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Lagrangian formulation of a generalised coupled hyperbolic system

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ABSTRACT

We perform Noether classification of the generalised system of coupled (2 + 1)-dimensional hyperbolic equations, namely $u_{tt} - u_{xx} - u_{yy} + P(v) = 0,\ v_{tt} - v_{xx} - v_{yy} + Q(u) = 0$. Besides this we compute conservation laws corresponding to cases that have Noether symmetries for the underlying coupled hyperbolic system.

1. Introduction

In Escobedo and Herrero (1991), system of equations

$$u_t - \triangle u = v^q, \tag{1.11}$$
$$v_t - \triangle v = u^p, \tag{1.12}$$

with $x \in \mathbb{R}^N, N \geqslant 1, t, p, q > 0$ was studied in which boundedness properties and blow-up of its solutions were analysed. After a few years same authors studied global existence and uniqueness of solutions for system (1.11), (1.12) in Escobedo and Herrero (1993). Similar problems of parabolic nature arise in numerous areas of applied mathematics and model many physical phenomena, for instance, population dynamics, chemical reactions or heat transfer.

Later in Gao and Gao (2013), blow-up and existence of solutions to initial & boundary-value problems to non-linear hyperbolic and parabolic systems including variable exponents were investigated. Recently, the system of coupled (2 + 1)-D hyperbolic equations

$$u_{tt} - u_{xx} - u_{yy} + \alpha v^q = 0, \tag{1.13}$$
$$v_{tt} - v_{xx} - v_{yy} + \beta u^p = 0, \tag{1.14}$$

with q, p constants and $\alpha, \beta \neq 0$ constants, has been studied in Muatjetjeja and Khalique (2015) from the symmetry stand point. Noether and Lie symmetry classification of (1.13)–(1.14) were performed.

In this work, we examine a generalisation of system (1.13)–(1.14), which we obtain by substituting arbitrary functions $P(v)$ and $Q(u)$ for v^q and u^p, respectively in (1.13)–(1.14). Thus, we analyze generalised system of two coupled (2 + 1)-D hyperbolic equations

$$u_{tt} - u_{xx} - u_{yy} + P(v) = 0, \tag{1.15}$$
$$v_{tt} - v_{xx} - v_{yy} + Q(u) = 0, \tag{1.16}$$

with $P(v), Q(u)$ being arbitrary elements. The purpose of our study is to classify Noether symmetries for system (1.15)–(1.16) and construct conservation laws corresponding to Noether operators admitted by system (1.15)–(1.16).

Lie symmetry analysis, originally developed by Sophus Lie (1842–1899) in the latter half of the nineteenth century, is one of the most systematic methods for studying differential equations. Recently it has attracted a lot of attention from the scientists and has been applied to different areas of research. See for example

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Ovsiannikov (1982), Ibragimov (1994-1996), Butt and Ahmad (2020), Wang et al. (2016), Yildirim and Mohyud-Din (2010),

The paper is planned in following manner. In Section 2 we give a few salient features regarding Noether point symmetries. Section 3 establishes Noether operators and corresponding conservation laws for system (1.15)–(1.16) are obtained. In Section 4 we present Concluding remarks.

2. Preliminaries and notations

We give some notations and results concerning the Noether operators, which will be used later. See for example Noether (1918) and Muatjetjeja and Khalique (2013) for details.

Let

$$Y=\tau\frac{\partial}{\partial t}+\xi^1\frac{\partial}{\partial x}+\xi^2\frac{\partial}{\partial y}+\eta^1\frac{\partial}{\partial u}+\eta^2\frac{\partial}{\partial v} \quad (2.7)$$

be a vector field, where τ,ξ^1,ξ^2,η^1 and η^2 are functions of (t,x,y,u,v). The first prolongation is defined as

$$\begin{aligned}Y^{[1]}=\;&\tau\tfrac{\partial}{\partial t}+\xi^1\tfrac{\partial}{\partial x}+\xi^2\tfrac{\partial}{\partial y}+\eta^1\tfrac{\partial}{\partial u}+\eta^2\tfrac{\partial}{\partial v}+\zeta^1_t\tfrac{\partial}{\partial u_t}+\zeta^2_t\tfrac{\partial}{\partial v_t}+\zeta^1_x\tfrac{\partial}{\partial u_x}\\&+\zeta^2_x\tfrac{\partial}{\partial v_x}+\zeta^1_y\tfrac{\partial}{\partial u_y}+\zeta^2_y\tfrac{\partial}{\partial v_y},\end{aligned}$$

where

$$\begin{aligned}\zeta^1_t&=D_t(\eta^1)-u_tD_t(\tau)-u_xD_t(\xi^1)-u_yD_t(\xi^2),\\ \zeta^1_x&=D_x(\eta^1)-u_tD_x(\tau)-u_xD_x(\xi^1)-u_yD_x(\xi^2),\\ \zeta^1_y&=D_y(\eta^1)-u_tD_y(\tau)-u_xD_y(\xi^1)-u_yD_y(\xi^2),\\ \zeta^2_t&=D_t(\eta^2)-v_tD_t(\tau)-v_xD_t(\xi^1)-v_yD_t(\xi^2),\\ \zeta^2_x&=D_x(\eta^2)-v_tD_x(\tau)-v_xD_x(\xi^1)-v_yD_x(\xi^2),\\ \zeta^2_y&=D_y(\eta^2)-v_tD_y(\tau)-v_xD_y(\xi^1)-v_yD_y(\xi^2)\end{aligned}$$

and

$$\begin{aligned}D_t=\;&\tfrac{\partial}{\partial t}+u_t\tfrac{\partial}{\partial u}+v_t\tfrac{\partial}{\partial v}+u_{tt}\tfrac{\partial}{\partial u_x}+v_{tt}\tfrac{\partial}{\partial v_x}+u_{tx}\tfrac{\partial}{\partial u_x}+v_{tx}\tfrac{\partial}{\partial v_x}\\&+u_{ty}\tfrac{\partial}{\partial u_y}+v_{ty}\tfrac{\partial}{\partial v_y}+\cdots,\\ D_x=\;&\tfrac{\partial}{\partial x}+u_x\tfrac{\partial}{\partial u}+v_x\tfrac{\partial}{\partial v}+u_{xx}\tfrac{\partial}{\partial u_x}+v_{xx}\tfrac{\partial}{\partial v_x}+u_{tx}\tfrac{\partial}{\partial u_t}+v_{tx}\tfrac{\partial}{\partial v_t}\\&+u_{xy}\tfrac{\partial}{\partial u_y}+v_{xy}\tfrac{\partial}{\partial v_y}+\cdots,\\ D_y=\;&\tfrac{\partial}{\partial y}+u_y\tfrac{\partial}{\partial u}+v_y\tfrac{\partial}{\partial v}+u_{yy}\tfrac{\partial}{\partial u_y}+v_{yy}\tfrac{\partial}{\partial v_y}+u_{ty}\tfrac{\partial}{\partial u_t}+v_{ty}\tfrac{\partial}{\partial v_t}\\&+u_{xy}\tfrac{\partial}{\partial u_x}+v_{xy}\tfrac{\partial}{\partial v_x}+\cdots.\end{aligned}$$

Recall, from calculus of variations, Euler–Lagrange operators

$$\begin{aligned}\tfrac{\delta}{\delta u}=\;&\tfrac{\partial}{\partial u}-D_t\tfrac{\partial}{\partial u_t}-D_x\tfrac{\partial}{\partial u_x}-D_y\tfrac{\partial}{\partial u_y}+D_t^2\tfrac{\partial}{\partial u_{tt}}+D_x^2\tfrac{\partial}{\partial u_{xx}}+D_y^2\tfrac{\partial}{\partial u_{yy}}+\cdots,\\ \tfrac{\delta}{\delta v}=\;&\tfrac{\partial}{\partial v}-D_t\tfrac{\partial}{\partial v_t}-D_x\tfrac{\partial}{\partial v_x}-D_y\tfrac{\partial}{\partial v_y}+D_t^2\tfrac{\partial}{\partial v_{tt}}+D_x^2\tfrac{\partial}{\partial v_{xx}}+D_y^2\tfrac{\partial}{\partial v_{yy}}+\cdots.\end{aligned}$$

Definition 1. A function $\mathcal{L}(t,x,y,u,v,u_t,v_t,u_x,v_x,u_y,v_y)$ is a first order Lagrangian of second order system of PDEs

$$E_1=0, \quad (2.8)$$

$$E_2=0, \quad (2.9)$$

if the system (2.8)–(2.9) is identical to Euler–Lagrange equations

$$\frac{\delta\mathcal{L}}{\delta u}=0,\quad \frac{\delta\mathcal{L}}{\delta v}=0. \quad (2.10)$$

Definition 2. The operator Y of (2.7), is a Noether operator corresponding to first order Lagrangian $\mathcal{L}$ of system (2.8)–(2.9) if there exists potential functions B^1, B^2 and B^3 that depend on (t,x,y,u,v) such that

$$\begin{aligned}&Y^{[1]}(\mathcal{L})+\mathcal{L}\{D_x(\xi^1)+D_y(\xi^2)+D_t(\tau)\}\\&=D_t(B^1)+D_x(B^2)+D_y(B^3).\end{aligned} \quad (2.11)$$

We state the acclaimed Noether theorem.

Theorem 1. (Noether Noether, 1918) If operator Y given by (2.7) is Noether corresponding to first order Lagrangian $\mathcal{L}$ of system (2.8)–(2.9), then $T=(T^1,T^2,T^3)$, where

$$\begin{aligned}T^1=\;&\tau\mathcal{L}+(\eta^1-u_t\tau-u_x\xi^1-u_y\xi^2)\tfrac{\partial\mathcal{L}}{\partial u_t}\\&+(\eta^2-v_t\tau-v_x\xi^1-v_y\xi^2)\tfrac{\partial\mathcal{L}}{\partial v_t}-B^1,\end{aligned} \quad (2.12)$$

$$\begin{aligned}T^2=\;&\xi^1\mathcal{L}+(\eta^1-u_t\tau-u_x\xi^1-u_y\xi^2)\tfrac{\partial\mathcal{L}}{\partial u_x}\\&+(\eta^2-v_t\tau-v_x\xi^1-v_y\xi^2)\tfrac{\partial\mathcal{L}}{\partial v_x}-B^2,\end{aligned} \quad (2.13)$$

$$\begin{aligned}T^3=\;&\xi^2\mathcal{L}+(\eta^1-u_t\tau-u_x\xi^1-u_y\xi^2)\tfrac{\partial\mathcal{L}}{\partial u_y}\\&+(\eta^2-v_t\tau-v_x\xi^1-v_y\xi^2)\tfrac{\partial\mathcal{L}}{\partial v_y}-B^3,\end{aligned} \quad (2.14)$$

represents a conserved vector of system (2.8)–(2.9) corresponding to the operator Y.

3. Noether symmetries and conservation laws of the system (1.15)–(1.16)

It can ready be confirmed that the system (1.15)–(1.16) possess a first-order Lagrangian given by

$$\mathcal{L}=u_yv_y+u_xv_x-u_tv_t+\int P(v)dv+\int Q(u)du. \quad (3.15)$$

Substituting the above value of $\mathcal{L}$ in the determining Eq. (2.11) and splitting on different derivatives with respect to u and v, one obtains a linear homogeneous overdetermined system of PDEs

$$\begin{aligned}&\tau_v=0,\ \tau_u=0,\ \xi^1_u=0,\ \xi^1_v=0,\ \xi^2_v=0,\ \xi^2_u=0,\ \xi^2_t-\tau_y=0,\\&\xi^1_t-\tau_x=0,\ \xi^1_y+\xi^2_x=0,\ \eta^1_u+\eta^2_v-\tau_t+\xi^1_x+\xi^2_y=0,\\&\eta^1_u+\eta^2_v+\tau_t-\xi^1_x+\xi^2_y=0,\\&\eta^1_t=-B^1_v,\ \eta^2_t=-B^1_u,\ \eta^1_x=B^2_v,\ \eta^2_x=B^2_u,\ \eta^1_y=B^3_v,\ \eta^2_y=B^3_u,\\&\eta^1Q(u)+\eta^2P(v)+\tau_t\left(\int P(v)dv+\int Q(u)du\right)+\xi^1_x\left(\int P(v)dv\right.\\&\left.+\int Q(u)du\right)+\xi^2_y\left(\int P(v)dv+\int Q(u)du\right)=B^1_t+B^2_x+B^3_y.\end{aligned}$$

The solution of above system is

$$\begin{aligned}\xi^1&=a(t,x,y),\\ \xi^2&=b(t,x,y),\\ \tau&=c(t,x,y),\\ \eta^1&=-b_yu-j(t,x,y)u+h(t,x,y),\\ \eta^2&=jv+k(t,x,y),\\ B^1&=-j_tuv-h_tv-k_tu+n(t,x,y),\\ B^2&=j_xuv+k_xu+h_xv+r(t,x,y),\\ B^3&=j_yuv+k_yu+h_yv+s(t,x,y),\end{aligned}$$

$$\begin{aligned}&(-b_yu-ju+h)Q(u)+(jv+k)P(v)+c_t\left(\int P(v)dv+\int Q(u)du\right)\\&+a_x\left(\int P(v)dv+\int Q(u)du\right)+b_y\left(\int P(v)dv+\int Q(u)du\right)\\&=(j_{xx}+j_{yy}-j_{tt})uv+(k_{xx}+k_{yy}-k_{tt})u+(h_{xx}+h_{yy}-h_{tt})v\\&+n_t+r_x+s_y.\end{aligned} \quad (3.16)$$

The study of Eq. (3.16) results in 5 cases:

Case 1. P, Q arbitrary, but not as contained in cases 2–5 below.

We have following six Noether point symmetries

$$X_1 = \frac{\partial}{\partial t}, B^1 = n, B^2 = r, B^3 = s,\ n_t + r_x + s_y = 0, \tag{3.17}$$

$$X_2 = \frac{\partial}{\partial x}, B^1 = n, B^2 = r, B^3 = s,\ n_t + r_x + s_y = 0, \tag{3.18}$$

$$X_3 = \frac{\partial}{\partial y}, B^1 = n, B^2 = r, B^3 = s,\ n_t + r_x + s_y = 0, \tag{3.19}$$

$$X_4 = y\frac{\partial}{\partial t} + t\frac{\partial}{\partial y}, B^1 = n, B^2 = r, B^3 = s,\ n_t + r_x + s_y = 0, \tag{3.20}$$

$$X_5 = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x}, B^1 = n, B^2 = r, B^3 = s,\ n_t + r_x + s_y = 0, \tag{3.21}$$

$$X_6 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, B^1 = n, B^2 = r, B^3 = s,\ n_t + r_x + s_y = 0. \tag{3.22}$$

The invocation of the celebrated Noether theorem gives the six conserved vectors corresponding to these six Noether symmetries:

$$\begin{aligned} T_1^1 =&\ u_y v_y + u_x v_x + u_t v_t + \int P(v)dv + \int Q(u)du - n,\\ T_1^2 =&\ -u_t v_x - u_x v_t - r,\\ T_1^3 =&\ -u_t v_y - u_y v_t - s; \end{aligned}$$

$$\begin{aligned} T_2^1 =&\ u_x v_t + u_t v_x - n,\\ T_2^2 =&\ u_y v_y - u_x v_x - u_t v_t + \int P(v)dv + \int Q(u)du - r,\\ T_2^3 =&\ -u_x v_y - u_y v_x - s; \end{aligned}$$

$$\begin{aligned} T_3^1 =&\ u_y v_t + u_t v_y - n,\\ T_3^2 =&\ -u_y v_x - u_x v_y - r,\\ T_3^3 =&\ u_x v_x - u_y v_y - u_t v_t + \int P(v)dv + \int Q(u)du - s; \end{aligned}$$

$$\begin{aligned} T_4^1 =&\ yu_x v_x + yu_y v_y + tu_t v_y + yu_t v_t + tu_y v_t + y\int P(v)dv\\ &+ y\int Q(u)du - n,\\ T_4^2 =&\ -yu_t v_x - tu_y v_x - yu_x v_t - tu_x v_y - r,\\ T_4^3 =&\ tu_x v_x - tu_t v_t - yu_t v_y - tu_y v_y - yu_y v_t + t\int P(v)dv\\ &+ t\int Q(u)du - s; \end{aligned}$$

$$\begin{aligned} T_5^1 =&\ xu_x v_x + xu_y v_y + xu_t v_t + tu_x v_t + tu_t v_x + x\int P(v)dv\\ &+ x\int Q(u)du - n,\\ T_5^2 =&\ tu_y v_y - tu_t v_t - xu_x v_t - tu_x v_x - xu_t v_x + t\int P(v)dv\\ &+ t\int Q(u)du - r,\\ T_5^3 =&\ -xu_t v_y - tu_x v_y - xu_y v_t - t v_x u_y - s; \end{aligned}$$

$$\begin{aligned} T_6^1 =&\ yu_x v_t - xu_y v_t + yu_t v_x - x v_y u_t - n,\\ T_6^2 =&\ yu_y v_y - yu_t v_t + xu_x v_y - yu_x v_x + xu_y v_x + y\int P(v)dv\\ &+ y\int Q(u)du - r,\\ T_6^3 =&\ xu_t v_t - xu_x v_x - yu_y v_x - yu_x v_y + xu_y v_y - x\int P(v)dv\\ &- x\int Q(u)du - s. \end{aligned}$$

Case 2. $P = \alpha v + \beta, Q = \gamma u + \lambda,\ \ \alpha, \beta, \gamma, \lambda$ constants, $\alpha, \gamma \neq 0$.

Here we have two sub-cases, viz.,

2.1. $\beta, \lambda \neq 0$. This subcase yields six Noether symmetries, $X_1, X_2, X_3, X_4, X_5, X_6$ given by operators (3.17)–(3.22) and (2.12)–(2.14) yields six conserved vectors corresponding to six Noether symmetries given by

$$\begin{aligned} T_1^1 =&\ u_y v_y + u_x v_x + u_t v_t + \tfrac{\alpha}{2}v^2 + \beta v + \tfrac{\gamma}{2}u^2 + \lambda u - n,\\ T_1^2 =&\ -u_t v_x - u_x v_t - r,\\ T_1^3 =&\ -u_t v_y - u_y v_t - s; \end{aligned}$$

$$\begin{aligned} T_2^1 =&\ u_x v_t + u_t v_x - n,\\ T_2^2 =&\ u_y v_y - u_x v_x - u_t v_t + \tfrac{\alpha}{2}v^2 + \beta v + \tfrac{\gamma}{2}u^2 + \lambda u - r,\\ T_2^3 =&\ -u_x v_y - u_y v_x - s; \end{aligned}$$

$$\begin{aligned} T_3^1 =&\ u_y v_t + u_t v_y - n,\\ T_3^2 =&\ -u_y v_x - u_x v_y - r,\\ T_3^3 =&\ u_x v_x - u_y v_y - u_t v_t + \tfrac{\alpha}{2}v^2 + \beta v + \tfrac{\gamma}{2}u^2 + \lambda u - s; \end{aligned}$$

$$\begin{aligned} T_4^1 =&\ yu_x v_x + yu_y v_y + tu_t v_y + yu_t v_t + tu_y v_t + \tfrac{\alpha}{2}yv^2 + \beta yv\\ &+ \tfrac{\gamma}{2}yu^2 + \lambda yu - n,\\ T_4^2 =&\ -yu_t v_x - tu_y v_x - yu_x v_t - tu_x v_y - r,\\ T_4^3 =&\ tu_x v_x - tu_t v_t - yu_t v_y - tu_y v_y - yu_y v_t + \tfrac{\alpha}{2}tv^2 + \beta tv\\ &+ \tfrac{\gamma}{2}tu^2 + \lambda tu - s; \end{aligned}$$

$$\begin{aligned} T_5^1 =&\ xu_x v_x + xu_y v_y + xu_t v_t + tu_x v_t + tu_t v_x + x\tfrac{\alpha}{2}v^2 + \beta xv\\ &+ x\tfrac{\gamma}{2}u^2 + \lambda xu - n,\\ T_5^2 =&\ tu_y v_y - tu_t v_t - xu_x v_t - tu_x v_x - xu_t v_x + \tfrac{\alpha}{2}tv^2 + \beta tv\\ &+ \tfrac{\gamma}{2}tu^2 + \lambda tu - r,\\ T_5^3 =&\ -xu_t v_y - tu_x v_y - xu_y v_t - t v_x u_y - s; \end{aligned}$$

$$\begin{aligned} T_6^1 =&\ yu_x v_t - xu_y v_t + yu_t v_x - x v_y u_t - n,\\ T_6^2 =&\ yu_y v_y - yu_t v_t + xu_x v_y - yu_x v_x + xu_y v_x + \tfrac{\alpha}{2}yv^2 + \beta yv\\ &+ \tfrac{\gamma}{2}yu^2 + \lambda yu - r,\\ T_6^3 =&\ xu_t v_t - xu_x v_x - yu_y v_x - yu_x v_y + xu_y v_y - \tfrac{\alpha}{2}xv^2 + \beta xv\\ &- \tfrac{\gamma}{2}xu^2 + \lambda xu - s. \end{aligned}$$

2.2. $\beta, \lambda = 0$. Here we have seven Noether symmetry operators, namely, $X_1, X_2, X_3, X_4, X_5, X_6$ given by operators (3.17)-(3.22) and X_7 given by

$$X_7 = h(t,x,y)\tfrac{\partial}{\partial u} + k(t,x,y)\tfrac{\partial}{\partial v},\ B^1 = -h_t v - k_t u + n,$$

$$B^2 = k_x u + h_x v + r,\ B^3 = k_y u + h_y v + r,\ n_t + r_x + s_y = 0, \tag{3.23}$$

where $h(t,x,y), k(t,x,y)$ (arbitrary functions) satisfy $k_{tt} - k_{xx} - k_{yy} + \gamma h = 0, h_{tt} - h_{xx} - h_{yy} + \alpha k = 0$. Consequently, Theorem 1 yields seven conserved vectors given by

$$\begin{aligned} T_1^1 =&\ u_y v_y + u_x v_x + u_t v_t + \tfrac{\alpha}{2}v^2 + \beta v + \tfrac{\gamma}{2}u^2 + \lambda u - n,\\ T_1^2 =&\ -u_t v_x - u_x v_t - r,\\ T_1^3 =&\ -u_t v_y - u_y v_t - s; \end{aligned}$$

$$\begin{aligned} T_2^1 =&\ u_x v_t + u_t v_x - n,\\ T_2^2 =&\ u_y v_y - u_x v_x - u_t v_t + \tfrac{\alpha}{2}v^2 + \beta v + \tfrac{\gamma}{2}u^2 + \lambda u - r,\\ T_2^3 =&\ -u_x v_y - u_y v_x - s; \end{aligned}$$

$$\begin{aligned} T_3^1 =&\ u_y v_t + u_t v_y - n,\\ T_3^2 =&\ -u_y v_x - u_x v_y - r,\\ T_3^3 =&\ u_x v_x - u_y v_y - u_t v_t + \tfrac{\alpha}{2}v^2 + \beta v + \tfrac{\gamma}{2}u^2 + \lambda u - s; \end{aligned}$$

$$T_4^1 = yu_x v_x + yu_y v_y + tu_t v_y + yu_t v_t + tu_y v_t + \tfrac{\alpha}{2} y v^2 + \beta y v + \tfrac{\gamma}{2} yu^2 + \lambda yu - n,$$
$$T_4^2 = -yu_t v_x - tu_y v_x - yu_x v_t - tu_x v_y - r,$$
$$T_4^3 = tu_x v_x - tu_t v_t - yu_t v_y - tu_y v_y - yu_y v_t + \tfrac{\alpha}{2} t v^2 + \beta t v + \tfrac{\gamma}{2} tu^2 + \lambda tu - s;$$

$$T_5^1 = xu_x v_x + xu_y v_y + xu_t v_t + tu_x v_t + tu_t v_x + x\tfrac{\alpha}{2} v^2 + \beta x v + x\tfrac{\gamma}{2} u^2 + \lambda xu - n,$$
$$T_5^2 = tu_y v_y - tu_t v_t - xu_x v_t - tu_x v_x - xu_t v_x + \tfrac{\alpha}{2} t v^2 + \beta t v + \tfrac{\gamma}{2} tu^2 + \lambda tu - r,$$
$$T_5^3 = -xu_t v_y - tu_x v_y - xu_y v_t - t v_x u_y - s;$$

$$T_6^1 = yu_x v_t - xu_y v_t + yu_t v_x - x v_y u_t - n,$$
$$T_6^2 = yu_y v_y - yu_t v_t + xu_x v_y - yu_x v_x + xu_y v_x + \tfrac{\alpha}{2} y v^2 + \beta y v + \tfrac{\gamma}{2} yu^2 + \lambda yu - r,$$
$$T_6^3 = xu_t v_t - xu_x v_x - yu_y v_x - yu_x v_y + xu_y v_y - \tfrac{\alpha}{2} x v^2 + \beta x v - \tfrac{\gamma}{2} xu^2 + \lambda xu - s;$$

$$T_7^1 = -h v_t - ku_t + h_t v + k_t u - n,$$
$$T_7^2 = h v_x + ku_x - k_x u - h_x v - r,$$
$$T_7^3 = h v_y + ku_y - k_y u - h_y v - s.$$

Case 3. $P = \alpha v^q + \beta, Q = \gamma u^p + \lambda$, with $q, p, \alpha, \beta, \gamma, \lambda$ constants, $\alpha, \gamma \neq 0$.

This case has two subcases.

3.1. $\beta, \lambda \neq 0, p, q \neq -1, 2p + 2q + 5 - pq \neq 0$.

Six Noether operators $X_1, X_2, X_3, X_4, X_5, X_6$ given by generators (3.17)-(3.22) are obtained. The invocation of Theorem 1 yields the following corresponding conserved vectors:

$$T_1^1 = u_y v_y + u_x v_x + u_t v_t + \tfrac{\alpha}{q+1} v^{q+1} + \beta v + \tfrac{\gamma}{p+1} u^{p+1} + \lambda u - n,$$
$$T_2^1 = -u_t v_x - u_x v_t - r,$$
$$T_3^1 = -u_t v_y - u_y v_t - s;$$

$$T_1^2 = u_x v_t + u_t v_x - n,$$
$$T_2^2 = u_y v_y - u_x v_x - u_t v_t + \tfrac{\alpha}{q+1} v^{q+1} + \beta v + \tfrac{\gamma}{p+1} u^{p+1} + \lambda u - r,$$
$$T_3^2 = -u_x v_y - u_y v_x - s;$$

$$T_1^3 = u_y v_t + u_t v_y - n,$$
$$T_2^3 = -u_y v_x - u_x v_y - r,$$
$$T_3^3 = u_x v_x - u_y v_y - u_t v_t + \tfrac{\alpha}{q+1} v^{q+1} + \beta v + \tfrac{\gamma}{p+1} u^{p+1} + \lambda u - s;$$

$$T_1^4 = yu_x v_x + yu_y v_y + tu_t v_y + yu_t v_t + tu_y v_t + \tfrac{\alpha}{q+1} y v^{q+1} + \beta y v + \tfrac{\beta}{p+1} yu^{p+1} + \lambda yu - n,$$
$$T_2^4 = -yu_t v_x - tu_y v_x - yu_x v_t - tu_x v_y - r,$$
$$T_3^4 = tu_x v_x - tu_t v_t - yu_t v_y - tu_y v_y - yu_y v_t + \tfrac{\alpha}{q+1} t v^{q+1} + \beta t v + \tfrac{\gamma}{p+1} tu^{p+1} + \lambda tu - s;$$

$$T_1^5 = xu_x v_x + xu_y v_y + xu_t v_t + tu_x v_t + tu_t v_x + \tfrac{\alpha}{q+1} x v^{q+1} + \beta x v + \tfrac{\gamma}{p+1} xu^{p+1} + \lambda xu - n,$$
$$T_2^5 = tu_y v_y - tu_t v_t - xu_x v_t - tu_x v_x - xu_t v_x + \tfrac{\alpha}{q+1} t v^{q+1} + \beta t v + \tfrac{\gamma}{p+1} tu^{p+1} + \lambda tu - r,$$
$$T_3^5 = -xu_t v_y - tu_x v_y - xu_y v_t - t v_x u_y - s;$$

$$T_1^6 = yu_x v_t - xu_y v_t + yu_t v_x - x v_y u_t - n,$$
$$T_2^6 = yu_y v_y - yu_t v_t + xu_x v_y - yu_x v_x + xu_y v_x + \tfrac{\alpha}{q+1} y v^{q+1} + \beta y v + \tfrac{\gamma}{p+1} yu^{p+1} + \beta yu - r,$$
$$T_3^6 = xu_t v_t - xu_x v_x - yu_y v_x - yu_x v_y + xu_y v_y - \tfrac{\alpha}{q+1} x v^{q+1} + \beta x v - \tfrac{\gamma}{p+1} xu^{p+1} + \beta xu - s.$$

3.2. $\beta, \lambda = 0, p = 5, q = 5$.

In this case, we obtain four extra Noether symmetries namely

$$X_7 = (y^2 + x^2 + t^2)\tfrac{\partial}{\partial t} + 2tx\tfrac{\partial}{\partial x} + 2ty\tfrac{\partial}{\partial y} - ut\tfrac{\partial}{\partial u} - vt\tfrac{\partial}{\partial v}, \quad B^1 = uv, B^2 = 0, B^3 = 0,$$
$$X_8 = -2xt\tfrac{\partial}{\partial t} + (y^2 - x^2 - t^2)\tfrac{\partial}{\partial x} - 2xy\tfrac{\partial}{\partial y} + ux\tfrac{\partial}{\partial u} + vx\tfrac{\partial}{\partial v}, \quad B^1 = 0, B^2 = uv, B^3 = 0,$$
$$X_9 = -2yt\tfrac{\partial}{\partial t} - 2yx\tfrac{\partial}{\partial x} + (x^2 - t^2 - y^2)\tfrac{\partial}{\partial y} + uy\tfrac{\partial}{\partial u} + vy\tfrac{\partial}{\partial v}, \quad B^1 = 0, B^2 = 0, B^3 = uv,$$
$$X_{10} = 2t\tfrac{\partial}{\partial t} + 2x\tfrac{\partial}{\partial x} + 2y\tfrac{\partial}{\partial x} - u\tfrac{\partial}{\partial u} - v\tfrac{\partial}{\partial v}, B^1 = 0, B^2 = 0, B^3 = 0.$$

Employing the Noether theorem we obtain the following new extra four nontrivial conserved vectors corresponding to these extra four Noether point symmetries:

$$T_7^1 = \tfrac{1}{6}\{t^2\beta u^6 + x^2\beta u^6 + y^2\beta u^6 + 6t v_t u + t^2\alpha v^6 + x^2\alpha v^6 + y^2\alpha v^6 - 6uv + 6t^2 u_y v_y + 6x^2 u_y v_y + 6y^2 u_y v_y + 6t^2 u_x v_x + 6x^2 u_x v_x + 6y^2 u_x v_x + 6t v u_t + 12ty v_y u_t + 12tx v_x u_t + 12tyu_y v_t + 12txu_x v_t + 6t^2 u_t v_t + 6x^2 u_t v_t + 6y^2 u_t v_t\},$$
$$T_7^2 = \tfrac{1}{3}\{tx\beta u^6 - 3t v_x u + tx\alpha v^6 + 6txu_y v_y - 3t v u_x - 6ty v_y u_x - 6tyu_y v_x - 6txu_x v_x - 3t^2 v_x u_t - 3x^2 v_x u_t - 3y^2 v_x u_t - 3t^2 u_x v_t - 3x^2 u_x v_t - 3y^2 u_x v_t - 6txu_t v_t\},$$
$$T_7^3 = \tfrac{1}{3}\{ty\beta u^6 - 3t v_y u + ty\alpha v^6 - 3t v u_y - 6tyu_y v_y - 6tx v_y u_x - 6txu_y v_x + 6tyu_x v_x - 3t^2 v_y u_t - 3x^2 v_y u_t - 3y^2 v_y u_t - 3t^2 u_y v_t - 3x^2 u_y v_t - 3y^2 u_y v_t - 6tyu_t v_t\};$$

$$T_8^1 = \tfrac{1}{3}\{-tx\beta u^6 - 3x v_t u - tx\alpha v^6 - 6txu_y v_y - 6txu_x v_x - 3x v u_t - 6xy v_y u_t - 3t^2 v_x u_t - 3x^2 v_x u_t + 3y^2 v_x u_t - 6xyu_y v_t - 3t^2 u_x v_t - 3x^2 u_x v_t + 3y^2 u_x v_t - 6txu_t v_t\},$$
$$T_8^2 = \tfrac{1}{6}\{-t^2\beta u^6 - x^2\beta u^6 + y^2\beta u^6 + 6x v_x u - t^2\alpha v^6 - x^2\alpha v^6 + y^2\alpha v^6 - 6uv - 6t^2 u_y v_y - 6x^2 u_y v_y + 6y^2 u_y v_y + 6x v u_x + 12xy v_y u_x + 12xyu_y v_x + 6t^2 u_x v_x + 6x^2 u_x v_x - 6y^2 u_x v_x + 12tx v_x u_t + 12txu_x v_t + 6t^2 u_t v_t + 6x^2 u_t v_t - 6y^2 u_t v_t\},$$
$$T_8^3 = \tfrac{1}{3}\{-xy\beta u^6 + 3x v_y u - xy\alpha v^6 + 3x v u_y + 6xyu_y v_y + 3t^2 v_y u_x + 3x^2 v_y u_x - 3y^2 v_y u_x + 3t^2 u_y v_x + 3x^2 u_y v_x - 3y^2 u_y v_x - 6xyu_x v_x + 6tx v_y u_t + 6txu_y v_t + 6xyu_t v_t\};$$

$$T_9^1 = \tfrac{1}{3}\{-ty\beta u^6 - 3y v_t u - ty\alpha v^6 - 6tyu_y v_y - 6tyu_x v_x - 3y v u_t - 3t^2 v_y u_t + 3x^2 v_y u_t - 3y^2 v_y u_t - 6xy v_x u_t - 3t^2 u_y v_t + 3x^2 u_y v_t - 3y^2 u_y v_t - 6xyu_x v_t - 6tyu_t v_t\},$$
$$T_9^2 = \tfrac{1}{3}\{-xy\beta u^6 + 3y v_x u - xy\alpha v^6 - 6xyu_y v_y + 3y v u_x + 3t^2 v_y u_x - 3x^2 v_y u_x + 3y^2 v_y u_x + 3t^2 u_y v_x - 3x^2 u_y v_x + 3y^2 u_y v_x + 6xyu_x v_x + 6ty v_x u_t + 6tyu_x v_t + 6xyu_t v_t\},$$
$$T_9^3 = \tfrac{1}{6}\{-t^2\beta u^6 + x^2\beta u^6 - y^2\beta u^6 + 6y v_y u - t^2\alpha v^6 + x^2\alpha v^6 - y^2\alpha v^6 - 6uv + 6y v u_y + 6t^2 u_y v_y - 6x^2 u_y v_y + 6y^2 u_y v_y + 12xy v_y u_x + 12xyu_y v_x - 6t^2 u_x v_x + 6x^2 u_x v_x - 6y^2 u_x v_x + 12ty v_y u_t + 12tyu_y v_t + 6t^2 u_t v_t - 6x^2 u_t v_t + 6y^2 u_t v_t\};$$

$$T^1_{10} = \tfrac{1}{3}\{3v_t u + 3u_t v + \beta t u^6 + \alpha t v^6 + 6tu_x v_x + 6xu_t v_x + 6xv_t u_x + 6tu_y v_y + 6yu_t v_y + 6yv_t u_y + 6tu_t v_t\},$$
$$T^2_{10} = \tfrac{1}{3}\{-3v_x u - 3u_x v + \beta x u^6 + \alpha x v^6 - 6tu_t v_x - 6tv_t u_x - 6xu_t v_t + 6xu_y v_y - 6yu_x v_y - 6yu_y v_x - 6xu_x v_x\},$$
$$T^3_{10} = \tfrac{1}{3}\{-3v_y u - 3u_y v + \beta y u^6 + \alpha y v^6 - 6tu_t v_y - 6tv_t u_y - 6yu_t v_t - 6xu_x v_y - 6xu_y v_x + 6yu_x v_x - 6yu_y v_y\}.$$

3.3. $\beta, \lambda = 0$.
We get five subcases. See Muatjetjeja and Khalique (2015).

Case 4. $P = \alpha e^{\beta v} + \gamma, \quad Q = \delta e^{\lambda u} + \sigma, \quad \alpha, \beta, \gamma, \delta, \lambda, \sigma$ constants, $\alpha, \delta \neq 0$.

This case gives six Noether operators $X_1, X_2, X_3, X_4, X_5, X_6$ given by generators (3.17)-(3.22) and corresponding six conserved vectors are

$$T^1_1 = u_y v_y + u_x v_x + u_t v_t + \tfrac{\alpha}{\beta} e^{\beta v} + \tfrac{\delta}{\lambda} e^{\lambda u} + \gamma v + \sigma u - n,$$
$$T^2_1 = -u_t v_x - u_x v_t - r,$$
$$T^3_1 = -u_t v_y - u_y v_t - s;$$

$$T^1_2 = u_x v_t + u_t v_x - n,$$
$$T^2_2 = u_y v_y - u_x v_x - u_t v_t + \tfrac{\alpha}{\beta} e^{\beta v} + \tfrac{\delta}{\lambda} e^{\lambda u} + \gamma v + \sigma u - r,$$
$$T^3_2 = -u_x v_y - u_y v_x - s;$$

$$T^1_3 = u_y v_t + u_t v_y - n,$$
$$T^2_3 = -u_y v_x - u_x v_y - r,$$
$$T^3_3 = u_x v_x - u_y v_y - u_t v_t + \tfrac{\alpha}{\beta} e^{\beta v} + \tfrac{\delta}{\lambda} e^{\lambda u} + \gamma v + \sigma u - s;$$

$$T^1_4 = yu_x v_x + yu_y v_y + tu_t v_y + yu_t v_t + tu_y v_t + \tfrac{\alpha}{\beta} e^{\beta v} y + \tfrac{\delta}{\lambda} e^{\lambda u} y + \gamma y v + \sigma y u - n,$$
$$T^2_4 = -yu_t v_x - tu_y v_x - yu_x v_t - tu_x v_y - r,$$
$$T^3_4 = tu_x v_x - tu_t v_t - yu_t v_y - tu_y v_y - yu_y v_t + \tfrac{\alpha}{\beta} e^{\beta v} t + \tfrac{\delta}{\lambda} e^{\lambda u} t + \gamma t v + \sigma t u - s;$$

$$T^1_5 = xu_x v_x + xu_y v_y + xu_t v_t + tu_x v_t + tu_t v_x + \tfrac{\alpha}{\beta} e^{\beta v} x + \tfrac{\delta}{\lambda} e^{\lambda u} x + \gamma x v + \sigma x u - n,$$
$$T^2_5 = tu_y v_y - tu_t v_t - xu_x v_t - tu_x v_x - xu_t v_x + \tfrac{\alpha}{\beta} e^{\beta v} t + \tfrac{\delta}{\lambda} e^{\lambda u} t + \gamma t v + \sigma t u - r,$$
$$T^3_5 = -xu_t v_y - tu_x v_y - xu_y v_t - tv_x u_y - s;$$

$$T^1_6 = yu_x v_t - xu_y v_t + yu_t v_x - xv_y u_t - n,$$
$$T^2_6 = yu_y v_y - yu_t v_t + xu_x v_y - yu_x v_x + xu_y v_x + \tfrac{\alpha}{\beta} e^{\beta v} y + \tfrac{\delta}{\lambda} e^{\lambda u} y + \gamma y v + \sigma y u - r,$$
$$T^3_6 = xu_t v_t - xu_x v_x - yu_y v_x - yu_x v_y + xu_y v_y - \tfrac{\alpha}{\beta} e^{\beta v} x + \tfrac{\delta}{\lambda} e^{\lambda u} x + \gamma x v + \sigma x u - s.$$

Case 5. $P = \alpha \ln v + \beta, Q = \gamma \ln u + \lambda, \alpha, \beta, \gamma, \lambda$ constants with $\alpha, \gamma \neq 0$.

For this case we obtain six Noether operators given by (3.17)-(3.22). The invocation of Noether theorem yields

$$T^1_1 = u_y v_y + u_x v_x + u_t v_t + \alpha v \ln v + \gamma u \ln u + \beta v + \gamma u - \alpha v - \gamma u - n,$$
$$T^2_1 = -u_t v_x - u_x v_t - r,$$
$$T^3_1 = -u_t v_y - u_y v_t - s;$$

$$T^1_2 = u_x v_t + u_t v_x - n,$$
$$T^2_2 = u_y v_y - u_x v_x - u_t v_t + \alpha v \ln v + \gamma u \ln u + \beta v + \gamma u - \alpha v - \gamma u - r,$$
$$T^3_2 = -u_x v_y - u_y v_x - s;$$

$$T^1_3 = u_y v_t + u_t v_y - n,$$
$$T^2_3 = -u_y v_x - u_x v_y - r,$$
$$T^3_3 = u_x v_x - u_y v_y - u_t v_t + \alpha v \ln v + \gamma u \ln u + \beta v + \gamma u - \alpha v - \gamma u - s;$$

$$T^1_4 = yu_x v_x + yu_y v_y + tu_t v_y + yu_t v_t + tu_y v_t + y(\alpha v \ln v + \gamma u \ln u + \beta v + \gamma u - \alpha v - \gamma u) - n,$$
$$T^2_4 = -yu_t v_x - tu_y v_x - yu_x v_t - tu_x v_y - r,$$
$$T^3_4 = tu_x v_x - tu_t v_t - yu_t v_y - tu_y v_y - yu_y v_t + t(\alpha v \ln v + \gamma u \ln u + \beta v + \gamma u - \alpha v - \gamma u) - s;$$

$$T^1_5 = xu_x v_x + xu_y v_y + xu_t v_t + tu_x v_t + tu_t v_x + x(\alpha v \ln v + \gamma u \ln u + \beta v + \gamma u - \alpha v - \gamma u) - n,$$
$$T^2_5 = tu_y v_y - tu_t v_t - xu_x v_t - tu_x v_x - xu_t v_x + t(\alpha v \ln v + \gamma u \ln u + \beta v + \gamma u - \alpha v - \gamma u) - r,$$
$$T^3_5 = -xu_t v_y - tu_x v_y - xu_y v_t - tv_x u_y - s;$$

$$T^1_6 = yu_x v_t - xu_y v_t + yu_t v_x - xv_y u_t - n,$$
$$T^2_6 = yu_y v_y - yu_t v_t + xu_x v_y - yu_x v_x + xu_y v_x + y(\alpha v \ln v + \gamma u \ln u + \beta v + \gamma u - \alpha v - \gamma u) - r,$$
$$T^3_6 = xu_t v_t - xu_x v_x - yu_y v_x - yu_x v_y + xu_y v_y - x(\alpha v \ln v + \gamma u \ln u + \beta v + \gamma u - \alpha v - \gamma u) - s.$$

Remark. It should be pointed out that Noether symmetry classification of (1.15)–(1.16) prompted five main cases and few subcases for the functions $P(v)$ and $Q(u)$. So altogether we obtained eight different cases for the functions $P(v)$ and $Q(u)$. It can be clearly seen that only one case, namely 3.3, was studied in Muatjetjeja and Khalique (2015). All the other seven cases are new results in the paper. .

4. Concluding remarks

In this work, a complete Noether symmetry classification for the generalised system of coupled (2 + 1)-dimensional hyperbolic Eqs. (1.15)–(1.16) was performed with respect to the first order Lagrangian. This resulted in obtaining five cases and several subcases for functions $P(v)$ and $Q(u)$ which gave Noether symmetries. Thereafter, corresponding to each of these Noether operators found, we presented the conservation laws. The work on the underlying problem was motivated by the recent work done in Muatjetjeja and Khalique (2015). However, the results obtained therein were not complete because the work in Muatjetjeja and Khalique (2015) was restricted to self-interaction power functions. Thus, in the present work no essential restriction on non-zero arbitrary self-interaction functions was placed. Also, one can see that all the results related to Noether classification obtained in Muatjetjeja and Khalique (2015) can be derived from the present work. Therefore, the results of the present work are new and more general.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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