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A new weighted Ostrowski type inequality on arbitrary time scale

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Abstract In this paper, we prove a new weighted generalized Montgomery identity and then use it to obtain a weighted Ostrowski type inequality for parameter function on an arbitrary time scale. In addition, the real, discrete and quantum cases are considered.

1. Introduction

The following result is known in the literature as Ostrowski's inequality (see for example page 468 of Dragomir (1999)).

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with the property that $|f'(t)|\leqslant M$ for all $t\in(a,b)$. Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leqslant\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)M,$$

for all $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller constant.

Hilger (1988) initiated the theory of time scales (see Section 2 for definition) which unifies the difference and differential calculus in a consistent way. In the bid to continue in the development of this theory, Bohner and Matthews (2008) extended Theorem 1 to time scales by proving

Theorem 2. *Let $a,b,s,t\in\mathbb{T}, a<b$ and $f:[a,b]\to\mathbb{R}$ be a differentiable. Then*

$$\left|f(t)-\frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s\right|\leqslant\frac{M}{b-a}(h_2(t,a)+h_2(t,b)),\tag{1}$$

where $M=\sup_{a<t<b}|f^\Delta(t)|$ and $h_2(\cdot,\cdot)$ is defined in item (a) of Remark 11.

Since the advent of the above result, many Ostrowski and weighted Ostrowski type results on time scales have been published. In order to prove Theorem 2, one needs the so-called Montgomery identity. In the literature, there exist a lot of generalizations of this identity, see for example Karpuz and Özkan (2008), Liu and Tuna (2012), Liu et al. (2014) and Liu et al. (2014). Lately, Liu and Ngô (2009) investigated Theorem 2 by introducing a parameter λ. Inspired by the later, Xu and Fang (2016) recently proved the following new generalization of the Montgomery identity.

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Theorem 3. *Suppose that* $a, b, s, t \in \mathbb{T}$, $a < b$, $f : [a, b] \to \mathbb{R}$ *is differentiable, and* ψ *is a function of* $[0, 1]$ *into* $[0, 1]$. *Then*

$$\frac{1 + \psi(1-\lambda) - \psi(\lambda)}{2} f(t) + \frac{\psi(\lambda)f(a) + (1 - \psi(1-\lambda))f(b)}{2} = \frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s + \frac{1}{b-a}\int_a^b K(s,t) f^\Delta(s)\Delta s,$$

where

$$K(s,t) = \begin{cases} s - \left(a + \psi(\lambda)\frac{b-a}{2}\right), & s \in [a, t), \\ s - \left(a + (1 + \psi(1-\lambda))\frac{b-a}{2}\right), & s \in [t, b). \end{cases} \quad (2)$$

Using the above result, Xu and Fang (2016) *also proved the following Ostrowski type inequality.*

Theorem 4. *Suppose that* $a, b, s, t \in \mathbb{T}$, $a < b$, $f : [a, b] \to \mathbb{R}$ *are differentiable, and* ψ *is a function of* $[0, 1]$ *into* $[0, 1]$. *Then the following inequality holds*

$$\left|\frac{1 + \psi(1-\lambda) - \psi(\lambda)}{2} f(t) + \frac{\psi(\lambda)f(a) + (1 - \psi(1-\lambda))f(b)}{2} - \frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s\right| \leqslant \frac{M}{b-a}\left[h_2\left(a, a + \psi(\lambda)\frac{b-a}{2}\right) + h_2\left(t, a + \psi(\lambda)\frac{b-a}{2}\right) + h_2\left(t, a + (1 + \psi(1-\lambda))\frac{b-a}{2}\right) + h_2\left(b, a + (1 + \psi(1-\lambda))\frac{b-a}{2}\right)\right],$$

for all $\lambda \in [0, 1]$ *such that* $a + \psi(\lambda)\frac{b-a}{2}$ *and* $a + (1 + \psi(1-\lambda))\frac{b-a}{2}$ *are in* $\mathbb{T}$, *and* $t \in \left[a + \psi(\lambda)\frac{b-a}{2}, a + (1 + \psi(1-\lambda))\frac{b-a}{2}\right]$, *where* $M = \sup_{a<t<b}|f^\Delta(t)| < \infty$.

In this paper, we prove a new weighted generalized Montgomery identity and then use it to obtain a weighted Ostrowski type inequality for parameter function on an arbitrary time scale. Theorems 3 and 4 are special cases of our results.

The paper is organized as follows. In Section 2, we recall necessary results and definitions in time scale theory. Our results are formulated and proved in Section 3.

2. Time scale essentials

To make this paper self contained, we collect the following results that will be of importance in the sequel. For more on the theory of time scales, we refer the reader to the books of Bohner and Peterson (2001) and Bohner and Peterson (2003). We start with the following definition.

Definition 5. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ and backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ are defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ for $t \in \mathbb{T}$ and $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ for $t \in \mathbb{T}$, respectively. Clearly, we see that $\sigma(t) \geqslant t$ and $\rho(t) \leqslant t$ for all $t \in \mathbb{T}$. If $\sigma(t) > t$, then we say that t is right-scattered, while if $\rho(t) < t$, then we say that t is left-scattered. If $\sigma(t) = t$, then t is called right dense, and if $\rho(t) = t$ then t is called left dense. Points that are both right dense and left dense are called dense. The set $\mathbb{T}^k$ is defined as follows: if $\mathbb{T}$ has a left scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - m$; otherwise, $\mathbb{T}^k = \mathbb{T}$. For $a, b \in \mathbb{T}$ with $a \leqslant b$, we define the interval $[a, b]$ in $\mathbb{T}$ by $[a, b] = \{t \in \mathbb{T} : a \leqslant t \leqslant b\}$. Open intervals and half-open intervals are defined in the same manner.

Definition 6. The function $f : \mathbb{T} \to \mathbb{R}$, is called differentiable at $t \in \mathbb{T}^k$, with delta derivative $f^\Delta(t) \in \mathbb{R}$, if for any given $\epsilon > 0$ there exist a neighborhood U of t such that

$$\left|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)\right| \leqslant \epsilon|\sigma(t) - s|, \quad \forall s \in U.$$

If $\mathbb{T} = \mathbb{R}$, then $f^\Delta(t) = \frac{df(t)}{dt}$, and if $\mathbb{T} = \mathbb{Z}$, then $f^\Delta(t) = f(t+1) - f(t)$.

Definition 7. The function $f : \mathbb{T} \to \mathbb{R}$ is said to be *rd*-continuous if it is continuous at all dense points $t \in \mathbb{T}$ and its left-sided limits exist at all left dense points $t \in \mathbb{T}$.

Definition 8. Let f be a *rd*-continuous function. Then $g : \mathbb{T} \to \mathbb{R}$ is called the antiderivative of f on $\mathbb{T}$ if it is differentiable on $\mathbb{T}$ and satisfies $g^\Delta(t) = f(t)$ for any $t \in \mathbb{T}^k$. In this case, we have

$$\int_a^b f(s)\Delta s = g(b) - g(a).$$

Theorem 9. *If* $a, b, c \in \mathbb{T}$ *with* $a < c < b$, $\alpha \in \mathbb{R}$ *and* f, g *are rd-continuous, then*

(i) $\int_a^b [f(t) + g(t)]\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t$.
(ii) $\int_a^b \alpha f(t)\Delta t = \alpha \int_a^b f(t)\Delta t$
(iii) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$
(iv) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$.
(v) $\left|\int_a^b f(t)\Delta t\right| \leqslant \int_a^b |f(t)|\Delta t$ for all $t \in [a, b]$.
(vi) $\int_a^b f(t)g^\Delta(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t)g(\sigma(t))\Delta t$.

Definition 10. Let $h_k : \mathbb{T}^2 \to \mathbb{T}$, $k \in \mathbb{N}$ be functions that are recursively defined as

$$h_0(t, s) = 1$$

and

$$h_{k+1}(t, s) = \int_s^t h_k(\tau, s)\Delta\tau, \quad \textit{for all} \quad s, t \in \mathbb{T}.$$

In view of the above definition, we make the following remarks (see Example 1.102 in the book (Bohner and Peterson, 2001)).

Remark 11. (a) Using the fact that for all $s, t \in \mathbb{T}$, $h_1(t, s) = t - s$, we get that

$$h_2(t, s) = \int_s^t (\tau - s)\Delta\tau.$$

(b) When $\mathbb{T} = \mathbb{R}$, then for all $s, t \in \mathbb{T}$,

$$h_k(t, s) = \frac{(t-s)^k}{k!}.$$

(c) When $\mathbb{T} = \mathbb{Z}$, then for all $s, t \in \mathbb{T}$,

$$h_k(t,s) = \binom{t-s}{k} = \prod_{i=1}^{k} \frac{t-s+1-i}{i}.$$

(d) When $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$, then for all $s, t \in \mathbb{T}$,

$$h_k(t,s) = \frac{(t-s)_q^k}{[k]!} \text{ for } k \in \mathbb{N}_0,$$

where $[k]_q := \frac{q^k - 1}{q-1}$ for $q \in \mathbb{R} \setminus \{1\}$ and $k \in \mathbb{N}_0$, $[k]! := \prod_{j=1}^{k} [j]_q$ for $k \in \mathbb{N}_0$,

$$(t-s)_q^k := \prod_{j=0}^{k-1} (t - q^j s) \text{ for } k \in \mathbb{N}_0.$$

3. Main results

For the proof of our main result, we will need the following lemma.

Lemma 12. (A weighted generalized Montgomery Identity)). *Let $v : [a,b] \to [0,\infty)$ be rd−continuous and positive and $w : [a,b] \to \mathbb{R}$ be differentiable such that $w^\Delta(t) = v(t)$ on $[a,b]$. Suppose also that $a, b, s, t \in \mathbb{T}$, $a < b$, $f : [a,b] \to \mathbb{R}$ is differentiable, and ψ is a function of $[0,1]$ into $[0,1]$. Then we have the following equation*

$$\left[\frac{1+\psi(1-\lambda)-\psi(\lambda)}{2} f(t) + \frac{\psi(\lambda)f(a) + (1-\psi(1-\lambda))f(b)}{2}\right] \int_a^b v(t)\Delta t = \int_a^b K(s,t) f^\Delta(s)\Delta s + \int_a^b v(s) f(\sigma(s))\Delta s, \tag{3}$$

where

$$K(s,t) = \begin{cases} w(s) - \left(w(a) + \psi(\lambda)\frac{w(b)-w(a)}{2}\right), & s \in [a,t), \\ w(s) - \left(w(a) + (1+\psi(1-\lambda))\frac{w(b)-w(a)}{2}\right), & s \in [t,b]. \end{cases} \tag{4}$$

Proof. Using item (vi) of Theorem 9, we obtain

$$\begin{aligned} &\int_a^t \left[w(s) - \left(w(a) + \psi(\lambda)\frac{w(b)-w(a)}{2}\right)\right] f^\Delta(s)\Delta s + \int_a^t v(s) f(\sigma(s))\Delta s \\ &= \left[w(t) - \left(w(a) + \psi(\lambda)\frac{w(b)-w(a)}{2}\right)\right] f(t) \\ &- \left[w(a) - \left(w(a) + \psi(\lambda)\frac{w(b)-w(a)}{2}\right)\right] f(a) \end{aligned} \tag{5}$$

and

$$\begin{aligned} &\int_t^b \left[w(s) - \left(w(a) + (1+\psi(1-\lambda))\frac{w(b)-w(a)}{2}\right)\right] f^\Delta(s)\Delta s \\ &+ \int_t^b v(s) f(\sigma(s))\Delta s \\ &= \left[w(b) - \left(w(a) + (1+\psi(1-\lambda))\frac{w(b)-w(a)}{2}\right)\right] f(b) \\ &- \left[w(t) - \left(w(a) + (1+\psi(1-\lambda))\frac{w(b)-w(a)}{2}\right)\right] f(t). \end{aligned} \tag{6}$$

Adding Eqs. (5) and (6), and using item (iv) of Theorem 9, gives

$$\begin{aligned} \int_a^b K(s,t) f^\Delta(s)\Delta s + \int_a^b v(s) f(\sigma(s))\Delta s &= \frac{1+\psi(1-\lambda)-\psi(\lambda)}{2} f(t) \int_a^b v(t)\Delta t \\ &+ \frac{\psi(\lambda)f(a) + (1-\psi(1-\lambda))f(b)}{2} \int_a^b v(t)\Delta t. \end{aligned} \tag{7}$$

Hence, Eq. (3) follows. □

Remark 13. If $w(t) = t$, then Lemma 12 reduces to Theorem 3.

Corollary 14. *For the case when $\mathbb{T} = \mathbb{R}$ in Lemma 12, we get*

$$\left[\frac{1+\psi(1-\lambda)-\psi(\lambda)}{2} f(t) + \frac{\psi(\lambda)f(a) + (1-\psi(1-\lambda))f(b)}{2}\right] \int_a^b v(t)dt = \int_a^b K(s,t) f'(s)ds + \int_a^b v(s) f(s)ds, \tag{8}$$

where $v(t) = w'(t)$ on $[a,b]$ and

$$K(s,t) = \begin{cases} w(s) - \left(w(a) + \psi(\lambda)\frac{w(b)-w(a)}{2}\right), & s \in [a,t), \\ w(s) - \left(w(a) + (1+\psi(1-\lambda))\frac{w(b)-w(a)}{2}\right), & s \in [t,b]. \end{cases} \tag{9}$$

Corollary 15. *If we consider $\psi(\lambda) = \lambda$ in Corollary 14, then the equation becomes*

$$\left[(1-\lambda)f(t) + \lambda\frac{f(a)+f(b)}{2}\right] \int_a^b v(t)dt = \int_a^b K(s,t) f'(s)ds + \int_a^b v(s) f(s)ds,$$

where $v(t) = w'(t)$ on $[a,b]$ and

$$K(s,t) = \begin{cases} w(s) - \left(w(a) + \lambda\frac{w(b)-w(a)}{2}\right), & s \in [a,t), \\ w(s) - \left(w(b) - \lambda\frac{w(b)-w(a)}{2}\right), & s \in [t,b]. \end{cases} \tag{10}$$

Corollary 16. *For the case when $\mathbb{T} = \mathbb{Z}, a = 0, b = n, f(k) = x_k, s = j$ and $t = i$, Lemma 12 becomes*

$$\left[\frac{1+\psi(1-\lambda)-\psi(\lambda)}{2} x_i + \frac{\psi(\lambda)x_0 + (1-\psi(1-\lambda))x_n}{2}\right] \sum_{j=0}^{n-1} v(j) = \sum_{j=0}^{n-1} K(j,i)\Delta x_j + \sum_{j=0}^{n-1} v(j) x_{j+1}, \tag{11}$$

where $v(j) = w(j+1) - w(j)$ on $[0, n-1]$ and

$$K(j,i) = \begin{cases} w(j) - \left(w(0) + \psi(\lambda)\frac{w(n)-w(0)}{2}\right), & j \in [0,i), \\ w(j) - \left(w(0) + (1+\psi(1-\lambda))\frac{w(n)-w(0)}{2}\right), & j \in [i, n-1]. \end{cases} \tag{12}$$

Corollary 17. *Let $\mathbb{T} = q^{\mathbb{N}_0}$, with $q > 1$, $a = q^m$ and $b = q^n$ with $m < n$. For this case, $\sigma(t) = qt$ and $f^\Delta(t) = D_q f(t) := \frac{f(qt)-f(t)}{(q-1)t}$. Using this information, Lemma 12 becomes*

$$\left[\frac{1+\psi(1-\lambda)-\psi(\lambda)}{2} f(t) + \frac{\psi(\lambda)f(a) + (1-\psi(1-\lambda))f(b)}{2}\right] \times \int_{q^m}^{q^n} v(t)d_q t = \sum_{j=m}^{n-1} K(q^j, t)\frac{f(q^{j+1}) - f(q^j)}{(q-1)q^j} + \int_{q^m}^{q^n} v(t) f(qt) d_q t, \tag{13}$$

where $v(t) = \frac{w(qt)-w(t)}{(q-1)t}$ *on* $[q^m, q^n]$ *and*

$$K(q^j,t) = \begin{cases} w(q^j) - \left(w(q^m) + \psi(\lambda)\frac{w(q^n)-w(q^m)}{2}\right), & q^j \in [q^m, t), \\ w(q^j) - \left(w(q^m) + (1+\psi(1-\lambda))\frac{w(q^n)-w(q^m)}{2}\right), \\ \quad q^j \in [t, q^n]. \end{cases} \tag{14}$$

Theorem 18. *Let* $v : [a,b] \to [0,\infty)$ *be rd-continuous and positive and* $w : [a,b] \to \mathbb{R}$ *be differentiable such that* $w^\Delta(t) = v(t)$ *on* $[a,b]$. *Suppose also that* $a,b,s,t \in \mathbb{T}$, $a < b$, $f : [a,b] \to \mathbb{R}$ *is differentiable, and* ψ *is a function of* $[0,1]$ *into* $[0,1]$. *Then we have the following inequality*

$$\left| \left[\frac{1+\psi(1-\lambda)-\psi(\lambda)}{2}f(t) + \frac{\psi(\lambda)f(a)+(1-\psi(1-\lambda))f(b)}{2}\right] \times \int_a^b v(t)\Delta t - \int_a^b v(s)f(\sigma(s))\Delta s \right| \leqslant M\int_a^b |K(s,t)|\Delta s, \tag{15}$$

where $K(s,t)$ *is defined by (4) and* $M = \sup_{a<t<b}|f^\Delta(t)| < \infty$.

Proof. The proof easily follows by applying the absolute value on both sides of Eq. (3) in Lemma 12 and then using item (v) of Theorem 9. □

Remark 19. Setting $w(t) = t$ in Theorem 18 reduces to Theorem 4 where the equation

$$\begin{aligned} \int_a^b |K(s,t)|\Delta s = {} & h_2\left(a, a+\psi(\lambda)\frac{b-a}{2}\right) \\ & + h_2\left(t, a+\psi(\lambda)\frac{b-a}{2}\right) \\ & + h_2\left(t, a+(1+\psi(1-\lambda))\frac{b-a}{2}\right) \\ & + h_2\left(b, a+(1+\psi(1-\lambda))\frac{b-a}{2}\right). \end{aligned}$$

holds for all $\lambda \in [0,1]$ such that $a+\psi(\lambda)\frac{b-a}{2}$ and $a+(1+\psi(1-\lambda))\frac{b-a}{2}$ are in $\mathbb{T}$, and $t \in \left[a+\psi(\lambda)\frac{b-a}{2}, a+(1+\psi(1-\lambda))\frac{b-a}{2}\right]$.

We obtain the following corollary by taking $w(t) = t^2 + c$, $c \in \mathbb{R}$ in Theorem 18. For this, $v(t) = \sigma(t) + t$, for $t \in [a,b]$.

Corollary 20. *Let* $a,b,s,t \in \mathbb{T}, a < b$, $f : [a,b] \to \mathbb{R}$ *is differentiable, and* ψ *is a function of* $[0,1]$ *into* $[0,1]$. *Then we have the following inequality*

$$\left| \left[\frac{1+\psi(1-\lambda)-\psi(\lambda)}{2}f(t) + \frac{\psi(\lambda)f(a)+(1-\psi(1-\lambda))f(b)}{2}\right] \times \int_a^b (\sigma(t)+t)\Delta t - \int_a^b (\sigma(s)+s)f(\sigma(s))\Delta s \right| \leqslant M\int_a^b |K(s,t)|\Delta s, \tag{16}$$

where

$$K(s,t) = \begin{cases} s^2 - \left(a^2 + \psi(\lambda)\frac{b^2-a^2}{2}\right), & s \in [a,t), \\ s^2 - \left(a^2 + (1+\psi(1-\lambda))\frac{b^2-a^2}{2}\right), & s \in [t,b] \end{cases} \tag{17}$$

and $M = \sup_{a<t<b}|f^\Delta(t)| < \infty$.

Corollary 21. *For the case when* $\mathbb{T} = \mathbb{R}$ *in Theorem 18, we get*

$$\left| \left[\frac{1+\psi(1-\lambda)-\psi(\lambda)}{2}f(t) + \frac{\psi(\lambda)f(a)+(1-\psi(1-\lambda))f(b)}{2}\right] \times \int_a^b v(t)dt - \int_a^b v(s)f(s)ds \right| \leqslant M\int_a^b |K(s,t)|ds, \tag{18}$$

where $v(t) = w'(t)$,

$$K(s,t) = \begin{cases} w(s) - \left(w(a) + \psi(\lambda)\frac{w(b)-w(a)}{2}\right), & s \in [a,t), \\ w(s) - \left(w(a) + (1+\psi(1-\lambda))\frac{w(b)-w(a)}{2}\right), & s \in [t,b]. \end{cases} \tag{19}$$

and $M = \sup_{a<t<b}|f'(t)| < \infty$.

Corollary 22. *For the case when* $\mathbb{T} = \mathbb{Z}$, $a = 0, b = n, f(k) = x_k$, $s = j$ *and* $t = i$, *Theorem 18 amounts to*

$$\left| \left[\frac{1+\psi(1-\lambda)-\psi(\lambda)}{2}x_i + \frac{\psi(\lambda)x_0+(1-\psi(1-\lambda))x_n}{2}\right]\sum_{j=0}^{n-1}v(j) - \sum_{j=0}^{n-1}v(j)x_{j+1} \right| \leqslant M\sum_{j=0}^{n-1}|K(j,i)|, \tag{20}$$

where $v(j) = w(j+1) - w(j)$,

$$K(j,i) = \begin{cases} w(j) - \left(w(0) + \psi(\lambda)\frac{w(n)-w(0)}{2}\right), & j \in [0,i), \\ w(j) - \left(w(0) + (1+\psi(1-\lambda))\frac{w(n)-w(0)}{2}\right), & j \in [i, n-1]. \end{cases} \tag{21}$$

and $M = \sup_{0<i<n}|\Delta x_i| < \infty$.

Corollary 23. *Let* $\mathbb{T} = q^{\mathbb{N}_0}$, *with* $q > 1$, $a = q^m$ *and* $b = q^n$ *with* $m < n$. *Then we have*

$$\left| \left[\frac{1+\psi(1-\lambda)-\psi(\lambda)}{2}f(t) + \frac{\psi(\lambda)f(a)+(1-\psi(1-\lambda))f(b)}{2}\right] \times \int_{q^m}^{q^n} v(t)d_qt - \int_{q^m}^{q^n} v(t)f(qt)d_qt \right| \leqslant M\sum_{j=m}^{n-1}|K(q^j,t)|, \tag{22}$$

where $v(t) = \frac{w(qt)-w(t)}{(q-1)t}$ on $[q^m, q^n]$,

$$K(q^j,t) = \begin{cases} w(q^j) - \left(w(q^m) + \psi(\lambda)\frac{w(q^n)-w(q^m)}{2}\right), & q^j \in [q^m, t), \\ w(q^j) - \left(w(q^m) + (1+\psi(1-\lambda))\frac{w(q^n)-w(q^m)}{2}\right), \\ \quad q^j \in [t, q^n]. \end{cases} \tag{23}$$

and $M = \sup_{q^m<q^j<q^n}\left|\frac{f(q^{j+1})-f(q^j)}{(q-1)q^j}\right| < \infty$.

Corollary 24. *For $\psi(\lambda) = \lambda^2$, the inequality in Theorem 18 boils down to*

$$\left|\left[(1-\lambda)f(t) + \frac{\lambda^2 f(a) + (2\lambda - \lambda^2)f(b)}{2}\right]\int_a^b v(t)\Delta t - \int_a^b v(s)f(\sigma(s))\Delta s\right| \leqslant M\int_a^b |K(s,t)|\Delta s, \tag{24}$$

where

$$K(s,t) = \begin{cases} w(s) - \left(w(a) + \lambda^2 \frac{w(b)-w(a)}{2}\right), & s \in [a,t), \\ w(s) - \left(w(a) + (\lambda^2 - 2\lambda + 2)\frac{w(b)-w(a)}{2}\right), & s \in [t,b] \end{cases} \tag{25}$$

and $M = \sup_{a<t<b}|f^\Delta(t)| < \infty$.

Remark 25.

1. Putting $\lambda = 0$ in Eq. (24) gives

$$|f(t)\int_a^b v(t)\Delta t - \int_a^b v(s)f(\sigma(s))\Delta s| \leqslant M\int_a^b |K(s,t)|\Delta s,$$

where

$$K(s,t) = \begin{cases} w(s) - w(a), & s \in [a,t), \\ w(s) - w(b), & s \in [t,b]. \end{cases}$$

2. For $\lambda = 1/2$ in Eq. (24) we obtain the following inequality

$$\left|\left[\frac{f(t)}{2} + \frac{f(a) + 3f(b)}{8}\right]\int_a^b v(t)\Delta t - \int_a^b v(s)f(\sigma(s))\Delta s\right| \leqslant M\int_a^b |K(s,t)|\Delta s,$$

where

$$K(s,t) = \begin{cases} w(s) - \frac{7w(a)+w(b)}{8}, & s \in [a,t), \\ w(s) - \frac{3w(a)+5w(b)}{8}, & s \in [t,b]. \end{cases}$$

3. If we take $\lambda = 1$ in Eq. (24), we obtain

$$\left|\frac{f(a)+f(b)}{2}\int_a^b v(t)\Delta t - \int_a^b v(s)f(\sigma(s))\Delta s\right| \leqslant M\int_a^b |K(s,t)|\Delta s,$$

where

$$K(s,t) = w(s) - \frac{w(a)+w(b)}{2} \quad s \in [a,b].$$

4. Conclusion

In this work, a new weighted Montgomery identity is established. Using this identity, a new weighted Ostrowski type inequality is also obtained. Our results reduce to results due to Xu and Fang (2016) if $w(t) = t$. In addition, the continuous, discrete and quantum cases are considered as drop-outs of our results.

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