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Szász-Beta operators via Hermite Polynomial

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ABSTRACT

The aim of present article is to introduce the Szász-Beta operators in terms of Hermite Polynomial. We calculate some estimates and then discuss convergence theorems and order of approximation in terms of Korovkin theorem and first order modulus of smoothness respectively. Next, we study pointwise approximation results in terms of Peetre's K-functional, second order modulus of smoothness, Lipschitz type space and rth order Lipschitz type maximal function. Lastly, weighted approximation results and statistical approximation theorems are proved.

1. Introduction and preliminaries

The approximation in operator theory is a significant area of mathematical analysis, emerged in the 19th century and continues to be studied by mathematicians worldwide. Its relevance extends beyond mathematics to various fields, including the basic sciences and engineering. The primary objective of approximation theory is to represent a complex function using simpler functions with more elementary properties, such as differentiability and integrability. It has applications in computational aspects like describing the shapes of geometric objects as well as in applied and pure mathematics, including fixed point theory and numerical analysis. Control nets and control points are used to study parametric surfaces and curves, respectively. The theory has widespread applications in other scientific branches, such as data structures, computer graphics, computer algebra and numerical analysis. In 1885, Weierstrass (1885) gave an elegant result in approximation theory named as Weierstrass approximation theorem. Several renowned mathematicians have worked on providing simpler and more understandable proofs for this theorem.

In order to provide a succinct proof of the Weierstrass approximation theorem using binomial distribution, Bernstein (1913) invented a sequence of polynomials known as Bernstein polynomials in 1912 as follows:

$$B_n(f;y)=\sum_{l=0}^{n} f\left(\frac{l}{n}\right)\left(\begin{array}{c} n \\ l \end{array}\right) y^l(1-y)^{n-l}, \quad y\in[0,1], \tag{1.1}$$

where f is a bounded function defined on $[0,1]$. The approximation with the sequences of operators given in (1.1) are restricted for bounded function on $[0,1]$. To approximate on $[0,\infty)$, Szász (1950) gave modification of the sequences given in (1.1) which play an important role in the development of operator theory as below:

$$S_n(f;y)=e^{-ny}\sum_{l=0}^{\infty}\frac{(ny)^l}{l!}f\left(\frac{l}{n}\right), n\in\mathbb{N}, \tag{1.2}$$

where the real valued function $f \in C[0,\infty)$. The linear positive operators introduced in (1.2) are restricted for the space of continuous functions only. To approximate in longer class of functions, i.e., space of functions which are measurable in Lebesgue sense, several integral versions of these sequences of operators are introduced, e.g., Szász–Kantorovich and Szász–Durremeyer operators etc. (see Szász, 1950;

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Durrmeyer, 1967). Many mathematicians, e.g., Aslan and Rathour (2022), Acu et al. (2011, 2019), Mohiuddine et al. (2017, 2021), Mursaleen et al. (2020, 2019), Khan et al. (2021), Nasiruzzaman (2021), Wafi and Rao (2016) and Wafi and Rao (2017) gave various generalizations for such type of sequences. Grazyna (2016) presented a class of sequence of operators $G_n^\alpha(.;.), n \in \mathbb{N}, \alpha \geq 0$, given by the formula

$$G_n^\alpha(f;y) = e^{-(ny+\alpha y^2)} \sum_{l=0}^{\infty} \frac{y^l}{l!} H_l(n,\alpha) f\left(\frac{l}{n}\right), \quad y \in \mathbb{R}_0^+, \tag{1.3}$$

where H_l is the two variable Hermite polynomial (see Appell et al., 1926) given by

$$H_l(n,\alpha) = l! \sum_{m=0}^{[\frac{l}{2}]} \frac{n^{l-2m}\alpha^m}{(l-2\ m)!m!}. \tag{1.4}$$

The Hermite polynomials and their properties were investigated in many papers, for example in Babusci et al. (2012), Dattoli et al. (2005). Integrals of these polynomials are ubiquitous in problems involving classical and quantum optics as well as quantum physics (see Dattoli et al., 2005). The operators (1.3) are linear and positive. Basic facts on positive linear operators, their generalizations and applications can be found in DeVore and Lorentz (1993).

The sequences of operators presented in (1.3) are restricted for continuous function only. Motivated with the above development, we introduce a sequences of positive linear operators to give approximations in bigger class of function, i.e., the space of Lebesgue measurable functions which is named as Szász-Beta operators in view of Hermite Polynomial as:

$$H_s^\alpha(f;y) = \sum_{l=0}^{\infty} P_{s,l}^\alpha(y) \int_0^\infty Q_s(v) f(v) dv, \quad for \quad y \in \mathbb{R}_0^+, \tag{1.5}$$

where

$$P_{s,l}^\alpha(y) = e^{-(sy+\alpha y^2)} \frac{y^l}{l!} H_l(s,\alpha) \quad and \quad Q_s(v) = \frac{1}{\beta(l+1,s)}\left[\frac{v^l}{(1+v)^{l+1+s}}\right],$$

with β (Beta) function, $\beta(l+1,s) = \int_0^\infty \frac{v^l}{(1+v)^{l+1+s}} dv$.

1.1. Properties

(Whittaker and Watson, 1990) Here we recall some properties of β (Beta) function as:

$$\begin{aligned}
\beta(p,q) &= \beta(q,p),\\
\beta(p,q) &= \frac{(p-1)!(q-1)!}{(p+q-1)!},\\
\beta(p,q) &= \beta(p,q+1) + \beta(p+1,q),\\
\beta(p+1,q) &= \frac{p}{p+q}\beta(p,q),\\
\beta(p,q+1) &= \frac{q}{p+q}\beta(p,q),\\
\beta(p+1,q) &= \frac{p}{q}\beta(p,q+1).
\end{aligned}$$

Lemma 1.1 (*Grazyna, 2016*). *Let $G_n^\alpha(.;.)$ be the sequence of operators presented by (1.3). Then, we have*

$$\begin{aligned}
G_n^\alpha(1;y) &= 1,\\
G_n^\alpha(e_1;y) &= y + \frac{2\alpha y^2}{n},\\
G_n^\alpha(e_2;y) &= y^2 + \frac{4\alpha y^3 + y}{n} + \frac{4\alpha^2 y^4 + 4\alpha y^2}{n^2},\\
G_n^\alpha(e_p;y) &= y^p + O(n^{-1}), \quad where \quad n > 2.
\end{aligned}$$

Lemma 1.2. *Let $H_s^\alpha(.;.)$ be the sequence of operators given by (1.5) and $e_i(t) = t^i$, $i \in \{0,1,\dots,4\}$. Then, one get*

$$\begin{aligned}
H_s^\alpha(1;y) &= 1,\\
H_s^\alpha(e_1;y) &= \frac{1}{s-1}(sy + 2\alpha y^2 + 1); \quad s > 1,\\
H_s^\alpha(e_2;y) &= \frac{1}{(s-2)(s-1)}\left[s^2y^2 + 4s(\alpha y^3 + y) + 4\alpha^2 y^4 + 10\alpha y^2 + 2\right];\\
&\quad s > 2,\\
H_s^\alpha(e_3;y) &= \frac{1}{(s-3)(s-2)(s-1)}\Big[s^3y^3 + s^2(6\alpha y^4 + 9y^2) + s\{14\alpha^2 y^5\\
&\quad + 36\alpha y^3 + 18y\} + 8\alpha^2 y^6 + 24\alpha^2 y^4 + 48\alpha y^2 + 6\Big]; \quad s > 3,\\
H_s^\alpha(e_4;y) &= \frac{1}{(s-4)(s-3)(s-2)(s-1)}\Big[s^4y^4 + 8s^3y(\alpha + 2y^2) + s^2\\
&\quad (24\alpha^2 y2 + 96\alpha y^4 + 12\alpha + 72y^2) + s\{32\alpha^3 y^3 + 4\alpha^2 y(53y^4 + 4)\\
&\quad + 324\alpha y^3 + 95y\} + 16\alpha^4 y^4 + 48\alpha^3 y^2(y^4 + 1) + 4\alpha^2(20y^6 + 25y^4\\
&\quad + 3) + 136\alpha y^2 + 24\Big]; \quad s > 4,
\end{aligned}$$

for each $y \in \mathbb{R}_0^+$.

Proof. From the Eq. (1.5), we have

$$H_s^\alpha(f;y) = \sum_{l=0}^{\infty} P_{s,l}^\alpha(y) \int_0^\infty Q_s(v) f(v) dv.$$

Now, for $i = 0$,

$$\begin{aligned}
H_s^\alpha(e_0;y) = &= \sum_{l=0}^{\infty} P_{s,l}^\alpha(y) \frac{1}{\beta(l+1,s)} \int_0^\infty \frac{v^l}{(1+v)^{l+1+s}} dv\\
&= \sum_{l=0}^{\infty} P_{s,l}^\alpha(y) \frac{1}{\beta(l+1,s)} \beta(l+1,s)\\
&= 1.
\end{aligned}$$

For $i = 1$,

$$\begin{aligned}
H_s^\alpha(e_1;y) &= \sum_{l=0}^{\infty} P_{s,l}^\alpha(y) \frac{1}{\beta(l+1,s)} \int_0^\infty \frac{v^{l+1}}{(1+v)^{l+1+s}} dv\\
&= \sum_{l=0}^{\infty} P_{s,l}^\alpha(y) \frac{1}{\beta(l+1,s)} \beta(l+2,s-1)\\
&= \sum_{l=0}^{\infty} P_{s,l}^\alpha(y) \frac{(l+1)}{(s-1)}\\
&= \frac{s}{s-1}\left[G_s^\alpha(e_1;y) + \frac{1}{s}\right]\\
&= \frac{1}{s-1}(sy + 2\alpha y^2 + 1).
\end{aligned}$$

For $i = 2$,

$$\begin{aligned}
H_s^\alpha(e_2;y) &= \sum_{l=0}^{\infty} P_{s,l}^\alpha(y) \frac{1}{\beta(l+1,s)} \int_0^\infty \frac{v^{l+2}}{(1+v)^{l+1+s}} dv\\
&= \sum_{l=0}^{\infty} P_{s,l}^\alpha(y) \frac{1}{\beta(l+1,s)} \beta(l+3,s-2)\\
&= \sum_{l=0}^{\infty} P_{s,l}^\alpha(y) \frac{(l+2)}{(s-2)} \frac{(l+1)}{(s-1)}\\
&= \frac{1}{(s-2)(s-1)}\left[s^2 G_s^\alpha(e_2;y) + 3sG_s^\alpha(e_1;y) + 2G_s^\alpha(e_0;y)\right]\\
&= \frac{1}{(s-2)(s-1)}\left(s^2y^2 + 4s(\alpha y^3 + y) + 4\alpha^2 y^4 + 10\alpha y^2 + 2\right).
\end{aligned}$$

Similarly, we may demonstrate the last portion of the Lemma.

Lemma 1.3. *Let $H_s^\alpha(.;.)$ be the operators given by (1.5) and central moments $\eta_i(t;y) = (t-y)^i$, $i \in \{0,1,2,4\}$. Then, one get*

$$\begin{aligned}
H_s^\alpha(\eta_0;y) &= 1,\\
H_s^\alpha(\eta_1;y) &= -\frac{1}{s-1}(2\alpha y^2 + y + 1), \quad s > 1,\\
H_s^\alpha(\eta_2;y) &= \frac{1}{(s-2)(s-1)}\left[s(y^2 + 2y) + 4\alpha^2 y^4 + 2\alpha y^2(4y + 5) + 2(y+1)^2\right],
\end{aligned}$$

$$s > 2,$$

$$H_s^\alpha(\eta_4;y) = o\left(\frac{1}{s^2}\right), \quad s > 4,$$

for each $y \in \mathbb{R}_0^+$.

Proof. Using the definition of $H_s^\alpha(.;.)$, we get for $i = 0$, it is obvious that

$$H_s^\alpha(\eta_0;y) = 1.$$

Now, we consider for $i = 1$, that is $H_s^\alpha(\eta_1;y)$ as follows:

$$\begin{aligned}
H_s^\alpha(\eta_1;y) &= \sum_{l=0}^{\infty} P_{s,l}^\alpha(y)\frac{1}{\beta(l+1,s)}\int_0^\infty \frac{v^l}{(1+v)^{l+1+s}}(y-v)dv \\
&= y\sum_{l=0}^{\infty} P_{s,l}^\alpha(y)\frac{1}{\beta(l+1,s)}\int_0^\infty \frac{v^l}{(1+v)^{l+1+s}}dv \\
&\quad - \sum_{l=0}^{\infty} P_{s,l}^\alpha(y)\frac{1}{\beta(l+1,s)}\int_0^\infty \frac{v^{l+1}}{(1+v)^{l+1+s}}dv \\
&= yH_s^\alpha(e_0;y) - H_s^\alpha(e_1;y) \\
&= -\frac{1}{s-1}(2\alpha y^2 + y + 1).
\end{aligned}$$

Further, for $i = 2$, that is $H_s^\alpha(\eta_2;y)$ as follows:

$$\begin{aligned}
H_s^\alpha(\eta_2;y) &= \sum_{l=0}^{\infty} P_{s,l}^\alpha(y)\frac{1}{\beta(l+1,s)}\int_0^\infty \frac{v^l}{(1+v)^{l+1+s}}(y-v)^2dv \\
&= y^2H_s^\alpha(e_0;y) - 2yH_s^\alpha(e_1;y) + H_s^\alpha(e_2;y) \\
&= y^2(1) - 2y\frac{1}{s-1}(sy + 2\alpha y^2 + 1) + \frac{1}{(s-2)(s-1)} \\
&\quad \left[s^2y^2 + 4s(\alpha y^2 + y) + 4\alpha^2y^4 + 10\alpha y^2 + 2\right] \\
&= \frac{1}{(s-2)(s-1)}\Big[(s-2)(s-1)y^2 - 2y(s-2)(sy + 2\alpha y^2 + 1) \\
&\quad + s^2y^2 + 4s(\alpha y^2 + y) + 4\alpha^2y^4 + 10\alpha y^2 + 2\Big] \\
&= \frac{1}{(s-2)(s-1)}\Big[s\{-4\alpha y^3 + y^2(4\alpha+1) + 2y\} + 4\alpha^2y^4 + 8\alpha y^3 \\
&\quad + y^2(10\alpha+2) + 4y + 2\Big].
\end{aligned}$$

Similarly, we can prove the rest part of this Lemma. ☐

In subsequent sections, we deal with convergence rate of operators and order of approximation. Fuhrer, direct results are discussed as locally and globally in different spaces. In the last section, A-Statistical approximation results are investigated in several functional spaces.

2. Convergence rate and approximation order

Definition 2.1. Let g be a continuous function defined on non-negative axes. Then the modulus of smoothness is given by

$$\omega(g;\delta) = \sup_{|y_1-y_2|\le\delta} |g(y_1) - g(y_2)|, \qquad y_1, y_2 \in [0,\infty).$$

Theorem 2.1. *Let* $H_s^\alpha(.;.)$ *be a sequence of operator introduced in Eq.* (1.5)*. Then, for all* $g \in C_B[0,\infty)$, $H_s^\alpha(g;y) \rightrightarrows g$ *on each closed and bounded subset of* $[0,\infty)$ *where* $\rightrightarrows$ *represents uniform convergent.*

Proof. In view of Korovkin type theorem which regard the uniform convergence of the sequence of linear positive operators, it is enough to see that

$$\lim_{s\to\infty} H_s^\alpha(t^i;y) = y^i, \ i = 0,1,2,$$

uniformly on every closed and bounded subset of $[0,\infty)$. In the light of Lemma 1.2, this result can easily be proved. ☐

In view of result given by Shisha and Mond (1968), we can prove the order of convergence in terms of Ditzian–Totik the modulus of continuity.

Theorem 2.2. *For* $g \in C_B[0,\infty)$ *and the operators* $H_s^\alpha(.;.)$ *introduced in Eq.* (1.5)*, we have*

$$|H_s^\alpha(g;y) - g(y)| \le 2\omega(g;\delta),$$

where $\delta = \sqrt{H_s^\alpha((t-y)^2;y)}$.

3. Locally approximation results

In this section, we think back to some functional spaces and functional relation as: $C_B[0,\infty)$: Represent a space of bounded and continuous real valued functions. Now, Peetre's K-functional is given by

$$K_2(g,\delta) = \inf_{h\in C_B^2[0,\infty)}\left\{\|g-h\|_{C_B[0,\infty)} + \delta\|h''\|_{C_B^2[0,\infty)}\right\},$$

where $C_B^2[0,\infty) = \{h \in C_B[0,\infty) : h', h'' \in C_B[0,\infty)\}$ provided with the norm $\|g\| = \sup_{0\le y<\infty}|g(y)|$ and Ditzian–Totik modulus of smoothness of second order is given by

$$\omega_2(g;\sqrt{\delta}) = \sup_{0<k\le\sqrt{\delta}}\ \sup_{y\in[0,\infty)} |f(y+2k) - 2f(y+k) + f(y)|.$$

We recall a relation from DeVore and Lorentz (1993) page no. 177, Theorem 2.4 as:

$$K_2(g;\delta) \le C\omega_2(g;\sqrt{\delta}), \tag{3.1}$$

where C is a constant absolute. Now in view to prove the further result, we take the auxiliary operator as:

$$\hat{H}_s^\alpha(g;y) = H_s^\alpha(g;y) + g(y) - g\left(\frac{sy + 2\alpha y^2 + 1}{s-1}\right) \tag{3.2}$$

where $g \in C_B[0,\infty)$, $y \ge 0$, $s > 1$ and $n > 2$. From Eq. (3.2), one can yield

$$\hat{H}_s^\alpha(1;y) = 1, \ \hat{H}_s^\alpha(\eta_1;x) = 0 \text{ and } |\hat{H}_s^\alpha(g;y)| \le 3\|g\|. \tag{3.3}$$

Lemma 3.1. *For* $n > 2$ *and* $y \ge 0$*, one yield*

$$|\hat{H}_s^\alpha(h;y) - h(y)| \le \theta(y)\|h''\|,$$

where $h \in C_B^2[0,\infty)$ *and* $\theta(y) = \hat{H}_s^\alpha(\eta_2;y) + (\hat{H}_s^\alpha(\eta_1;y))^2$.

Proof. For $h \in C_B^2[0,\infty)$ and in view of relation Taylor expansion, we get

$$h(t) = h(y) + (t-y)h'(y) + \int_y^t (t-v)h''(v)dv. \tag{3.4}$$

Now, applying the auxiliary operators $\hat{H}_s^\alpha(.;.)$ given in Eq. (3.2) on both the sides in above Eq. (3.4), we get

$$\hat{H}_s^\alpha(h;y) - h(y) = h'(y)\hat{H}_s^\alpha(\eta_1;y) + \hat{H}_s^\alpha\left(\int_y^t (t-v)h''(v)dv;y\right).$$

Using the Eqs. (3.3) and (3.4), we get

$$\begin{aligned}
\hat{H}_s^\alpha(h;y) - h(y) &= \hat{H}_s^\alpha\left(\int_y^t (t-v)h''(v)dv;y\right) \\
&= H_s^\alpha\left(\int_y^t (t-v)h''(v)dv;y\right) \\
&\quad - \int_y^{\frac{sy+2\alpha y^2+1}{s-1}} \left(\frac{sy+2\alpha y^2+1}{s-1} - v\right)h''(v)dv,
\end{aligned}$$

$$|\hat{H}_s^\alpha(h;y) - h(y)| \le \left|H_s^\alpha\left(\int_y^t (t-v)h''(v)dv;y\right)\right|$$

$$+\left|\int_y^{\frac{sy+2\alpha y^2+1}{s-1}}\left(\frac{sy+2\alpha y^2+1}{s-1}-v\right)h''(v)dv\right|. \tag{3.5}$$

Since,

$$\left|\int_y^t (t-v)h''(v)dv\right| \le (t-y)^2\|h''\|, \tag{3.6}$$

then

$$\left|\int_y^{\frac{sy+2\alpha y^2+1}{s-1}}\left(\frac{sy+2\alpha y^2+1}{s-1}-v\right)h''(v)dv\right| \le \left(\frac{sy+2\alpha y^2+1}{s-1}-y\right)^2\|h''\|. \tag{3.7}$$

In view of (3.5), (3.6) and (3.7), we find

$$\begin{aligned}|\widehat{H}_s^\alpha(h;y)-h(y)| &\le \left\{\widehat{H}_s^\alpha(\eta_2;y)+\left(\frac{sy+2\alpha y^2+1}{s-1}-y\right)^2\right\}\|h''\|\\ &= \theta(y)\|h''\|.\end{aligned}$$

Which proves the required result. ☐

Theorem 3.2. *Let $g \in C_B^2[0,\infty)$. Then, there corresponds a non-negative constant $\tilde{C} > 0$ such that*

$$|H_s^\alpha(g;y)-g(y)| \le \tilde{C}\omega_2\big(g;\sqrt{\theta(y)}\big)+\omega(g;H_s^\alpha(\eta_1;y)),$$

where $\theta(y)$ is given by in Lemma 3.1.

Proof. For $h \in C_B^2[0,\infty)$ and $g \in C_B[0,\infty)$ and with the definition of $\widehat{H}_s^\alpha(.;.)$, we get

$$\begin{aligned}|H_s^\alpha(g;y)-g(y)| &\le |\widehat{H}_s^\alpha(g-h;y)|+|(g-h)(y)|+|\widehat{H}_s^\alpha(h;y)-h(y)|\\ &+\left|g\left(\frac{sy+2\alpha y^2+1}{(s-1)}\right)-g(y)\right|.\end{aligned}$$

In the light of Lemma 3.1 and inequalities in Eq. (3.3), one get

$$\begin{aligned}|H_s^\alpha(g;y)-g(y)| &\le 4\|g-h\|+|\widehat{H}_s^\alpha(h;y)-h(y)|+\left|g\left(\frac{sy+2\alpha y^2+1}{s-1}\right)-g(y)\right|\\ &\le 4\|g-h\|+\theta(y)\|h''\|+\omega\Big(g;H_s^\alpha((t-y);y)\Big).\end{aligned}$$

Using Eq. (3.1), we yield the desired result. ☐

Now, we discuss the next result in Lipschitz type space (Özarslan and Aktuğlu, 2013), which is given as:

$$Lip_{\tilde{M}}^{\zeta_1,\zeta_2}(\gamma) := \left\{g \in C_B[0,\infty) : |g(t)-g(y)| \le \tilde{M}\frac{|t-y|^\gamma}{(t+\zeta_1 y+\zeta_2 y^2)^{\frac{\gamma}{2}}} : y,t \in (0,\infty)\right\},$$

where $\tilde{M} > 0$, $0 < \gamma \le 1$ and $\zeta_1, \zeta_2 > 0$.

Theorem 3.3. *For the sequence of positive linear operators (1.5) and $g \in Lip_M^{\zeta_1,\zeta_2}(\gamma)$, one has*

$$|H_s^\alpha(g;y)-g(y)| \le \tilde{M}\left(\frac{\lambda(y)}{\zeta_1 y+\zeta_2 y^2}\right)^{\frac{\gamma}{2}}, \tag{3.8}$$

where $0 < \gamma \le 1$, $\zeta_1, \zeta_2 \in (0,\infty)$ and $\lambda(y) = H_s^\alpha(\eta_2;y)$.

Proof. For $\gamma = 1$ and $y \ge 0$, we get

$$\begin{aligned}|H_s^\alpha(g;y)-g(y)| &\le H_s^\alpha(|g(t)-g(y)|;y)\\ &\le \tilde{M}H_s^\alpha\left(\frac{|t-y|}{(t+\zeta_1 y+\zeta_2 y^2)^{\frac{1}{2}}};y\right).\end{aligned}$$

Since $\frac{1}{t+\zeta_1 y+\zeta_2 y^2} < \frac{1}{\zeta_1 y+\zeta_2 y^2}$, for all $y \in (0,\infty)$, we yield

$$\begin{aligned}|H_s^\alpha(g;y)-g(y)| &\le \frac{\tilde{M}}{(\zeta_1 y+\zeta_2 y^2)^{\frac{1}{2}}}(H_s^\alpha(\eta_2;y))^{\frac{1}{2}}\\ &\le \tilde{M}\left(\frac{\lambda(y)}{\zeta_1 y+\zeta_2 y^2}\right)^{\frac{1}{2}},\end{aligned}$$

which implies that Theorem 3.3 works for $\gamma = 1$. Next, we consider for $\gamma \in (0,1)$ and in view of Hölder's inequality using $p = \frac{2}{\gamma}$ and $q = \frac{2}{2-\gamma}$, we have

$$\begin{aligned}|H_s^\alpha(g;y)-g(y)| &\le \big(H_s^\alpha(|g(t)-g(y)|^{\frac{2}{\gamma}};y)\big)^{\frac{\gamma}{2}}\\ &\le \tilde{M}\left(H_s^\alpha\left(\frac{|t-y|^2}{(t+\zeta_1 y+\zeta_2 y^2)};y\right)\right)^{\frac{\gamma}{2}}.\end{aligned}$$

Since $\frac{1}{t+\zeta_1 y+\zeta_2 y^2} < \frac{1}{\zeta_1 y+\zeta_2 y^2}$, for all $y \in (0,\infty)$, one get

$$|H_s^\alpha(g;y)-g(y)| \le \tilde{M}\left(\frac{H_s^\alpha(|t-y|^2;y)}{\zeta_1 y+\zeta_2 y^2}\right)^{\frac{\gamma}{2}} \le \tilde{M}\left(\frac{\lambda(y)}{\zeta_1 y+\zeta_2 y^2}\right)^{\frac{\gamma}{2}}.$$

Hence, we yield the required result. ☐

Next, we deal the approximation locally in view of rth order modulus of smoothness then, Lipschitz-type maximal function which is introduced by Lenze (1988) as:

$$\tilde{\omega}_r(g;y) = \sup_{t\ne y, t\in(0,\infty)}\frac{|g(t)-g(y)|}{|t-y|^r}, \quad y \in [0,\infty) \text{ and } r \in (0,1]. \tag{3.9}$$

Theorem 3.4. *Let $g \in C_B[0,\infty)$ and $r \in (0,1]$. Then, for all $y \in [0,\infty)$, we have*

$$|H_s^\alpha(g;y)-g(y)| \le \tilde{\omega}_r(g;y)\Big(\lambda(y)\Big)^{\frac{r}{2}}.$$

Proof. It is noted that

$$|H_s^\alpha(g;y)-g(y)| \le H_s^\alpha(|g(t)-g(y)|;y).$$

Using Eq. (3.9), one get

$$|H_s^\alpha(g;y)-g(y)| \le \tilde{\omega}_s(g;y)H_s^\alpha(|t-y|^r;y).$$

then by using Hölder's inequality using $p = \frac{2}{r}$ and $q = \frac{2}{2-r}$, we have

$$|H_s^\alpha(g;y)-g(y)| \le \tilde{\omega}_r(g;y)\big(H_s^\alpha(|t-y|^2;y)\big)^{\frac{r}{2}}.$$

Hence, we completes the proof. ☐

4. Approximation properties globally

Suppose that $v(y) = 1+y^4, 0 \le y < \infty$ be the weight function. Then, $B_v[0,\infty) = g(y) : |g(y)| \le \tilde{M}_g(1+y^4)$, here $\tilde{M}_g$ is a constant based on g and $C_v[0,\infty)$ denotes space of continuous function in $B_v[0,\infty)$ equipped with the norm $\|g(y)\|_v = \sup_{y\in[0,\infty)}\frac{|g(y)|}{v(y)}$ and $C_v^{\tilde{k}}[0,\infty) = \{g \in C_v[0,\infty) : \lim_{y\to\infty}\frac{g(y)}{v(y)} = \tilde{k}$, where $\tilde{k}$ is a constant depending on $g\}$. Ditzian–Totik modulus of continuity for the function g defined on the closed interval $[0,b]$ with $b > 0$ is defined by

$$\omega_b(g,\delta) = \sup_{|t-y|\le\delta}\ \sup_{y,t\in[0,b]}|g(t)-g(y)|. \tag{4.1}$$

One can easily note that for any $g \in C_v[0,\infty)$, the modulus of smoothness given by in the Eq. (4.1) approaches to zero.

Theorem 4.1. *Let $g \in C_v[0,\infty)$ and modulus of smoothness $\omega_{b+1}(g;\delta)$ given on $[0,b+1] \subset [0,\infty)$. Then, for any $y \in [0,b]$, we get*

$$\|H_s^\alpha(.;.)-g\|_{C[0,b]} \le 4\tilde{M}_g(1+b^2)\delta_s(b)+2\omega_{b+1}(g;\sqrt{\delta_s(b)}),$$

where $\delta_s(b) = \max_{y\in[0,b]}H_s^\alpha(\eta_2;y)$.

Proof. For any $y \in [0,b]$ and $t \in [0,\infty)$, we get

$$|g(t)-g(y)| \le 4\tilde{M}_g(1+b^2)(t-y)^2+\left(1+\frac{|t-y|}{\delta}\right)\omega_{b+1}(g;\delta).$$

Using operator $H_s^\alpha(.;.)$ on both sides, we have

$$|H_s^\alpha(g;y)-g(y)| \le 4\tilde{M}_g(1+b^2)H_s^\alpha(\eta_2;y) + \left(1+\frac{H_s^\alpha(|t-y|;y)}{\delta}\right)\omega_{b+1}(g;\delta).$$

Next, in view of Lemma 1.3 and $y \in [0,b]$, we get

$$|H_s^\alpha(.;.)-g| \le 4\tilde{M}_g(1+b^2)\delta_s(b) + \left(1+\frac{\sqrt{\delta_s(b)}}{\delta}\right)\omega_{b+1}(g;\delta).$$

Taking $\delta = \delta_s(b)$, we can easily prove the required result. □

Remark 4.2. In this paper, we use the test function, which is given by $e_i(t) = t^i$, $i = 0, 1, 2$.

Theorem 4.3 (Gadjiev, 1974, Gadjiev, 1976). *Suppose that the sequence of positive linear operators $(L_s)_{s\ge 1}$ acting from $C_\nu[0,\infty)$ to $B_\nu[0,\infty)$ satisfies the conditions*

$$\lim_{s\to\infty} \|L_s(e_i;.)-e_i\|_\nu = 0, \quad where\ i = 0,1,2,$$

then, for $g \in C_\nu^{\tilde{k}}[0,\infty)$, we have

$$\lim_{s\to\infty} \|L_s g - g\|_\nu = 0.$$

Theorem 4.4. *Let $g \in C_\nu^{\tilde{k}}[0,\infty)$. Then, we have*

$$\lim_{s\to\infty} \|H_s^\alpha(g;y) - g\|_\nu = 0.$$

Proof. In order to prove Theorem 4.4, it is sufficient to check that

$$\lim_{s\to\infty} \|H_s^\alpha(e_i;.)-e_i\|_\nu = 0, for\ i = 0,1,2.$$

In the light of Lemma 1.2, it is obvious $\|H_s^\alpha(e_0;.)-1\|_\nu = 0$, here $s \to \infty$ and restrict with $s > 2$, also

$$\|H_s^\alpha(e_1;.)-e_1\|_{\nu(y)} = \sup_{y\in[0,\infty)} \frac{1}{\nu(y)}\left|\frac{sy+2\alpha y^2+1}{s-1} - y\right| = \frac{1}{s-1}\sup_{y\in[0,\infty)}\frac{2\alpha y2}{1+y^4} + \frac{1}{s-1}\sup_{y\in[0,\infty)}\frac{2}{1+y^4}.$$

For a large value of s, we get $\|H_s^\alpha(e_1;.)-e_1\|_\nu \to 0$. Also,

$$\begin{aligned}\|H_s^\alpha(e_2;.)-e_2\|_\nu \le{}& \left(\frac{4\alpha^2}{(s-2)(s-1)}\right)\sup_{y\in[0,\infty)}\frac{y^4}{1+y^4} \\ &+ \left(\frac{4s\alpha}{(s-2)(s-1)}\right)\sup_{y\in[0,\infty)}\frac{y^3}{1+y^4} \\ &+ \left(\frac{10\alpha+3s-2}{(s-2)(s-1)}\right)\sup_{y\in[0,\infty)}\frac{y^2}{1+y^4} \\ &+ \left(\frac{4sy}{(s-2)(s-1)}\right)\sup_{y\in[0,\infty)}\frac{y}{1+y^4} \\ &+ \left(\frac{2}{(s-2)(s-1)}\right)\sup_{y\in[0,\infty)}\frac{1}{1+y^4}.\end{aligned}$$

Which implies that $\|H_s^\alpha(e_2;.)-e_2\|_\nu \to 0$ as $s \to \infty$. Hence, we completes the proof of Theorem 4.4 □

5. A-statistical approximation

In this section, we recall some notations from Gadjiev and Orhan (2007). Suppose that $A = (a_{s\mu})$ $(s, \mu \in \mathbb{N})$ represents a non-negative infinite summability matrix. Then, a sequence $y := (y_\mu)$ is called to be A-statistically convergent to L, that is $st_A - \lim y = L$, if for every $\epsilon > 0$

$$\lim_s \sum_{\mu:|y_k-L|\ge\epsilon} a_{s\mu} = 0.$$

Let $q = (q_s)$ be a sequence with following assertions holds

$$st_A - \lim_s q_s = 1 \text{ and } st_A - \lim_s q_s^s = a,\ 0 \le a < 1. \tag{5.1}$$

Theorem 5.1. *Consider $A = (a_{s\mu})$ be a non-negative regular suitability matrix and the sequence $q = (q_s)$ with condition (5.1) with $q_s \in (0,1)$, $s \in \mathbb{N}$. Then, for each $g \in C_\nu^0[0,\infty)$, $st_A - \lim_s \|H_s^\alpha(g;y)-g\|_\nu = 0$.*

Proof. By using Lemma 1.2, we have

$$st_A - \lim_s \|H_s^\alpha(e_0;y)-e_0\|_\nu = 0$$

and

$$\|H_s^\alpha(e_1;y)-y\|_\nu = \sup_{y\in[0,\infty)}\frac{1}{1+y^4}\left|\frac{sy+2\alpha y^2+1}{s-1} - y\right| = \frac{1}{1+y^4}\sup_{y\in[0,\infty)}\frac{2\alpha y2}{s-1} + \frac{1}{1+y^4}\sup_{y\in[0,\infty)}\frac{2}{s-1}.$$

Now

$$\tilde{I}_1 := \left\{s : \|H_s^\alpha(e_1;y)-y\| \ge \epsilon\right\},$$

$$\tilde{I}_2 := \left\{s : \frac{2\alpha}{s-1} \ge \frac{\epsilon}{2}\right\},$$

$$\tilde{I}_3 := \left\{s : \frac{2}{s-1} \ge \frac{\epsilon}{2}\right\}.$$

Which implies that $\tilde{I}_1 \subseteq \tilde{I}_2 \cup \tilde{I}_3$, this shows that $\sum_{\mu\in\tilde{I}_1} a_{s\mu} \le \sum_{\mu\in\tilde{I}_2} a_{s\mu} + \sum_{\mu\in\tilde{I}_3} a_{s\mu}$. Therefore, we get

$$st_A - \lim_s \|H_s^\alpha(e_1;y)-y\|_\nu = 0. \tag{5.2}$$

Now by using Lemma 1.2, we have

$$\|H_s^\alpha(e_2;y)-y^2\|_{1+y^4} \le \sup_{y\in[0,\infty)}\frac{1}{\nu(y)}\left|\frac{1}{(s-2)(s-1)}\left\{s^2y^2+4s(\alpha y^3+y) + 4\alpha^2y^4+10\alpha y^2+2\right\} - y^2\right|.$$

For a given $\varepsilon > 0$, we have the following sets

$$\tilde{G}_1 := \left\{s : \left\|H_s^\alpha(e_2;y)-y^2\right\|_\nu \ge \epsilon\right\}$$

$$\tilde{G}_2 := \left\{s : \frac{4\alpha^2}{(s-2)(s-1)} \ge \frac{\epsilon}{5}\right\}$$

$$\tilde{G}_3 := \left\{s : \frac{4s\alpha}{(s-2)(s-1)} \ge \frac{\epsilon}{5}\right\}$$

$$\tilde{G}_4 := \left\{s : \frac{10\alpha+3s-2}{(s-2)(s-1)} \ge \frac{\epsilon}{5}\right\}$$

$$\tilde{G}_5 := \left\{s : \frac{4s}{(s-2)(s-1)} \ge \frac{\epsilon}{5}\right\}$$

$$\tilde{G}_6 := \left\{s : \frac{2}{(s-2)(s-1)} \ge \frac{\epsilon}{5}\right\}.$$

One can note that $\tilde{G}_1 \subseteq \tilde{G}_2 \bigcup \tilde{G}_3 \bigcup \tilde{G}_4 \bigcup \tilde{G}_5 \bigcup \tilde{G}_6$. Thus, we have

$$\sum_{\mu\in\tilde{G}_1} a_{m\mu} \le \sum_{\mu\in\tilde{G}_2} a_{m\mu} + \sum_{\mu\in\tilde{G}_3} a_{s\mu} + \sum_{\mu\in\tilde{G}_4} a_{s\mu} + \sum_{\mu\in\tilde{G}_5} a_{s\mu} + \sum_{\mu\in\tilde{G}_6} a_{s\mu}.$$

As $s \to \infty$, we have

$$st_A - \lim_n \|H_s^\alpha(e_2;.)-e_2\|_\nu = 0. \tag{5.3}$$

Hence, we completes the proof of Theorem 5.1. □

Now, we discuss the rate of A-Statistical approximation convergence in view of the Peetre's K-functional for operators $H_s^\alpha(.;.)$.

Theorem 5.2. *Let $g \in C_B^2[0,\infty)$. Then,*

$$st_A - \lim_s \|H_s^\alpha(g;\cdot) - f\|_{C_B[0,\infty)} = 0.$$

Proof. In view of Taylor's result, we get

$$g(t)=g(y)+g'(y)(t-y)+\frac{1}{2}g''(\eta)(t-y)^2,$$

where $t \le \eta \le y$. Operating $H_s^\alpha(.;.)$, both the sides in the above equation, we get

$$H_s^\alpha(g;y)-g(y)=g'(y)H_s^\alpha(\eta_1;y)+\frac{1}{2}g''(\eta)H_s^\alpha(\eta_2;y),$$

which yields that

$$\begin{aligned}\|H_s^\alpha(g;\cdot)-g\|_{C_B[0,\infty)} &\le \|g'\|_{C_B[0,\infty)}\|H_s^\alpha(e_1-,.)\|_{C_B[0,\infty)}\\ &\quad+\|g''\|_{C_B[0,\infty)}\|H_s^\alpha(e_1-,.)^2\|_{C_B[0,\infty)}\\ &=\tilde{W}_1+\tilde{W}_2, \quad say. \end{aligned} \tag{5.4}$$

From the Eqs. (5.2) and (5.3), one has

$$\lim_s \sum_{\mu\in\mathbb{N}:\tilde{W}_1\ge\frac{\epsilon}{2}} a_{s\mu}=0,$$

$$\lim_s \sum_{\mu\in\mathbb{N}:\tilde{W}_2\ge\frac{\epsilon}{2}} a_{s\mu}=0.$$

From Eq. (5.4), we have

$$\lim_s \sum_{\mu\in\mathbb{N}:\|H_s^\alpha(g;\cdot)-g\|_{C_B[0,\infty)}\ge\epsilon} a_{s\mu} \le \lim_s \sum_{\mu\in\mathbb{N}:\tilde{W}_1\ge\frac{\epsilon}{2}} a_{s\mu}+\lim_s \sum_{\mu\in N:\tilde{W}_2\ge\frac{\epsilon}{2}} a_{s\mu}.$$

Thus $st_A-\lim_s \|H_s^\alpha(g;\cdot)-g\|_{C_B[0,\infty)} \to 0.$ as $s\to\infty$. Hence, we arrive the proof. □

Theorem 5.3. *Let* $g\in C_B^2[0,\infty)$. *Then,*

$$\|H_s^\alpha(g;\cdot)-g\|_{C_B[0,\infty)} \le M\omega_2(g;\sqrt{\delta}),$$

where $\delta=\|H_s^\alpha(e_1-\cdot;\cdot)\|_{C_B[0,\infty)}+\|H_s^\alpha((e_1-\cdot)^2;\cdot)\|_{C_B[0,\infty)}$, *and* $\|g\|_{C_B^2[0,\infty)}=\|g\|_{C_B[0,\infty)}+\|g'\|_{C_B[0,\infty)}+\|g''\|_{C_B[0,\infty)}$.

Proof. Let $h\in C_B^2[0,\infty)$. Using Eq. (5.4), one get

$$\begin{aligned}\|H_s^\alpha(h)-h\|_{C_B[0,\infty)} &\le \|h'\|_{C_B[0,\infty)}\|H_s^\alpha((e_1-\cdot);\cdot)\|_{C_B[0,\infty)}\\ &\quad+\frac{1}{2}\|h''\|_{C_B[0,\infty)}H_s^\alpha((e_1-\cdot)^2;\cdot)\|_{C_B[0,\infty)}\\ &\le \delta\|h\|_{C_B^2[0,\infty)}. \end{aligned} \tag{5.5}$$

For every $g\in C_B[0,\infty)$ and $h\in C_B^2$, from Eq. (5.5), we obtain

$$\begin{aligned}\|H_s^\alpha(g;\cdot)-g\|_{C_B[0,\infty)} &\le \|H_s^\alpha(g;\cdot)-H_s^\alpha(h;\cdot)\|_{C_B[0,\infty)}\\ &\quad+\|H_s^\alpha(h;\cdot)-h\|_{C_B[0,\infty)}+\|h-g\|_{C_B[0,\infty)}\\ &\le 2\|h-g\|_{C_B[0,\infty)}+\|H_s^\alpha(h;\cdot)-h\|_{C_B[0,\infty)}\\ &\le 2\|h-g\|_{C_B[0,\infty)}+\delta\|h\|_{C_B^2}. \end{aligned}$$

In view of Peetre's K-functional, one get

$$\|H_s^\alpha(g;\cdot)-g\|_{C_B[0,\infty)} \le 2K_2(g;\delta)$$

and

$$\|H_s^\alpha(g;\cdot)-g\|_{C_B[0,\infty)} \le \tilde{M}\left\{\omega_2(g;\sqrt{\delta})+\min(1,\delta)\|g\|_{C_B[0,\infty)}\right\}.$$

Using Eq. (5.3), we obtain that

$$st_A-\lim_s \delta=0, \text{ thus } st_A-\lim_s \omega(g;\sqrt{\delta})=0,$$

which completes the proof of required result. □

6. Conclusion

In this paper, we introduce a sequence of linear positive operators In this paper, we introduce a sequence of linear positive operators in integral form via Hermite Polynomial to approximate the functions which belongs to Lebesgue measurable space named as Szász-Beta type operators defined by (1.5). Further, we calculate the some estimates which are used to prove convergence rate and approximation order. Moreover, the various approximation results, e.g., locally and globally approximation results and A-Statistical approximation are investigated using these sequences of operators to achieve better approximations in several functional spaces.

Declaration of competing interest

None.

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