{\{2\}}$	0.08021$^{\{3\}}$	0.08069$^{\{4\}}$	0.07580$^{\{1\}}$
	$\sum$ Ranks	15$^{\{5\}}$	6$^{\{2\}}$	10$^{\{3\}}$	11$^{\{4\}}$	3$^{\{1\}}$
80	AVEs	0.44678	0.49888	0.49745	0.49242	0.49762
	ABBs	0.05524$^{\{5\}}$	0.03556$^{\{3\}}$	0.03569$^{\{4\}}$	0.03517$^{\{2\}}$	0.03347$^{\{1\}}$
	MSEs	0.00393$^{\{5\}}$	0.00196$^{\{3\}}$	0.00200$^{\{4\}}$	0.00191$^{\{2\}}$	0.00176$^{\{1\}}$
	MREs	0.11048$^{\{5\}}$	0.07111$^{\{3\}}$	0.07138$^{\{4\}}$	0.07035$^{\{2\}}$	0.06694$^{\{1\}}$
	$\sum$ Ranks	15$^{\{5\}}$	9$^{\{3\}}$	12$^{\{4\}}$	6$^{\{2\}}$	3$^{\{1\}}$
100	AVEs	0.44578	0.49805	0.49762	0.49246	0.49460
	ABBs	0.05516$^{\{5\}}$	0.03122$^{\{2\}}$	0.03192$^{\{4\}}$	0.03160$^{\{3\}}$	0.02976$^{\{1\}}$
	MSEs	0.00376$^{\{5\}}$	0.00153$^{\{2\}}$	0.00161$^{\{4\}}$	0.00154$^{\{3\}}$	0.00141$^{\{1\}}$
	MREs	0.11033$^{\{5\}}$	0.06245$^{\{2\}}$	0.06385$^{\{4\}}$	0.06319$^{\{3\}}$	0.05952$^{\{1\}}$
	$\sum$ Ranks	15$^{\{5\}}$	6$^{\{2\}}$	12$^{\{4\}}$	9$^{\{3\}}$	3$^{\{1\}}$

age absolute biases (ABBs), and average mean relative estimates (MREs) for all sample sizes and parametric combinations using the R software©.

These indices can be determined using the following equations

$$MSEs = \frac{1}{N}\sum_{i=1}^{N}\left(\widehat{\theta} - \theta\right)^2, \quad ABBs = \frac{1}{N}\sum_{i=1}^{N}|\widehat{\theta} - \theta|,$$

$$MREs = \frac{1}{N}\sum_{i=1}^{N}|\widehat{\theta} - \theta|/\theta,$$

where $\theta = \alpha$ or β.

Simulation results including AVEs, ABBs, MSEs and MREs for the two parameters α and β of the DPL and DL distributions using several estimation approaches are reported in Tables 3–6. Additionally, these tables illustrate the rank of each estimator among all the five estimators which is represented by superscript indicators in each row and the $\sum$ *Ranks* which is represented by the partial sum of ranks in each column for every sample size.

One can note that the parameter estimates for both distributions are entirely good, that is, these estimates are quite reliable and very close to the true parameter values, showing small ABBs, MSEs and MREs for all values of the parameters α and β. The five estimators achieve the consistency property, where the MSEs, ABBs and MREs decrease as n increases, for all considered parametric values.

The performance ordering of these estimators is determined based on the partial and overall rank of the introduced estimators for both DPL and DL distributions which are listed in Table 7. Based

Table 6
Simulation results of the AVEs, ABBs, MSEs and MREs of DL distribution.

n	Est.	MLE	CVME	LSE	PCE	WLSE
			$\beta = 1$			
20	AVEs	0.82182	0.97809	0.96423	0.38079	0.97326
	ABBs	$0.19066^{(5)}$	$0.13673^{(2)}$	$0.13624^{(1)}$	$0.14081^{(4)}$	$0.13880^{(3)}$
	MSEs	$0.04606^{(4)}$	$1.12290^{(5)}$	$0.02838^{(1)}$	$0.03222^{(3)}$	$0.03047^{(2)}$
	MREs	$0.19066^{(5)}$	$0.13673^{(2)}$	$0.13624^{(1)}$	$0.14081^{(4)}$	$0.13880^{(3)}$
	$\sum$ Ranks	$14^{\{5\}}$	$9^{\{3\}}$	$3^{\{1\}}$	$11^{\{4\}}$	$8^{\{2\}}$
40	AVEs	0.80929	0.96822	0.96475	0.38904	0.96779
	ABBs	$0.19194^{(5)}$	$0.09707^{(1)}$	$0.09854^{(2)}$	$0.10242^{(3)}$	$0.10261^{(4)}$
	MSEs	$0.04282^{(4)}$	$1.11120^{(5)}$	$0.01488^{(1)}$	$0.01615^{(3)}$	$0.01603^{(2)}$
	MREs	$0.19194^{(5)}$	$0.09707^{(1)}$	$0.09854^{(2)}$	$0.10242^{(3)}$	$0.10261^{(4)}$
	$\sum$ Ranks	$14^{\{5\}}$	$7^{\{2\}}$	$5^{\{1\}}$	$9^{\{3\}}$	$10^{\{4\}}$
60	AVEs	0.80397	0.96037	0.96231	0.39034	0.96423
	ABBs	$0.19624^{(5)}$	$0.08432^{(4)}$	$0.08163^{(1)}$	$0.08230^{(2)}$	$0.08331^{(3)}$
	MSEs	$0.04263^{(4)}$	$1.10966^{(5)}$	$0.01014^{(1)}$	$0.01056^{(2)}$	$0.01059^{(3)}$
	MREs	$0.19624^{(5)}$	$0.08432^{(4)}$	$0.08163^{(1)}$	$0.08230^{(2)}$	$0.08331^{(3)}$
	$\sum$ Ranks	$14^{\{5\}}$	$13^{\{4\}}$	$3^{\{1\}}$	$6^{\{2\}}$	$9^{\{3\}}$
80	AVEs	0.80331	0.96340	0.95640	0.39368	0.96558
	ABBs	$0.19673^{(5)}$	$0.07436^{(3)}$	$0.07544^{(4)}$	$0.07339^{(2)}$	$0.07205^{(1)}$
	MSEs	$0.04193^{(4)}$	$1.10632^{(5)}$	$0.00847^{(3)}$	$0.00828^{(2)}$	$0.00789^{(1)}$
	MREs	$0.19673^{(5)}$	$0.07436^{(3)}$	$0.07544^{(4)}$	$0.07339^{(2)}$	$0.07205^{(1)}$
	$\sum$ Ranks	$14^{\{5\}}$	$11^{\{3.5\}}$	$11^{\{3.5\}}$	$6^{\{2\}}$	$3^{\{1\}}$
100	AVEs	0.80240	0.96210	0.95805	0.39654	0.96737
	ABBs	$0.19760^{(5)}$	$0.06718^{(3)}$	$0.06812^{(4)}$	$0.06577^{(1)}$	$0.06624^{(2)}$
	MSEs	$0.04160^{(4)}$	$1.10346^{(5)}$	$0.00699^{(3)}$	$0.00664^{(2)}$	$0.00663^{(1)}$
	MREs	$0.19760^{(5)}$	$0.06718^{(3)}$	$0.06812^{(4)}$	$0.06577^{(1)}$	$0.06624^{(2)}$
	$\sum$ Ranks	$14^{\{5\}}$	$11^{\{3.5\}}$	$11^{\{3.5\}}$	$4^{\{1\}}$	$5^{\{2\}}$
			$\beta = 1.5$			
20	AVEs	1.12290	1.38100	1.36864	1.46240	1.40862
	ABBs	$0.38079^{(5)}$	$0.21436^{(3)}$	$0.21037^{(1)}$	$0.23352^{(4)}$	$0.21259^{(2)}$
	MSEs	$0.16817^{(5)}$	$0.06645^{(2)}$	$0.06416^{(1)}$	$0.08795^{(4)}$	$0.06797^{(3)}$
	MREs	$0.25386^{(5)}$	$0.14290^{(3)}$	$0.14025^{(1)}$	$0.15568^{(4)}$	$0.14173^{(2)}$
	$\sum$ Ranks	$15^{\{5\}}$	$8^{\{3\}}$	$3^{\{1\}}$	$12^{\{4\}}$	$7^{\{2\}}$
40	AVEs	1.11120	1.36654	1.36327	1.46978	1.41346
	ABBs	$0.38904^{(5)}$	$0.17306^{(3)}$	$0.17496^{(4)}$	$0.16748^{(2)}$	$0.16515^{(1)}$
	MSEs	$0.16322^{(5)}$	$0.04279^{(3)}$	$0.04275^{(2)}$	$0.04293^{(4)}$	$0.04105^{(1)}$
	MREs	$0.25936^{(5)}$	$0.11537^{(3)}$	$0.11664^{(4)}$	$0.11165^{(2)}$	$0.11010^{(1)}$
	$\sum$ Ranks	$15^{\{5\}}$	$9^{\{3\}}$	$10^{\{4\}}$	$8^{\{2\}}$	$3^{\{1\}}$
60	AVEs	1.10966	1.36881	1.36059	1.47028	1.41498
	ABBs	$0.39034^{(5)}$	$0.15704^{(3)}$	$0.15872^{(4)}$	$0.13801^{(1)}$	$0.14123^{(2)}$
	MSEs	$0.16099^{(5)}$	$0.03448^{(3)}$	$0.03479^{(4)}$	$0.02881^{(1)}$	$0.02943^{(2)}$
	MREs	$0.26023^{(5)}$	$0.10469^{(3)}$	$0.10581^{(4)}$	$0.09201^{(1)}$	$0.09415^{(2)}$
	$\sum$ Ranks	$15^{\{5\}}$	$9^{\{3\}}$	$12^{\{4\}}$	$3^{\{1\}}$	$6^{\{2\}}$
80	AVEs	1.10632	1.36635	1.36064	1.47588	1.41368
	ABBs	$0.39368^{(5)}$	$0.14766^{(3)}$	$0.15346^{(4)}$	$0.11639^{(1)}$	$0.12443^{(2)}$
	MSEs	$0.16079^{(5)}$	$0.02978^{(3)}$	$0.03201^{(4)}$	$0.02114^{(1)}$	$0.02258^{(2)}$
	MREs	$0.26245^{(5)}$	$0.09844^{(3)}$	$0.10231^{(4)}$	$0.07759^{(1)}$	$0.08295^{(2)}$
	$\sum$ Ranks	$15^{\{5\}}$	$9^{\{3\}}$	$12^{\{4\}}$	$3^{\{1\}}$	$6^{\{2\}}$
100	AVEs	1.10346	1.36260	1.36113	1.47760	1.41373
	ABBs	$0.39654^{(5)}$	$0.14799^{(4)}$	$0.14687^{(3)}$	$0.10493^{(1)}$	$0.11783^{(2)}$
	MSEs	$0.16058^{(5)}$	$0.02903^{(4)}$	$0.02836^{(3)}$	$0.01690^{(1)}$	$0.02007^{(2)}$
	MREs	$0.26436^{(5)}$	$0.09866^{(4)}$	$0.09791^{(3)}$	$0.06996^{(1)}$	$0.07856^{(2)}$
	$\sum$ Ranks	$15^{\{5\}}$	$12^{\{4\}}$	$9^{\{3\}}$	$3^{\{1\}}$	$6^{\{2\}}$

on Table 7, we conclude that the performance ordering of these estimators, for the DPL distribution, from best to worst is WLSE, LSE, PCE, CVME, and MLE for all the studied cases. Furthermore, the performance ordering of these estimators, for the DL model, from best to worst is PCE, WLSE, LSE, CVME, and MLE for all the studied cases.

In summary, we can conclude that the weighted least-squares method outperforms all other considered estimation methods with overall score of 36, hence this method is recommended to estimate the parameter of the DPL distribution, whereas the percentile method is recommended to estimate the DL parameter due to its superiority with overall rank of 42.

5. Modeling medicine data

In this section, we use three real-life data sets from medical science to show the superiority of the DPL and DL distributions by comparing them with some well-known discrete distributions such as discrete Pareto (DP) (Krishna and Pundir, 2009), discrete Burr (DB) (Krishna and Pundir, 2009) and discrete Burr-Hatke (DBH) (El-Morshedy et al., 2020) distributions.

The first data set contains 20 observations about the numbers of daily deaths in Saudi Arabia due to COVID-19 infections from 24 March to 12 April, 2020. The second data set contains 20 observations about the numbers of daily recover patients in Saudi Arabia

Table 7
Partial and overall ranks of all the methods of estimation of DPL and DL distributions by various values of α and β.

DPL	Parameter	n	MLE	CVME	LSE	PCE	WLSE
		20	1	4	5	3	2
		40	1	5	4	2	3
	$\alpha = 0.2$	60	1	5	4	3	2
		80	2	5	3	4	1
		100	5	3	4	1	2
		20	1	5	4	3	2
		40	4.5	4.5	2	3	1
	$\alpha = 0.5$	60	5	2	3	4	1
		80	5	3	4	2	1
		100	5	2	4	3	1
		20	4	4	1.5	4	1.5
		40	4.5	4.5	1	2.5	2.5
	$\alpha = 1$	60	3.5	3.5	2	5	1
		80	5	1	4	2.5	2.5
		100	5	4	1	3	2
		20	4.5	3	1	4.5	2
		40	5	2	1	4	3
	$\alpha = 1.5$	60	5	2	1	4	3
		80	5	4	2.5	2.5	1
		100	5	4	3	1.5	1.5
	$\sum$ Ranks		77	70.5	55	61.5	36
	Overall Rank		5	4	2	3	1
DL	Parameter	n	MLE	CVME	LSE	PCE	WLSE
		20	1	4.5	4.5	2	3
		40	1	5	4	2	3
	$\beta = 0.2$	60	1	4.5	4.5	2	3
		80	1	5	3.5	2	3.5
		100	1	4.5	4.5	2	3
		20	2	4	5	1	3
		40	5	4	3	1	2
	$\beta = 0.5$	60	5	2	3	4	1
		80	5	3	4	2	1
		100	5	2	4	3	1
		20	5	3	1	4	2
		40	5	2	1	3	4
	$\beta = 1$	60	5	4	1	2	3
		80	5	3.5	3.5	2	1
		100	5	3.5	3.5	1	2
		20	5	3	1	4	2
		40	5	3	4	2	1
	$\beta = 1.5$	60	5	3	4	1	2
		80	5	3	4	1	2
	100	5	4	3	1	2	
	$\sum$ Ranks		77	70.5	66	42	44.5
	Overall Rank		5	4	3	1	2

Table 8
Estimates, LL and UL of bootstrap CIs, KS and KS-PV of the DPL and DL distributions for the three data sets

Data	Distribution	WLS Estimates	LL	UL	KS	KS-PV
Data I	DPL	$\hat{\alpha}$=0.07278	0.05382	0.10649	0.18382	0.50870
	DB	$\hat{\alpha} = 5.75821$	1.41244	34.39454	0.25392	0.15164
		$\hat{\theta} = 0.94314$	0.79116	0.98987	––	––
	DP	$\hat{\theta} = 0.73538$	0.58620	0.81962	0.28408	0.07926
	DBH	$\hat{\lambda} = 0.99999$	0.99989	0.99999	0.70001	0.00000
Data II	DPL	$\hat{\alpha} = 0.00511$	0.00372	0.00722	0.25158	0.13310
	DB	$\hat{\alpha} = 5.07404$	0.75426	17.97669	0.38465	0.00358
		$\hat{\theta} = 0.97169$	0.82295	0.99039	––	––
	DP	$\hat{\theta} = 0.86488$	0.78640	0.91224	0.38665	0.00335
	DBH	$\hat{\lambda} = 0.99999$	0.99989	0.99999	0.96668	0.00000
Data III	DPL	$\hat{\alpha} = 0.09261$	0.06989	0.14024	0.09792	0.99080
	DB	$\hat{\alpha} = 5.18202$	1.34475	35.44724	0.29850	0.05665
		$\hat{\theta} = 0.93792$	0.79226	0.98992	––	––
	DP	$\hat{\theta} = 0.73546$	0.59074	0.82545	0.30059	0.05388
	DBH	$\hat{\lambda} = 0.99999$	0.99989	0.99999	0.75001	0.00000
Data	Distribution	PC Estimates	LL	UL	KS	KS-PV
Data I	DL	$\hat{\beta} = 0.07242$	0.05054	0.09936	0.18505	0.50001
	DB	$\hat{\alpha} = 90.65945$	80.76587	99.69480	0.51712	0.00005

(*continued on next page*)

Table 8 (*continued*)

Data	Distribution	WLS Estimates	LL	UL	KS	KS-PV
		$\hat{\theta} = 0.99230$	0.01396	0.99546	––	––
	DP	$\hat{\theta} = 0.49762$	0.32026	0.73038	0.53422	0.00002
	DBH	$\hat{\lambda} = 0.20863$	0.00100	0.92293	0.98549	0.00000
Data II	DL	$\hat{\beta} = 0.00524$	0.00366	0.00688	0.24941	0.13934
	DB	$\hat{\alpha} = 90.65945$	4.40536	99.69480	0.76872	0.00000
		$\hat{\theta} = 0.99517$	0.00150	0.99564	––	––
	DP	$\hat{\theta} = 0.63987$	0.47984	0.81331	0.77764	0.00000
	DBH	$\hat{\lambda} = 0.06873$	0.00010	0.92501	1.00000	0.00000
Data III	DL	$\hat{\beta} = 0.08983$	0.06421	0.11654	0.10653	0.97706
	DB	$\hat{\alpha} = 90.65711$	86.33405	96.47999	0.57943	0.00000
		$\hat{\theta} = 0.99198$	0.98601	0.99546	––	––
	DP	$\hat{\theta} = 0.48391$	0.31194	0.71604	0.60646	0.00000
	DBH	$\hat{\lambda} = 0.30223$	0.00010	0.92164	0.96955	0.00000

from COVID-19 infections from 24 March to 12 April, 2020. The first and second data sets are reported on https://www.kaggle.com/sudalairajkumar/novel-corona-virus-2019-dataset/data#.

The third data set refers to remission times in weeks of 20 leukemia patients randomly assigned to a certain treatment (Lawless, 2011), and it was analyzed by Al-Babtain et al. (2020).

Based on our study in Section 4, we conclude that the weighted least squares (WLS) method is recommended to estimate the DPL parameter, and the percentile (PC) method is recommended to estimate the DL parameter. Hence, the WLS and PC methods will be applied in this section to estimate the parameters of both distributions from the three real data sets.

Table 8 reports the WLS estimates of the DPL distribution, lower limits (LL) and upper limits (UL) of the bootstrap confidence intervals (CIs) and Kolmogorov–Smirnov (KS) statistics along with their associated p-values (KS-PV) for the three data sets. Further, the Pc estimates of the DL distribution, LL and UL of the bootstrap CIs, KS and its KS-PV for the three data sets were listed in Table 8.

Fig. 2 displays the probability-probability (PP) plots of the fitted DPL and DL models and other distributions for the three data sets, respectively.

Based on the KS and KS-PV, we conclude that the DPL and DL distributions provide adequate fit for the three data sets as compared with other discrete models.

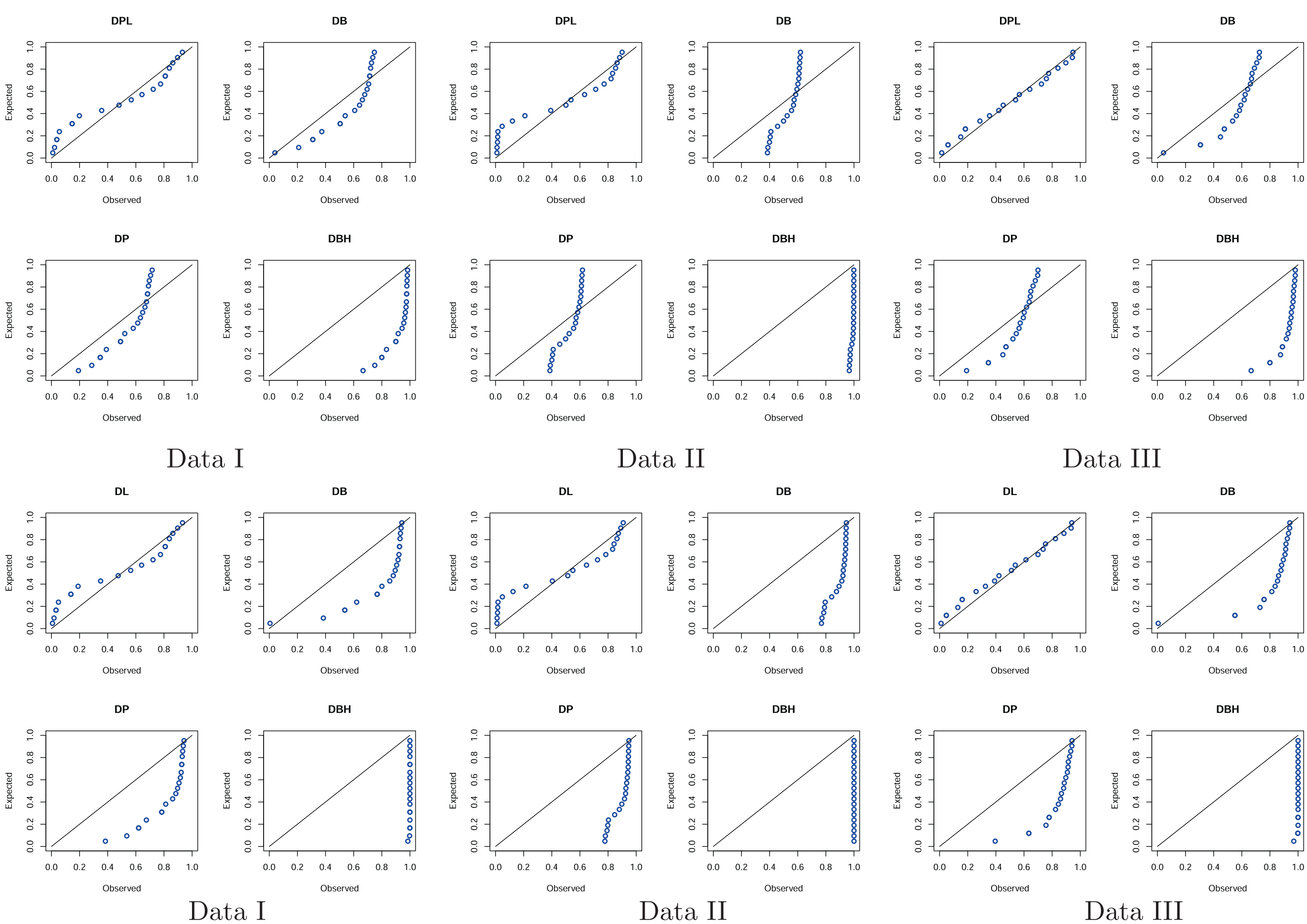

Fig. 2. PP plots of the DPL (top panel) and DP (bottom panel) models and other models for three data sets.

6. Concluding remarks

In this paper, we derive two risk measures called value at risk and tail value at risk for the discrete Poisson-Lindley (DPL) and discrete Lindley (DL) distributions, and study their behavior using numerical simulations. Further, we discuss the estimation of the parameters for the two discrete distributions using five classical methods of estimation namely, the maximum likelihood, least squares, weighted least squares, percentiles, and Cramér-von Mises. We present detailed simulation results to compare these estimators in terms of mean square errors, average absolute biases, mean relative estimates, and total absolute relative errors of the parameters. The simulation study illustrates that all classical estimators perform very well and their performance ordering, depending on overall ranks, from best to worst is WLSE, LSE, PCE, CVME, and MLE for the DPL distribution. Further, the performance ordering for the DL parameter is PCE, WLSE, LSE, CVME, and MLE. The practical importance of the two distributions is discussed using three real-life data sets form medicine field. The DPL and DL distributions provide better fit for the three analyzed data than some other discrete distributions.

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Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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