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Numerical computational solution of the linear Volterra integral equations system via rationalized Haar functions

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Abstract In this paper, we use rationalized Haar (RH) functions to solve the linear Volterra integral equations system. We convert the integral equations system, to a system of linear equations. We show that our estimates have a good degree of accuracy.

1. Introduction

This integral equation is a mathematical model of many evolutionary problems with memory arising from biology, chemistry, physics, engineering. In recent years, many different basic functions have been used to estimate the solution of integral equations, such as orthonormal bases and wavelets. In the recent paper, we apply RH functions to solve the linear Volterra integral equations system. The method is first applied to an equivalent integral equations system, where the solution is approximated by a RH functions with unknown coefficients. The operational matrix of product is given, this matrix is then used to evaluate the unknown coefficients and find an approximate solution for $y(t) = (y_1(t), y_2(t), \ldots, y_n(t))$.

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2. Properties of RH functions

2.1. Definition of RH functions

The RH functions $\mathrm{RH}(r,t)$, $r = 1, 2, 3, \ldots$, are composed of three values 1, -1 and 0 and can be defined on the interval $[0,1)$ as

$$\mathrm{RH}(r,t) = \begin{cases} 1, & J_1 \leqslant t < J_{\frac{1}{2}} \\ -1, & J_{\frac{1}{2}} \leqslant t < J_0\,, \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

where $J_u = \frac{j-u}{2^i}$ and $u = 0, \frac{1}{2}, 1$ (Ohkita and Kobayashi, 1986).
The value of r is defined by two parameters i and j as

$$r = 2^i + j - 1; \quad i = 0, 1, 2, \ldots, \quad j = 1, 2, 3, \ldots, 2^i, \tag{2}$$

$\mathrm{RH}(0,t)$ is defined for $i = j = 0$ and is given by

$$\mathrm{RH}(0,t) = 1; \quad 0 \leqslant t < 1.$$

The orthogonality property is given by

$$\langle \mathrm{RH}(r,t), \mathrm{RH}(v,t)\rangle = \int_0^1 \mathrm{RH}(r,t)\mathrm{RH}(v,t)\,dt = \begin{cases} 2^{-i}, & \text{for } r = v \\ 0, & \text{for } r \neq v \end{cases},$$

where v and r introduced in Eq. (2).

2.2. Function approximation

A function $f(t)$ defined over the space $L^2[0,1)$ may be expanded in RH functions as

$$f(t) = \sum_{r=0}^{\infty} a_r \mathrm{RH}(r,t), \tag{3}$$

where

$$a_r = \frac{\langle f(t), \mathrm{RH}(r,t)\rangle}{\langle \mathrm{RH}(r,t), \mathrm{RH}(r,t)\rangle}. \tag{4}$$

If we let $i = 0,1,2,3,\ldots,\alpha$ then the infinite series in Eq. (3) is truncated up to its first k terms as

$$f(t) \simeq \sum_{r=0}^{k-1} a_r \mathrm{RH}(r,t) = A^T\phi(t), \tag{5}$$

where $k = 2^{\alpha+1}, \alpha = 0,1,2,\ldots, \phi_r(t) = \mathrm{RH}(r,t)$,

$$A = [a_0, a_1, \ldots, a_{k-1}]^T, \tag{6}$$

$$\phi(t) = [\phi_0(t), \phi_1(t), \ldots, \phi_{k-1}(t)]^T. \tag{7}$$

If each waveform is divided into k intervals, the magnitude of the waveform can be represented as

$$\hat{\phi}_{k\times k} = \left[\phi\left(\frac{1}{2k}\right), \phi\left(\frac{3}{2k}\right), \ldots, \phi\left(\frac{2k-1}{2k}\right)\right], \tag{8}$$

where in Eq. (8) the row denotes the order of the RH functions (Maleknejad and Mirzaee, 2006).

By using Eqs. (8) and (5) we get

$$\left[f\left(\frac{1}{2k}\right), f\left(\frac{3}{2k}\right), \ldots, f\left(\frac{2k-1}{2k}\right)\right] = A^T\hat{\phi}_{k\times k}. \tag{9}$$

We can also approximate the function $k(t,s) \in L^2([0,1)\times[0,1))$ as follows:

$$k(t,s) \simeq \phi^T(t)H\phi(s), \tag{10}$$

where $H = [h_{ij}]_{k\times k}$ is an $k \times k$ matrix that:

$$h_{ij} = \frac{\langle \mathrm{RH}(i,t), \langle k(t,s), \mathrm{RH}(j,s)\rangle\rangle}{\langle \mathrm{RH}(i,t), \mathrm{RH}(i,t)\rangle\langle \mathrm{RH}(j,t), \mathrm{RH}(j,t)\rangle}, \tag{11}$$

for $i,j = 0,1,2,\ldots,k-1$.

From Eqs. (8) and (9) we have:

$$H = \left(\hat{\phi}^{-1}_{k\times k}\right)^T \hat{H}\hat{\phi}^{-1}_{k\times k}, \tag{12}$$

where

$$\hat{H} = [\hat{h}_{ij}]_{k\times k}, \quad \hat{h}_{ij} = k\left(\frac{2i-1}{2k}, \frac{2j-1}{2k}\right); \quad i,j = 1,2,\ldots,k,$$

and

$$\hat{\phi}^{-1}_{k\times k} = \left(\frac{1}{k}\right)\left(\hat{\phi}_{k\times k}\right)^T\left(\int_0^1 \phi(t)\phi^T(t)\,dt\right)^{-1}. \tag{13}$$

We also define the matrix $D = D_{k\times k}$ as follows:

$$D = \int_0^1 \phi(t)\phi^T(t)\,dt. \tag{14}$$

For the RH functions, D has the following form Maleknejad and Mirzaee (2006, 2003) and Razzaghi and Ordokhani (2002):

$$D = \mathrm{diag}\left(1, 1, \frac{1}{2}, \frac{1}{2}, \underbrace{\frac{1}{2^2}, \ldots, \frac{1}{2^2}}_{2^2}, \underbrace{\frac{1}{2^3}, \ldots, \frac{1}{2^3}}_{2^3}, \ldots, \underbrace{\frac{1}{2^\alpha}, \ldots, \frac{1}{2^\alpha}}_{2^\alpha}\right).$$

2.3. Operational matrix of integration

The integration of the $\phi(t)$ defined in Eq. (7) is given by

$$\int_0^t \phi(t')\,dt' = P\phi(t), \tag{15}$$

where $P = P_{k\times k}$ is the $k \times k$ operational matrix for integration and is given in Maleknejad and Mirzaee (2006, 2003) as

$$P_{k\times k} = \frac{1}{2k}\begin{pmatrix} 2kP_{(\frac{k}{2})\times(\frac{k}{2})} & -\hat{\phi}_{(\frac{k}{2})\times(\frac{k}{2})} \\ \hat{\phi}^{-1}_{(\frac{k}{2})\times(\frac{k}{2})} & 0 \end{pmatrix}, \tag{16}$$

where $\hat{\phi}_{1\times 1} = [1], P_{1\times 1} = \left[\frac{1}{2}\right]$.

2.4. The product operation matrix

The product operation matrix for RH functions is defined as follows:

$$\phi(t)\phi^T(t)A \simeq \widetilde{A}_{k\times k}\phi(t), \tag{17}$$

where A is given in Eq. (6) and $\widetilde{A}_{k\times k}$ in an $k \times k$ matrix, which is called the product operation matrix of RH functions.

In general we have

$$\widetilde{A}_{k\times k} = \begin{pmatrix} \widetilde{A}_{(\frac{k}{2})\times(\frac{k}{2})} & \widetilde{H}_{(\frac{k}{2})\times(\frac{k}{2})} \\ \widehat{H}_{(\frac{k}{2})\times(\frac{k}{2})} & \widetilde{D}_{(\frac{k}{2})\times(\frac{k}{2})} \end{pmatrix}, \tag{18}$$

with

$$\widetilde{A}_{1\times 1} = a_0,$$

$$\widetilde{H}_{(\frac{k}{2})\times(\frac{k}{2})} = \hat{\phi}_{(\frac{k}{2})\times(\frac{k}{2})} \cdot \mathrm{diag}\left[a_{\frac{k}{2}}, a_{\frac{k}{2}+1}, \ldots, a_{k-1}\right],$$

$$\widehat{H}_{(\frac{k}{2})\times(\frac{k}{2})} = \mathrm{diag}\left[a_{\frac{k}{2}}, a_{\frac{k}{2}+1}, \ldots, a_{k-1}\right] \cdot \hat{\phi}^{-1}_{(\frac{k}{2})\times(\frac{k}{2})},$$

and

$$\widetilde{D}_{(\frac{k}{2})\times(\frac{k}{2})} = \mathrm{diag}\left[\left[a_0, a_1, \ldots, a_{\frac{k}{2}-1}\right] \cdot \hat{\phi}_{(\frac{k}{2})\times(\frac{k}{2})}\right].$$

See Razzaghi and Ordokhani (2002).

3. Linear Volterra integral equations system

We consider the following linear integral equations system:

$$\sum_{j=1}^{n} g_{ij}(t)y_i(t) + \sum_{j=1}^{n}\int_0^1 k_{ij}(t,s)y_j(s)\,ds = x_i(t); \quad i = 1,2,\ldots,n, \tag{19}$$

where $x_i(t), g_{ij}(t) \in L^2[0,1), k_{ij}(t,s) \in L^2([0,1)\times[0,1))$ for $i,j = 1,2,\ldots,n$ and $y_i(t)$ for $i = 1,2,\ldots,n$ are unknown functions (Delves and Mohammed, 1983).

Table 1 Numerical results for Example 1.

Nodes t	Method of Saeed and Ahmed (2008) with $h = 0.01$	Presented method for $k = 32$	Exact solution
$t = 0$	$(1,0)$	$(1,0)$	$(1,0)$
$t = 0.1$	$(1.01020, 0.20305)$	$(1.00009, 0.20001)$	$(1, 0.2)$
$t = 0.2$	$(1.01021, 0.30712)$	$(1.00006, 0.40007)$	$(1, 0.4)$
$t = 0.3$	$(1.03011, 0.41223)$	$(1.00002, 0.60004)$	$(1, 0.6)$
$t = 0.4$	$(1.03992, 0.51840)$	$(1.00003, 0.80005)$	$(1, 0.8)$
$t = 0.5$	$(1.04962, 0.62562)$	$(1.00002, 1.00001)$	$(1, 1)$
$t = 0.6$	$(1.05922, 0.73392)$	$(1.00006, 1.20002)$	$(1, 1.2)$
$t = 0.7$	$(1.06872, 0.84330)$	$(1.00009, 1.40008)$	$(1, 1.4)$
$t = 0.8$	$(1.07811, 0.95378)$	$(1.00001, 1.60008)$	$(1, 1.6)$
$t = 0.9$	$(1.08740, 1.10653)$	$(1.00009, 1.80002)$	$(1, 1.8)$
$t = 1$	$(1.09657, 1.99806)$	$(1.00001, 1.99992)$	$(1, 2)$

Table 2 Numerical results for Example 2.

Nodes t	Method of Saeed and Ahmed (2008) with $h = 0.01$	Presented method for $k = 32$	Exact solution
$t = 0$	$(0,0)$	$(0,0)$	$(0,0)$
$t = 0.1$	$(0.09999, 0.01005)$	$(0.10008, 0.01008)$	$(0.1, 0.01)$
$t = 0.2$	$(0.19999, 0.04020)$	$(0.20008, 0.04002)$	$(0.2, 0.04)$
$t = 0.3$	$(0.29998, 0.09046)$	$(0.30008, 0.09002)$	$(0.3, 0.09)$
$t = 0.4$	$(0.039996, 0.16083)$	$(0.40008, 0.16008)$	$(0.4, 0.16)$
$t = 0.5$	$(0.049994, 0.25131)$	$(0.50009, 0.25005)$	$(0.5, 0.25)$
$t = 0.6$	$(0.059991, 0.36191)$	$(0.60008, 0.36009)$	$(0.6, 0.36)$
$t = 0.7$	$(0.69989, 0.49263)$	$(0.70008, 0.49005)$	$(0.7, 0.49)$
$t = 0.8$	$(0.79985, 0.64347)$	$(0.80008, 0.64005)$	$(0.8, 0.64)$
$t = 0.9$	$(0.89982, 0.81443)$	$(0.90008, 0.81009)$	$(0.9, 0.81)$
$t = 1$	$(0.99977, 1.00552)$	$(0.99999, 0.10003)$	$(1, 1)$

If we approximate $x_i(t), g_{ij}(t), y_i(t)$ and $k_{ij}(t,s)$ by Eqs. (5) and (10) as follows:

$$x_i(t) \simeq X_i^T\phi(t), \quad y_i(t) \simeq Y_i^T\phi(t),$$
$$g_{ij}(t) \simeq G_{ij}^T\phi(t), \quad k_{ij}(t,s) \simeq \phi^T(t)H_{ij}\phi(s).$$

With substituting in Eq. (19) we have:

$$\sum_{j=1}^{n}\left(G_{ij}^T\phi(t)\phi^T(t)Y_j\right) + \sum_{j=1}^{n}\int_0^t \phi^T(t)H_{ij}\phi(s)\phi^T(s)Y_j\,ds = \phi^T(t)X_i; \quad i = 1,2,\ldots,n.$$

By using Eq. (17) we have:

$$\sum_{j=1}^{n}\phi^T(t)\widetilde{G_{ij}}Y_j + \sum_{j=1}^{n}\int_0^t \phi^T(t)H_{ij}\widetilde{Y_j}\phi(s)\,ds = \phi^T(t)X_i; \quad i = 1,2,3,\ldots,n,$$
$$\Rightarrow \sum_{j=1}^{n}\phi^T(t)\widetilde{G_{ij}}Y_j + \sum_{j=1}^{n}\phi^T(t)H_{ij}\widetilde{Y_j}\int_0^t \phi(s)\,ds = \phi^T(t)X_i; \quad i = 1,2,3,\ldots,n,$$
$$\Rightarrow \sum_{j=1}^{n}\phi^T(t)\widetilde{G_{ij}}Y_j + \sum_{j=1}^{n}\phi^T(t)H_{ij}\widetilde{Y_j}P\phi(t) = \phi^T(t)X_i; \quad i = 1,2,3,\ldots,n,$$
$$\Rightarrow \sum_{j=1}^{n}\widetilde{G_{ij}}Y_j + \sum_{j=1}^{n}H_{ij}\widetilde{Y_j}P\phi(t) = X_i; \quad i = 1,2,3,\ldots,n. \tag{20}$$

In order to construct the approximations for $y(t) = (y_1(t), y_2(t), \ldots, y_n(t))$ we collocate $\phi(t)$ in k points. By using Eq. (8) and Newton–Cotes points given in Philips and Taylor (1937) as

$$t_p = \frac{2p-1}{2k}; \quad p = 1,2,\ldots,k, \tag{21}$$

we have

$$\phi(t_p) = \hat{\phi}_{k\times k}e_p; \quad p = 1,2,\ldots,k,$$

where

$$e_p = [0,0,\ldots,0,1,0,\ldots,0]^T \in R^k,$$

and 1 pth-component. Eq. (20) can be expressed as

$$\sum_{j=1}^{n}\widetilde{G_{ij}}Y_j + \sum_{j=1}^{n}H_{ij}\widetilde{Y_j}P\hat{\phi}_{k\times k}e_p = X_i; \quad i = 1,2,3,\ldots,n, \quad p = 1,2,3,\ldots,k.$$

By solving this system of linear equations we can find vectors Y_j so:

$$y_j(t) \simeq \phi^T(t)Y_j; \quad j = 1,2,\ldots,n. \tag{22}$$

4. Numerical examples

For computational purpose, we consider two test problems.

Example 1. Consider the integral equations system (Saeed and Ahmed, 2008):

$$\begin{cases} y_1(t) - \int_0^t y_2(s)\,ds = 1 - t^2 \\ y_2(t) - \int_0^t y_1(s)\,ds = t \end{cases},$$

and the exact solution $y(t) = (y_1(t), y_2(t)) = (1, 2t)$, Table 1 shows the numerical results and comparison with the exact solution and Monte-Carlo method (Saeed and Ahmed, 2008).

Example 2. Consider the integral equations system (Saeed and Ahmed, 2008):

$$\begin{cases} y_1(t) - \int_0^t (s^2 - t)(y_1(s) + y_2(s))\,ds = t + \frac{1}{2}t^3 + \frac{1}{12}t^4 - \frac{1}{5}t^5 \\ y_2(t) - \int_0^t s(y_1(s) + y_2(s))\,ds = t^2 - \frac{1}{3}t^3 - \frac{1}{4}t^4 \end{cases},$$

and exact solution $y(t) = (y_1(t), y_2(t)) = (t, t^2)$, Table 2 shows the numerical results and comparison with the exact solution and Monte-Carlo method (Saeed and Ahmed, 2008).

5. Conclusions

In this work, we applied an application of RH functions method for solving the linear Volterra integral equations system. According to the numerical results which obtaining from the illustrative examples, we conclude that for sufficiently large **k** we get a good accuracy, since by reducing step size length the least square error will be reduced.

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