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ORIGINAL ARTICLE

Coupling of homotopy perturbation and modified Lindstedt–Poincaré methods for traveling wave solutions of the nonlinear Klein–Gordon equation

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Abstract oupled method of He's homotopy perturbation method and modified Lindstedt–Poincaré method is applied to the search for traveling wave solutions of a variety of Klein–Gordon equations. The results obtained provide confirmation for the validity of the coupled method.

1. Introduction

Periodic wave-trains are found in many physical systems and, they are fundamental solutions related to the elementary solutions in the form of sinusoidal wave-trains in linear theory. In nonlinear theory, the solutions are no longer sinusoidal, but periodic solutions may still exist. For this class of periodic solutions, the main effect of non-linearity becomes visible on amplitude dependence in the dispersion relation. This, off course, leads to a new qualitative behavior, not merely to the correction of the dispersion relation in linear case. In this article, we will consider after Whitham (1965) nonlinear, Klein–Gordon, equation governed by

$$u_{tt} - u_{xx} + V'(u) = 0, \tag{1}$$

where $V(u)$ is any nonlinear potential function which yields oscillatory solutions and $V'(u)$ is chosen as the derivative of a potential energy. Klein–Gordon equation is not only a useful model but also arises in variety of physical situations (Knobel, 2000; Shen, 1994). It is worth to note that the dispersion relation of a periodic wave-train in (1) is dependent on its amplitude. For this class of problems with waves of small amplitude, perturbation methods based on small amplitude expansions infer the existence of the periodic wave-trains (Nayfeh, 1973). However, the perturbation methods, in principle, works only for the nonlinear problems with small parameters. In addition, analytical solution obtained by the perturbation method has a little range of validity in most cases.

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Consequently, new approximate-analytical techniques are considered necessary. For example, the sine-Gordon and the double sine-Gordon equations, the sinh-Gordon equation and the double sinh-Gordon equations were investigated by using the standard tanh method (Malfliet, 1992, 1996a,b; Wazwaz, 2005). Sirendaoreji and Sun Jiong (2002) applied a direct method for solving Sinh-Gordon type equation. Fu et al. (2004) have obtained exact solutions for double and triple sinh-Gordon equations. Wazwaz (2005) obtained traveling wave solutions for combined and double combined sine-cosine-Gordon equations by the variable separated ODE method.

More recently, Lim et al. (2001), by coupling linearization of Klein–Gordon equation with the method of harmonic balance, established two general analytical approximate formulas for the dispersion relation which depends on the amplitude of the wave train. In this paper, we will be coupling He's homotopy perturbation method (He, 2000; Öziş and Yıldırım, 2006) and modified Lindstedt–Poincaré method (He, 2002a,b, 2006; Öziş and Yıldırım, 2007, 2006; Liu, 2005) to obtain the periodic wave-trains in (1) and will use three examples to illustrate the applicability and the effectiveness of the proposed method. For latest developments in this field, the reader is referred to see (He et al., 2006; Noor and Mohyud-Din, 2008; Mohyud-Din et al., 2009a,b,c,d, 2010 Mohyud-Din, 2009; Mohyud-Din and Noor, 2007, 2009; Abdou, 2010a,,, 2009 Abdou et al., in press; El-Wakil and Abdou, 2008, 2010; El-Wakil et al., in press; He, 1999) and the references therein.

2. Brief concept of coupling of homotopy perturbation and modified Lindstedt–Poincaré methods

Recognizing He's homotopy perturbation method; the homotopy with and imbedding parameter $p \in [0, 1]$ is constructed, and the imbedding parameter is considered as a "small parameter", so the method is called homotopy perturbation method and is proceed as the standard perturbation method but taking the full advantage of the traditional perturbation methods and the homotopy techniques. The main merit of the homotopy perturbation method is that the perturbation equation can be easily constructed (therefore is problem dependent) by homotopy in topology and the initial approximation can also be freely selected. On the other hand, in He's modified Lindstedt–Poincaré method the coefficient of the second term, i.e., u is also expanded into a series besides the assumed solution. For further reading, refer to the comprehensive book by He et al. (2006) and the references therein. To our view, if He's modified Lindstedt–Poincaré method is applicable to the perturbed equation constructed by He's homotopy perturbation method then the obtained solution would be exceedingly accurate and may possibly be applicable to wide range of physical systems. For better illustration of coupled method of He's homotopy perturbation method and modified Lindstedt–Poincaré method, and making the underlying idea clear, we demonstrate three examples. By doing so, we will try to validate the applicability, accuracy and effectiveness of the proposed method.

Example 3.1. Consider the Klein–Gordon equation governed by

$$u_{tt} - \alpha^2 u_{xx} + \gamma^2 u = \beta u^3 \tag{2}$$

We first connect the independent variables x and t (Malfliet, 1992, 1996a,b) into one wave variable by

$$\xi = x - ct. \tag{3}$$

Substituting the transformation (3) into Eq. (2) gives

$$u'' + \frac{\gamma^2}{(c^2 - \alpha^2)} u - \frac{\beta}{(c^2 - \alpha^2)} u^3 = 0, \quad (c^2 \neq \alpha^2) \tag{4}$$

where prime denotes differentiation with respect to ξ. Physical considerations show that Eq. (4) has a periodic solution. In order to look for the periodic solution, we construct the following homotopy:

$$u'' + \frac{\gamma^2}{(c^2 - \alpha^2)} u - \frac{\beta}{(c^2 - \alpha^2)} p u^3 = 0, \tag{5}$$

It is apparent that when $p = 0$, (5) reduces to linear equation and when $p = 1$, it becomes the original nonlinear one. Therefore, the embedding parameter p monotonically increases from zero to unit as the linear operator is continuously deformed to the nonlinear problem given. We assume that the periodic solution to Eq. (5) may be written as a power series in p:

$$u = u_0 + p u_1 + p^2 u_2 + \cdots \tag{6}$$

If higher order approximate solution is required, the He's modified Lindstedt–Poincaré method can be applied. Hence, we expand the coefficient of the linear term into a series of p:

$$\frac{\gamma^2}{(c^2 - \alpha^2)} = \omega^2 + p\omega_1 + p^2\omega_2 + \cdots \tag{7}$$

Substituting (6) and (7) into Eq. (5), and processing as the standard perturbation method, we have:

$$u_0'' + \omega^2 u_0 = 0, \tag{8}$$

$$u_1'' + \omega^2 u_1 + \omega_1 u_0 - \frac{\beta}{(c^2 - \alpha^2)} u_0^3 = 0 \tag{9}$$

The initial approximation u_0 can be freely chosen, hereby we set

$$u_0 = A \cos \omega\xi, \tag{10}$$

which satisfies the Eq. (8). The substitution of (10) in (9) yields

$$u_1'' + \omega^2 u_1 + \omega_1 A \cos \omega\xi - \frac{\beta}{(c^2 - \alpha^2)} A^3 \cos^3 \omega\xi = 0 \tag{11}$$

or

$$u_1'' + \omega^2 u_1 + A \cos \omega\xi \left(\omega_1 - \frac{3\beta A^2}{4(c^2 - \alpha^2)}\right) - \frac{\beta A^3}{4(c^2 - \alpha^2)} \times \cos 3\omega\xi = 0 \tag{12}$$

No secular term in u_1 requires that

$$\omega_1 = \frac{3\beta A^2}{4(c^2 - \alpha^2)}. \tag{13}$$

Eq. (12) becomes

$$u_1'' + \omega^2 u_1 - \frac{\beta A^3}{4(c^2 - \alpha^2)} \cos 3\omega\xi = 0 \tag{14}$$

We write down its special solution:

$$u_1 = -\frac{\beta A^3}{32\omega^2(c^2 - \alpha^2)} (\cos 3\omega\xi - \cos \omega\xi) \tag{15}$$

Setting $p = 1$, in (7) gives

$$\frac{\gamma^2}{(c^2-\alpha^2)} = \omega^2 + \frac{3\beta A^2}{4(c^2-\alpha^2)}. \tag{16}$$

Hence, the frequency is:

$$\omega = \sqrt{\frac{4\gamma^2 - 3\beta A^2}{4(c^2-\alpha^2)}}. \tag{17}$$

If the first-order approximate solution is adequate, setting $p = 1$, we have

$$u = u_0 + u_1$$
$$= A\cos\omega\xi - \frac{\beta A^3}{32\omega^2(c^2-\alpha^2)}(\cos 3\omega\xi - \cos\omega\xi) \tag{18}$$

which is the solution of Klein–Gordon equation in (2).

Example 3.2. As a second example, consider the sine-Gordon equation governed by

$$u_{tt} - ku_{xx} + 2\alpha\sin u = 0 \tag{19}$$

Using the transformation (3), Eq. (19) gives:

$$(c^2-k)u'' + 2\alpha\sin u = 0. \tag{20}$$

Approximating $\sin u \cong u - \frac{u^3}{6}$, Eq. (20) reduces to:

$$u'' + \frac{2\alpha}{(c^2-k)}\left(u - \frac{u^3}{6}\right) = 0, \quad (c^2 \neq k) \tag{21}$$

We, in this case, construct the homotopy in the form:

$$u'' + \frac{2\alpha}{(c^2-k)}u - p\frac{\alpha}{3(c^2-k)}u^3 = 0. \tag{22}$$

Supposing the solution in the form of (6) and if the coefficient of second term of (22) can be expressed as

$$\frac{2\alpha}{(c^2-k)} = \omega^2 + p\omega_1 + p^2\omega_2 + \cdots \tag{23}$$

respectively, and substituting (6) and (23) into (22), collecting terms of the same powers of p, we have

$$u_0'' + \omega^2 u_0 = 0, \tag{24}$$

$$u_1'' + \omega^2 u_1 + \omega_1 u_0 - \frac{\alpha}{3(c^2-k)}u_0^3 = 0 \tag{25}$$

We start initial approximation with $u_0 = A\cos\omega\xi$, as in Example 1, hence Eq. (25) yields

$$u_1'' + \omega^2 u_1 + \omega_1 A\cos\omega\xi - \frac{\alpha}{3(c^2-k)}(A\cos\omega\xi)^3 = 0. \tag{26}$$

Making simplification, we obtain:

$$u_1'' + \omega^2 u_1 + A\cos\omega\xi\left(w_1 - \frac{\alpha A^2}{4(c^2-k)}\right) - \frac{\alpha A^3}{12(c^2-k)}\cos 3\omega\xi = 0. \tag{27}$$

Again, elimination of the secular terms requires

$$\omega_1 = \frac{\alpha A^2}{4(c^2-k)} \tag{28}$$

and Eq. (27) reduces to

$$u_1'' + \omega^2 u_1 - \frac{\alpha A^3}{12(c^2-k)}\cos 3\omega\xi = 0. \tag{29}$$

We obtain a particular solution of Eq. (29), which reads

$$u_1 = -\frac{\alpha A^3}{96\omega^2(c^2-k)}(\cos 3\omega\xi - \cos\omega\xi) \tag{30}$$

Setting $p = 1$, in (23) gives

$$\frac{2\alpha}{(c^2-k)} = \omega^2 + \frac{\alpha A^2}{4(c^2-k)} \tag{31}$$

and therefore, the frequency is

$$\omega = \sqrt{\frac{8\alpha - \alpha A^2}{4(c^2-k)}} \tag{32}$$

If, for example, first-order approximate solution is adequate, setting $p = 1$, we have

$$u = u_0 + u_1 = A\cos\omega\xi - \frac{\alpha A^3}{96\omega^2(c^2-k)}(\cos 3\omega\xi - \cos\omega\xi) \tag{33}$$

which is the solution of the sine-Gordon equation in (19).

Example 3.3. Consider, now, combined sine-cosine-Gordon equation:

$$u_{tt} - ku_{xx} + \alpha\sin u + \beta\cos u = 0 \tag{34}$$

Using the transformation (3) again, Eq. (34) gives

$$(c^2-k)u'' + \alpha\sin u + \beta\cos u = 0. \tag{35}$$

Approximating the above equation by $\sin u \cong u - \frac{u^3}{6}$ and $\cos u \cong 1 - \frac{u^2}{2}$, we obtain following equation:

$$u'' + \frac{\alpha}{(c^2-k)}\left(u - \frac{u^3}{6}\right) + \frac{\beta}{(c^2-k)}\left(1 - \frac{u^2}{2}\right) = 0, \quad (c^2 \neq k) \tag{36}$$

We, in a similar manner, construct the homotopy in the form:

$$u'' + \frac{\alpha}{(c^2-k)}u + p\left(\frac{\beta}{(c^2-k)} - \frac{\beta}{2(c^2-k)}u^2 - \frac{\alpha}{6(c^2-k)}u^3\right) = 0. \tag{37}$$

Supposing the solution in the form of (6) and if the coefficient of second term of (37) can be expressed as

$$\frac{\alpha}{(c^2-k)} = \omega^2 + p\omega_1 + p^2\omega_2 + \cdots \tag{38}$$

respectively, and substituting (6) and (38) into (37), collecting terms of the same powers of p, we have

$$u_0'' + \omega^2 u_0 = 0, \tag{39}$$

$$u_1'' + \omega^2 u_1 + \omega_1 u_0 + \left(\frac{\beta}{(c^2-k)} - \frac{\beta}{2(c^2-k)}u_0^2 \right. \tag{40}$$

$$\left. - \frac{\alpha}{6(c^2-k)}u_0^3\right) = 0. \tag{41}$$

We start initial approximation, again, with $u_0 = A\cos\omega\xi$, as in Examples 2 and 3, hence (41) yields

$$u_1'' + \omega^2 u_1 + \omega_1 A\cos\omega\xi + \left(\frac{\beta}{(c^2-k)} - \frac{\beta}{2(c^2-k)}(A\cos\omega\xi)^2 - \frac{\alpha}{6(c^2-k)}(A\cos\omega\xi)^3\right) = 0. \tag{42}$$

Making simplification, we obtain:

$$u_1'' + \omega^2 u_1 + A\cos\omega\xi\left(\omega_1 - \frac{\alpha A^2}{8(c^2-k)}\right) + \left(\frac{4\beta - \beta A^2}{4(c^2-k)}\right) - \frac{\beta A^2}{4(c^2-k)}\cos 2\omega\xi - \frac{\alpha A^3}{24(c^2-k)}\cos 3\omega\xi = 0 \quad (43)$$

No secular term in u_1 requires that

$$\omega_1 = \frac{\alpha A^2}{4(c^2-k)} \quad \text{and} \quad A = 2$$

and (43) reduces to

$$u_1'' + \omega^2 u_1 - \frac{\beta A^2}{4(c^2-k)}\cos 2\omega\xi - \frac{\alpha A^3}{24(c^2-k)}\cos 3\omega\xi = 0 \quad (44)$$

We obtain a particular solution of Eq. (44), which reads

$$u_1 = -\frac{16\beta A^2 + \alpha A^3}{192\omega^2(c^2-k)}\cos\omega\xi + \frac{\beta A^2}{12\omega^2(c^2-k)}\cos 2\omega\xi + \frac{\alpha A^3}{192\omega^2(c^2-k)}\cos 3\omega\xi \quad (45)$$

Setting $p = 1$, in (38) gives

$$\frac{\alpha}{(c^2-k)} = \omega^2 + \frac{\alpha A^2}{8(c^2-k)} \quad (46)$$

and the frequency is

$$\omega = \sqrt{\frac{8\alpha - \alpha A^2}{8(c^2-k)}} \quad (47)$$

If, for example, first-order approximate solution is adequate, setting $p = 1$, we have

$$u = u_0 + u_1 = A\cos\omega\xi - \frac{16\beta A^2 + \alpha A^3}{192\omega^2(c^2-k)}\cos\omega\xi + \frac{\beta A^2}{12\omega^2(c^2-k)} \times \cos 2\omega\xi + \frac{\alpha A^3}{192\omega^2(c^2-k)}\cos 3\omega\xi \ldots \quad (48)$$

3. Conclusion

In this paper, coupling of He's homotopy perturbation and modified Lindstedt–Poincaré methods is applied to solve a variety of Klein–Gordon equations. The advantage of the approach is that it does not need a small parameter in the physical system, leading to wide application in nonlinear wave equations. Moreover, the method is capable of significantly minimizing the size of computational labor compared to other existing techniques. The obtained results are entirely new.

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