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ORIGINAL ARTICLE

Modified fractional decomposition method having integral w.r.t $(d\xi)^{\alpha}$

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Abstract In this paper, a novel analytical method is proposed for differential equations with time-fractional derivative. This method is based on the famous Adomian decomposition method and the modified Riemann-Liouville derivative. The fractional derivatives are considered in the Jumarie sense. However, all the previous works avoid the term of fractional derivative and handle them as a restricted variation. In order to overcome this shortcoming, a fractional decomposition method is proposed with modified Riemann-Liouville derivative. This method is a more efficient approach to solve the fractional differential equations. Several illustrative examples are given to demonstrate the effectiveness of the present method.

1. Introduction

At present, a growing number of works by many authors from various fields of science and engineering deal with differential equations described by fractional-order equations which means equations involving derivatives and integrals of non-integer order. Fractional-order derivatives and integrals provide a powerful instrument for the description of memory and hereditary properties of different substances. This is the most significant advantage of the fractional-order models in comparison with integer-order models, in which, in fact, such effects are neglected. The fractional differential equations have been occurring in many physical problems such as electromagnetic, acoustics, viscoelasticity, electrochemistry and material science (Miller and Ross, 2003; Oldham and Spanier, 1999; Podlubry, 1999). A broad class of analytical solutions methods and numerical solutions methods have been used in to handle these problems, such as the Backlund transformation (Miura, 1978), the Laplace decomposition method (Khan, 2009; Khan and Faraz, 2011; Khan and Austin, 2010), the Adomian decomposition method (George and Chakrabarti, 1995; Arora and Abdelwahid, 1993; Shawagfeh, 1999, 2002; Saha Ray and Bera, 2004, 2005a,b,c, 2006; Momani and Odibat, 2006; Jafari and Daftardar-Gejji, 2006; Daftardar-Gejji and Jafari, 2007), the homotopy perturbation method (Abdulaziz et al., 2008; Yildirim, 2008, 2010), variational iteration method (Odibat

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and Momani, 2009; Mustafa Inc., 2008; Das, 2009; He, 1998; Faraz et al., 2011), and other asymptotic methods (He, 2006) have been used to solve differential equations.

Large classes of differential equations, both ordinary as well as partial, can be solved by the Adomian decomposition method (Adomian, 1988, 1989, 1991, 1994a,b, 1998; Adomian and Rach, 1991). A reliable modification of ADM has been done by Wazwaz (1999). This computational method yields analytical solutions and has certain advantages over standard numerical methods. It is free from rounding off errors as it does not involve discretization and does not require large computer-obtained memory or power.

The solution of fractional differential equations has been obtained through Adomian decomposition method by the researchers (George and Chakrabarti, 1995; Arora and Abdelwahid, 1993; Shawagfeh, 1999, 2002; Saha Ray and Bera, 2004, 2005a,b,c, 2006; Momani and Odibat, 2006; Jafari and Daftardar-Gejji, 2006; Daftardar-Gejji and Jafari, 2007). Instead of this variety of different methods, we introduce here a method which is free of disadvantages and suitable for a wide class of initial value problems for fractional differential equations. The method uses the Adomian decomposition method and is based on modified Riemann–Liouville fractional derivative. Recently, a new modified Riemann–Liouville left derivative is proposed by Jumarie (1993). Comparing with the classical Caputo derivative, the definition of the fractional derivative is not required to satisfy higher integer-order derivative than α. Secondly, α-th derivative of a constant is zero. For these merits, Jumarie's modified derivative was successfully applied in the probability calculus (Jumarie, 2006), fractional Laplace problems (Jumarie, 2009a). With the Jumarie's fractional derivative, we propose a new integral in Adomian decomposition method w.r.t $(d\xi)^\alpha$.

It is the purpose of this paper to introduce a new decomposition method for fractional differential equations. We aim to extend the works of Shawagfeh (1999, 2002) and Saha Ray and Bera (2005b, 2006) and make further progress beyond the achievements made so far in this regard. The main aim of the present analysis is to extend the idea of Adomian decomposition method for time-fractional differential equations by using a new modified Riemann–Liouville definition by involving integrals w.r.t $(d\xi)^\alpha$. Several examples are tested, and the obtained results suggest that this newly developed technique introduces a promising tool and powerful improvement for many applications in scientific fields.

2. Basic definitions

We give some basic definitions and properties of the fractional calculus theory which are used further in this paper.

Definition 2.1. Assume $f: R \to R, \quad x \to f(x)$ denote a continuous (but not necessarily differentiable) function and let the partition $h > 0$ in the interval [0, 1]. Jumarie's derivative is defined through the fractional difference (Jumarie, 2009):

$$\Delta^\alpha = (\mathrm{FW}-1)^\alpha f(x) = \sum_0^\infty (-1)^k \binom{\alpha}{k} f[x+(\alpha-k)h] \tag{2.1}$$

where $\mathrm{FW}f(x) = f(x+h)$. Then the fractional derivative (Jumarie, 2009) is defined as the following limit.

$$f^{(\alpha)} = \lim_{h\to 0} \frac{\Delta^\alpha[f(x)-f(0)]}{h^\alpha} \tag{2.2}$$

This definition is close to the standard definition of derivatives, and as a direct result, the α-th derivative of a constant, $0 < \alpha < 1$; is zero.

Definition 2.2. The Riemann–Liouville fractional integral operator of order $\alpha \geqslant 0$ is defined (Miller and Ross, 2003; Oldham and Spanier, 1999; Podlubny, 1999) as

$${}_0I_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\xi)^{\alpha-1} f(\xi)d\xi, \quad \alpha > 0, \tag{2.3}$$

Definition 2.3. The modified Riemann–Liouville derivative (Jumarie, 2009) is defined as

$${}_0D_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-\xi)^{n-\alpha}(f(\xi)-f(0))d\xi, \tag{2.4}$$

where $x \in [0, 1], \quad n-1 \leqslant \alpha < n \quad \text{and} \quad n \geqslant 1$.

The proposed modified Riemann–Liouville derivative as shown in Eq. (2.4) is strictly equivalent to Eq. (2.2). Meanwhile, we would introduce some properties of the fractional modified Riemann–Liouville derivative in Eqs. (2.5) and (2.6).

(a) Fractional Leibniz product law.

$${}_0D_x^{(\alpha)}(uv) = u^{(\alpha)}v + uv^{(\alpha)}. \tag{2.5}$$

(b) Fractional Leibniz Formulation

$${}_0I_x^\alpha D_x^\alpha f(x) = f(x) - f(0), \quad 0 < \alpha \leqslant 1. \tag{2.6}$$

Therefore, the integration by part can be used during the fractional calculus

$${}_aI_b^\alpha u^{(\alpha)}v = (uv)/_a^{\,b} - {}_aI_b^\alpha uv^{(\alpha)}. \tag{2.7}$$

Definition 2.4. Fractional derivative of compounded functions is defined as

$$d^\alpha f \cong \Gamma(1+\alpha)df, \quad 0 < \alpha < 1 \tag{2.8}$$

Definition 2.5. The integral with respect to $(dx)^\alpha$ is defined as the solution of fractional differential equation

$$dy \cong f(x)(dx)^\alpha, \quad x \geqslant 0, \quad y(0) = 0, \quad 0 < \alpha < 1 \tag{2.9}$$

Lemma 2.4. Let $f(x)$ denote a continuous function then the solution of the Eq. (2.9) is defined as

$$y = \int_0^x f(\xi)(d\xi)^\alpha = \alpha \int_0^x (x-\xi)^{\alpha-1} f(\xi)d\xi, \quad 0 < \alpha \leqslant 1 \tag{2.10}$$

For example $f(x) = x^\gamma$ in Eq. (2.10) one obtains

$$\int_0^x \xi^\gamma (d\xi)^\alpha = \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}, \quad 0 < \alpha \leqslant 1 \tag{2.11}$$

Definition 2.6. Assume that the continuous function $f: R \to R, \quad x \to f(x)$ has a fractional derivative of order $k\alpha$, for any positive integer k and any α, $0 < \alpha \leqslant 1$; then the following equality holds, which is

$$f(x+h)=\sum_{k=0}^{\infty}\frac{h^{\alpha k}}{\alpha k!}f^{\alpha k}(x),\quad 0<\alpha\leqslant 1.\quad 0<\alpha\leqslant 1 \tag{2.12}$$

On making the substitution $h\to x$ and $x\to 0$ we obtain the fractional Mc-Laurin series

$$f(x)=\sum_{k=0}^{\infty}\frac{x^{\alpha k}}{\alpha k!}f^{\alpha k}(0),\quad 0<\alpha\leqslant 1 \tag{2.13}$$

3. Modified fractional decomposition method (MFDM)

The principles of the decomposition method and its applicability for various kinds of differential equations are given in George and Chakrabarti (1995), Arora and Abdelwahid (1993), Shawagfeh (1999, 2002), Saha Ray and Bera (2004, 2005a,b,c, 2006), Momani and Odibat (2006), Jafari and Daftardar-Gejji (2006), Daftardar-Gejji and Jafari (2007) and the references cited therein. The decomposition method requires that the linear fractional differential equation be expressed in terms of operator form as

$$\begin{aligned}&D_t^{\alpha}u(x,t)=L[x]u(x,t)+q(x,t),\quad t>0,x\in\mathbb{R},\\&u(x,0)=f(x),\quad \alpha>0.\end{aligned} \tag{3.1}$$

where $D_t^{\alpha}=\frac{\partial^{\alpha}}{\partial t^{\alpha}}$, $L[x]$ is the linear operator in x, $f(x)$ and $q(x,t)$ are continuous functions.

According to Adomian decomposition method (George and Chakrabarti, 1995; Arora and Abdelwahid, 1993; Shawagfeh, 1999, 2002; Saha Ray and Bera, 2004, 2005a,b,c, 2006; Momani and Odibat, 2006; Jafari and Daftardar-Gejji, 2006; Daftardar-Gejji and Jafari, 2007), we apply the operator I^{α}, the inverse of the operator D_t^{α}, to both sides of Eq. (3.1) which yields

$$u(x,t)=\sum_{k=0}^{m-1}\frac{t^{\alpha k}}{\alpha k!}u^{\alpha k}(x,0^{+})+I^{\alpha}(L[x]u(x,t)+q(x,t)). \tag{3.2}$$

The initial condition implies

$$u(x,t)=f(x)+I^{\alpha}(L[x]u(x,t)+q(x,t)) \tag{3.3}$$

The Adomian decomposition method (George and Chakrabarti, 1995; Arora and Abdelwahid, 1993; Shawagfeh, 1999, 2002; Saha Ray and Bera, 2004, 2005a,b,c, 2006; Momani and Odibat, 2006; Jafari and Daftardar-Gejji, 2006; Daftardar-Gejji and Jafari, 2007) assumes a series solution for $u(x,t)$ given by

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t) \tag{3.4}$$

Using Eq. (3.4) into (3.3) we get

$$\sum_{n=0}^{\infty}u_n(x,t)=f(x)+I^{\alpha}(R[x]\sum_{n=0}^{\infty}u_n(x,t)+q(x,t)). \tag{3.5}$$

Matching both sides of Eq. (3.5), we have the following relation

$$\begin{aligned}&u_0(x,t)=f(x)+I^{\alpha}(q(x,t)),\\&u_1(x,t)=I^{\alpha}(L[x]u_0(x,t)),\\&u_2(x,t)=I^{\alpha}(L[x]u_1(x,t)),\\&u_3(x,t)=I^{\alpha}(L[x]u_2(x,t)),\end{aligned} \tag{3.6}$$

In general the recursive relation is given by

$$u_{j+1}(x,t)=I^{\alpha}(L[x]u_j(x,t))\quad j\geqslant 0. \tag{3.7}$$

Instead of iteration procedure, Eqs. (3.6) and (3.7), by using the above Eq. (2.10) we suggest the following modification

$$\begin{aligned}&u_0(x,t)=f(x)+I^{\alpha}(q(x,t))\\&u_{j+1}(x,t)=\frac{1}{\Gamma(\alpha+1)}\int_0^t(L[x]u_j(x,\xi))(d\xi)^{\alpha},\quad j\geqslant 0.\end{aligned} \tag{3.8}$$

4. Applications

In order to elucidate the solution procedure of the modified fractional decomposition method, we consider first the linear fractional diffusion equation.

Example 4.1. Consider the linear fractional diffusion equation

$$\begin{aligned}&\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}=\frac{\partial^2u(x,t)}{\partial x^2}+\frac{\partial(xu(x,t))}{\partial x},\quad 0<\alpha<1\\&u(x,0)=f(x)=x.\end{aligned} \tag{4.1}$$

In order to solve this equation by using the modified fractional decomposition method, we simply substitute the initial condition into Eq. (3.8) to obtain the following recursive relation:

$$\begin{aligned}&u_0(x,t)=x,\\&u_{n+1}(x,t)=\frac{1}{\Gamma(\alpha+1)}\int_0^t(\frac{\partial^2u_n(x,\xi)}{\partial x^2}+\frac{\partial(xu_n(x,\xi))}{\partial x})(d\xi)^{\alpha},\quad n\geqslant 0.\end{aligned} \tag{4.2}$$

Here source term $q(x,t)=0$.

In view of the recursive relation (4.2), the first few components are derived as follows:

$$\begin{aligned}&u_1(x,t)=\frac{1}{\Gamma(\alpha+1)}\int_0^t\left(\frac{\partial^2u_0(x,\xi)}{\partial x^2}+\frac{\partial(xu_0(x,\xi))}{\partial x}\right)(d\xi)^{\alpha}\\&u_1(x,t)=\frac{2xt^{\alpha}}{\Gamma(1+\alpha)},\\&u_2(x,t)=\frac{2^2xt^{2\alpha}}{\Gamma(1+2\alpha)},\\&u_3(x,t)=\frac{2^3xt^{3\alpha}}{\Gamma(1+3\alpha)},\\&\vdots\\&u_n(x,t)=\frac{2^rxt^{n\alpha}}{\Gamma(1+n\alpha)},\end{aligned} \tag{4.3}$$

Using the above terms, the solution $u(x,t)$ is

$$\begin{aligned}u(x,t)&=x+\frac{2xt^{\alpha}}{\Gamma(1+\alpha)}+\frac{2^2xt^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{2^3xt^{3\alpha}}{\Gamma(1+3\alpha)}+\dots\\&=\sum_{n=0}^{\infty}\frac{2^rxt^{n\alpha}}{\Gamma(n\alpha+1)}=xE_{\alpha}(2t^{\alpha}),\end{aligned} \tag{4.4}$$

With the property ${}_0D_t^{\alpha}E_{\alpha}(t)=E_{\alpha}(t)$, we can readily check $u(x,t)=xE_{\alpha}(2t^{\alpha})$ is an exact solution of Eq. (4.1).

Example 4.2. Let us consider $u(x,0)=f(x)=1$. We then obtain

$$
\begin{aligned}
u_0(x,t) &= 1,\\
u_1(x,t) &= \frac{t^{\alpha}}{\Gamma(1+\alpha)},\\
u_2(x,t) &= \frac{t^{2\alpha}}{\Gamma(1+2\alpha)},\\
u_3(x,t) &= \frac{t^{3\alpha}}{\Gamma(1+3\alpha)},\\
&\vdots\\
u_n(x,t) &= \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}
\end{aligned}
\tag{4.5}
$$

In view of the above terms,

$$
\begin{aligned}
u(x,t) &= 1+\frac{t^{\alpha}}{\Gamma(1+\alpha)}+\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}+\ldots\\
&= \sum_{n=0}^{\infty}\frac{t^{n\alpha}}{\Gamma(1+n\alpha)} = E_{\alpha}(t^{\alpha})
\end{aligned}
\tag{4.6}
$$

where $E_{\alpha}(t^{\alpha}) = \sum_{r=0}^{\infty}\frac{t^{r\alpha}}{\Gamma(1+r\alpha)}$ is the Mittag-Leffler function in one parameter.

Example 4.3. Consider the following one-dimensional linear inhomogeneous fractional wave equation

$$
\begin{aligned}
&\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}+\frac{\partial u(x,t)}{\partial x} = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\sin x + t\cos x,\\
&x\in\mathbb{R},\quad t>0,\quad \alpha>0
\end{aligned}
\tag{4.7}
$$

Subject to the initial condition

$$
u(x,0) = 0. \tag{4.8}
$$

According to iteration algorithm, Eq. (3.8) we obtain the following

$$
\begin{aligned}
u_0(x,t) &= 0 + I^{\alpha}\left[\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\sin x + t\cos x\right],\\
u_{n+1}(x,t) &= -\frac{1}{\Gamma(1+\alpha)}\int_0^t \frac{\partial u_n(x,\xi)}{\partial x}(d\xi)^{\alpha},\quad n\geqslant 0.
\end{aligned}
\tag{4.9}
$$

Using the above recursive relationship and Mathematica, the first few terms of the decomposition series are given by

$$
\begin{aligned}
u_1(x,t) &= -\frac{t^{1+\alpha}\cos x}{\Gamma(2+\alpha)}+\frac{t^{1+2\alpha}\sin x}{\Gamma(2+2\alpha)}\\
u_2(x,t) &= -\frac{t^{1+2\alpha}\sin x}{\Gamma(2+2\alpha)}-\frac{t^{1+3\alpha}\cos x}{\Gamma(2+3\alpha)}\\
&\vdots
\end{aligned}
\tag{4.10}
$$

Therefore the solution is

$$
\begin{aligned}
u(x,t) =& t\sin x + \frac{t^{1+\alpha}}{\Gamma(2+\alpha)}\cos x - \frac{t^{1+\alpha}\cos x}{\Gamma(2+\alpha)}+\frac{t^{1+2\alpha}\sin x}{\Gamma(2+2\alpha)}\\
&-\frac{t^{1+3\alpha}\cos x}{\Gamma(2+3\alpha)}-\frac{t^{1+2\alpha}\sin x}{\Gamma(2+2\alpha)}+\ldots
\end{aligned}
\tag{4.11}
$$

Canceling the noise terms and keeping the non-noise terms yield the exact solution of Eq. (4.7). If we begin with $u_0(x,t) = t\sin x$ then the exact solution follows immediately by using two iterations.

5. Conclusion

In this paper, the authors propose a very effective and convenient method called modified fractional decomposition method (MFDM) having integral w.r.t $(d\xi)^{\alpha}$ to solve any order of fractional differential equations. Three typical examples have been discussed as illustrations. In previous papers (George and Chakrabarti, 1995; Arora and Abdelwahid, 1993; Shawagfeh, 1999, 2002; Saha Ray and Bera, 2004, 2005a,b,c, 2006; Momani and Odibat, 2006; Jafari and Daftardar-Gejji, 2006; Daftardar-Gejji and Jafari, 2007), many authors have already established as well as successfully exhibited the applicability of Adomian decomposition method to obtain the solutions of different types of fractional differential equations. In this work, we demonstrate that modified decomposition method is also well suited to solve fractional differential equation. The fractional differential equations are described in the Jumarie sense. The modified version is valid for other fractional differential equations, and this paper can be used as a standard paradigm for other applications.

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