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ORIGINAL ARTICLE

Differential transform method for special systems of integral equations

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Abstract In this article, differential transform method has been employed to solve special kinds of systems of integral equations. To illustrate the method three examples of different forms in this class of functional equations have been prepared. The method for these systems has been lead to an exact solution. The reliability, effectively, and simplicity of the method are confirmed by the results of applying the method on the different forms of systems of these kinds of integral equations.

1. Introduction

The differential transform scheme is a method for solving a wide range of problems whose mathematical models yield equations or systems of equations involving algebraic, differential, integral and integro-differential (e.g. see Chen and Ho, 1996; Jang et al., 1997; Chen and Ho, 1999; Arikhoglu and Ozkol, 2006; Arikhoglu and Ozkol, 2007; Ayaz, 2003; Ayaz, 2004; Hassan, 2008; Biazar and Eslami, 2010; Biazar and Eslami, 2010). The concept of the differential transform was first proposed by Zhou (1986), and its main applications therein is solved both linear and non-linear initial value problems in electric circuit analysis. This method constructs an analytical solution in the form of polynomials. It is different from the high-order Taylor series method, which requires symbolic computation of the necessary derivatives of the data functions. The Taylor series method is computationally expansive for large orders. The differential transform is an iterative procedure for obtaining analytic Taylor series solutions of differential equations. In recent years the application of differential transform theory has been appeared in many researches.

The theory and application of integral equation is an important subject within applied mathematics. Integral equations are used as mathematical models for many physical situations, and integral equations also occur as reformulations of other mathematical problems such as partial differential equations and ordinary differential equations. These are motivations to solve this kind of equations.

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In this paper, we present differential transform method for solving systems of integral equations:

$$\begin{cases} f_1(x,t) + \int_0^x k_1(x,t,u_1(x,t),\dots u_1^{(i)}(x,t),\dots, \\ \qquad u_n(x,t)\dots,u_n^{(i)}(x,t))dx = u_1(x,t), \\ f_2(x,t) + \int_0^x k_2(x,t,u_1(x,t),\dots u_1^{(i)}(x,t),\dots, \\ \qquad u_n(x,t)\dots,u_n^{(i)}(x,t))dx = u_2(x,t), \qquad i = 1\dots l. \\ \vdots \\ f_n(x,t) + \int_0^x k_n(x,t,u_1(x,t),\dots u_1^{(i)}(x,t),\dots, \\ \qquad u_n(x,t)\dots,u_n^{(i)}(x,t))dx = u_n(x,t). \end{cases} \tag{1}$$

where $u_k(x,t),\ k = 1,\dots n$, are n functions to be found, $f_k(x,t)$ are given known functions, and $k_k(x,t)$ are known functions, called integral kernels.

2. Basic idea of differential transform method

The basic definitions and fundamental operations of two-dimensional differential transform are defined in Chen and Ho (1999). The differential transform of the function $u(x,y)$ is defined as the following

$$U(k,h) = \frac{1}{k!h!}\left[\frac{\partial^{k+h}u(x,y)}{\partial x^k \partial y^h}\right]_{(x_0,y_0)}, \tag{2}$$

where $u(x,y)$ is the original function and $U(k,h)$ is the transformed function.

The inverse differential transform of $U(k,h)$ is defined as follows:

$$u(x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h)(x-x_0)^k(y-y_0)^h. \tag{3}$$

In a real application, and when (x_0,y_0) are taken as (0,0), then the function $u(x,y)$ is expressed by a finite series and Eq. (3) can be written as

$$u(x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} \frac{1}{k!h!}\left[\frac{\partial^{k+h}u(x,y)}{\partial x^k \partial y^h}\right]x^k y^h. \tag{4}$$

Eq. (4) implies that the concept of two-dimensional differential transform is derived from two-dimensional Taylor series expansion. In this study we use the lower case letters to represent the original functions and upper case letters to stand for the transformed functions (T-functions).

From the definitions of Eqs. (2) and (3), it is readily proved that the transformed functions comply with the following basic mathematical operations.

The following theorems can be proved easily.

1. If $u(x,y) = g(x,y) \pm h(x,y)$, then $U(k,h) = G(k,h) \pm H(k,h)$.
2. If $u(x,y) = \lambda g(x,y)$, then $U(k,h) = \lambda G(k,h)$, where, λ is a constant.
3. If $u(x,y) = \frac{\partial g(x,y)}{\partial x}$, then $U(k,h) = (k+1)G(k+1,h)$.
4. If $u(x,y) = \frac{\partial g(x,y)}{\partial y}$, then $U(k,h) = (h+1)G(k,h+1)$.
5. If $u(x,y) = \frac{\partial^{r+s} g(x,y)}{\partial x^r \partial y^s}$, then
$$U(k,h) = (k+1)(k+2)+\cdots(k+r)(h+1)(h+2)\cdots(h+s)G(k+r,h+s).$$
6. If $u(x,y) = x^m y^n$, then $U(k,h) = \delta(k-m,h-n) = \delta(k-m)\delta(h-n)$, where
$$\delta(k-m,h-n) = \begin{cases} 1, & k = m, h = n. \\ o & otherwise. \end{cases}$$
7. If $u(x,y) = g(x,y)h(x,y)$, then $U(k,h) = \sum_{r=0}^{k}\sum_{s=0}^{h} G(r,h-s)H(k-r,s)$.
8. If $u(x,y) = f(x,y)g(x,y)h(x,y)$, then
$$U(k,h) = \sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s} (r,h-s-p)G(t,s)H(k-r-t,p).$$
9. If $u(x,y) = f(x,y)g(x,y)h(x,y)v(x,y)$, then
$$U(k,h) = \sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{d=0}^{k-r-t}\sum_{s=0}^{h}\sum_{p=0}^{h-s}\sum_{q=0}^{h-s-p} F(r,h-s-p-q)G(t,s)H(d,q) \times V(k-r-t-d,q).$$
10. If $u(x,y) = \int_0^x g(x,y)h(x,y)dx$, then $U(k,h) = \frac{1}{k}\sum_{r=0}^{k-1}\sum_{s=0}^{h} G(r,h-s)H(k-r-1,s)$, where $k \geqslant 1$ and $U(0,h) = 0$.
11. If $u(x,y) = \int_0^x f(x,y)g(x,y)h(x,y)dx$, then
$$U(k,h) = \frac{1}{k}\sum_{r=0}^{k-1}\sum_{t=0}^{k-r-1}\sum_{s=0}^{h}\sum_{p=0}^{h-s} (r,h-s-p)G(t,s)H(k-r-t-1,p),$$
where $k \geqslant 1$ and $U(0,h) = 0$.
12. If $u(x,y) = \int_0^x f(x,y)g(x,y)h(x,y)v(x,y)dx$, then
$$U(k,h) = \frac{1}{k}\sum_{r=0}^{k-1}\sum_{t=0}^{k-r-1}\sum_{d=0}^{k-r-t-1}\sum_{s=0}^{h}\sum_{p=0}^{h-s}\sum_{q=0}^{h-s-p} (r,h-s-p-q)G(t,s)H(d,q)V(k-r-t-d-1,q),$$
where $k \geqslant 1$ and $U(0,h) = 0$.

3. Numerical case studies

Three examples are presented to demonstrate the procedure of solving system (1) by differential transform method. The results reveal that the proposed technique is very effective and simple and leads to an exact solutions.

Example 1. Consider the following non-linear system of integral equations (Wazwaz, 2003)

$$\begin{cases} u(x,t) = x + e^{x-t} - \int_0^x uv dx, \\ v(x,t) = -x + e^{-x+t} + \int_0^x u^2v^2 dx. \end{cases} \tag{5}$$

Taking the differential transform of (5), leads to

$$\begin{cases} U(k,h)=\delta(k-1)\delta(h)+\frac{(-1)^h}{h!k!}-\frac{1}{k}\sum_{r=0}^{k-1}\sum_{s=0}^{h}U(r,h-s)V(k-1-r,s),\\ V(k,h)=-\delta(k-1)\delta(h)+\frac{(-1)^k}{h!k!}\\ \quad +\frac{1}{k}\sum_{r=0}^{k-1}\sum_{t=0}^{k-r-1}\sum_{d=0}^{k-r-t-1}\sum_{s=0}^{h}\sum_{p=0}^{h-s}\sum_{q=0}^{h-s-p}U(r,h-s-p-q)U(t,s)\\ \quad \times V(d,p)V(k-r-t-d-1,q). \end{cases}$$

By recursive method, the results are found as follows:

$$\begin{array}{lll} U(0,0)=1, & U(0,1)=-1, & U(0,3)=-\frac{1}{6},\\ V(0,0)=1, & V(0,1)=1, & V(0,3)=\frac{1}{6},\\ U(1,0)=1, & U(0,2)=\frac{1}{2}, & U(1,1)=-1,\\ V(1,0)=-1, & V(0,2)=\frac{1}{2}, & V(1,1)=-1,\\ \vdots & & \end{array}$$

Therefore, the solutions of the integral Eq. (5) are given by

$$u(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U(k,h)x^k t^h=(1+x+\frac{x^2}{2}+\cdots)\times(1-t+\frac{t^2}{2}-\cdots)=e^{x-t},$$
$$v(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}V(k,h)x^k t^h=(1-x+\frac{x^2}{2}+\cdots)\times(1+t+\frac{t^2}{2}+\cdots)=e^{-x+t}.$$

Which are exact solutions of Example 1.

Example 2. Consider the following non-linear system of integral differential equations (Wazwaz, 2003)

$$\begin{cases} u(x,y)=4x-2x^2+2x^2y^2-v+\int_0^x uv_x dx,\\ v(x,y)=-4xy-2x^2+2x^2y^2+u+\int_0^x u_x v dx. \end{cases} \tag{6}$$

Eq. (6) is transformed by using theorems 1, 2, 4 and 8 as follows.

$$U(k,h)=4\delta(k-1)\delta(h)-2\delta(k-2)\delta(h)+2\delta(k-2)\delta(h-2)$$
$$-V(k,h)+\frac{1}{k}\sum_{r=0}^{k-1}\sum_{s=0}^{h}(k-r)U(r,h-s)V(k-r,s),$$
$$V(k,h)=-4\delta(k-1)\delta(h-1)-2\delta(k-2)\delta(h)+2\delta(k-2)\delta(h-2)$$
$$+U(k,h)+\frac{1}{k}\sum_{r=0}^{k-1}\sum_{s=0}^{h}(k-r)V(r,h-s)U(k-r,s).$$

Consequently, we find

$$\begin{array}{lll} U(0,0)=0, & U(0,j)=0, j=1\ldots\infty & U(1,2)=0,\\ V(0,0)=0, & V(0,j)=0, j=1\ldots\infty & V(1,2)=0,\\ U(1,0)=2, & U(1,1)=2, & U(2,1)=0,\\ V(1,0)=2, & V(1,1)=-2, & V(2,1)=0,\\ \vdots & & \end{array}$$

Therefore (3), the solutions of system of integral differential Eq. (6), are given by

$$u(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U(k,h)x^k y^h=2x+2xy,$$
$$v(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}V(k,h)x^k y^h=2x-2xy.$$

Which are exact solutions.

Example 3. Consider the following system

$$\begin{cases} u(x,y)=2x+2y-2x^2-4xy+\int_0^x uv_x dx,\\ v(x,y)=2x-2y-2x^2+4xy+\int_0^x u_x v dx. \end{cases} \tag{7}$$

With the exact solutions $u(x,y)=2x+2y, v(x,y)=2x-2y$ (Wazwaz, 2003).

Taking the differential transform of (7), leads to

$$U(k,h)=2\delta(k-1)\delta(h)+2\delta(k)\delta(h-1)-2\delta(k-2)\delta(h)$$
$$-4\delta(k-1)\delta(h-1)$$
$$+\frac{1}{k}\sum_{r=0}^{k-1}\sum_{s=0}^{h}(k-r)U(r,h-s)V(k-r,s),$$
$$V(k,h)=2\delta(k-1)\delta(h)-2\delta(k)\delta(h-1)-2\delta(k-2)\delta(h)$$
$$+4\delta(k-1)\delta(h-1)$$
$$+\frac{1}{k}\sum_{r=0}^{k-1}\sum_{s=0}^{h}(k-r)V(r,h-s)U(k-r,s).$$

Consequently, we find

$$\begin{array}{lll} U(0,0)=0, & U(0,1)=2, & U(1,2)=0,\\ V(0,0)=0, & V(0,1)=-2, & V(1,2)=0,\\ U(1,0)=2, & U(1,1)=0, & V(0,j)=0,\ j=1\ldots\infty\\ V(1,0)=2, & V(1,1)=0, & U(0,j)=0,\ j=1\ldots\infty\\ \vdots & & \end{array}$$

So, the solutions of system (7) are given by

$$u(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U(k,h)x^k y^h=2x+2y,$$
$$v(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}V(k,h)x^k y^h=2x-2y.$$

In this example, exact solutions are found we have also derived the exact solutions.

4. Conclusion

In this work differential transform method, has been used successfully for solving special systems of integral equations. The results reveal the efficiency of this method for solving these systems. The present method reduces the computational difficulties of other usual methods and the calculations are manipulated simply. In this paper, the Maple 13 Package, is used to perform calculations.

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