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On general quasi-variational inequalities

Muhammad Aslam Noor*

Mathematics Department, COMSATS Institute of Information Technology, Islamabad, Pakistan
Mathematics Department, College of Science, King Saud University, Riyadh, Saudi Arabia

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Abstract A new class of general quasi-variational inequalities involving two operators is introduced and studied. Using essentially the projection operator technique, we establish the equivalence between the general quasi-variational inequalities and the fixed-point problem and the Wiener–Hopf equations. These alternative equivalent formulations have been used to suggest and analyze several iterative methods for solving the general quasi-variational inequalities. We also discuss the convergence criteria of these iterative methods under some suitable conditions. Several special cases are also discussed.

1. Introduction

Quasi-variational inequalities, which were introduced and studied in the early 1960s, are being used to consider a wide class of unrelated problems in a unified and general framework, see Borwein and Lewis (2006), Cristescu and Lupsa (2002), Glowinski et al. (1981), Noor (1975, 1988a, 1993, 1997c, 2004), Noor et al. (1993). It has been shown that the variational inequalities provide a general, natural, simple, unified and efficient framework for a general treatment of a wide class of unrelated linear and nonlinear problems. This theory combines theoretical and algorithmic advances with novel domain of applications. Analysis of these problems requires a blend of techniques from convex analysis, functional analysis and numerical analysis, see Baiocchi and Capelo (1984), Bensoussan and Lions (1978), Borwein and Lewis (2006), Cristescu and Lupsa (2002), Giannessi et al. (2001), Gilbert et al. (2001), Glowinski et al. (1981), Kravchuk and Neittaanmaki (2007), Noor (1975, 1985, 1988a,b, 1993, 1997a,b,c, 1998, 1999, 2000, 2004, 2007, 2008a, 2010a, 2009a,b,c, 2010b, 2008b), Noor et al. (1993, 2010), Robinson (1992), Shi (1991), Stampacchia (1964) and the references therein. There are significant developments of these problems related to nonconvex optimization, iterative method and structural analysis. Bensoussan and Lions (1978) have shown that a class of impulse control problems can be formulated as quasi-variational inequality problem. Kravchuk and Neittaanmaki (2007), and Noor (1998) have shown that a wide class of problems, which arises in mechanics can be studied in the general framework of quasi-variational inequalities. Noor (1985) proved that a class of quasi-variational inequalities is equivalent to the fixed-point problem using the projection technique.

* Address: Mathematics Department, COMSATS Institute of Information Technology, Islamabad, Pakistan.
E-mail address: noormaslam@hotmail.com.

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This equivalent formulation has been used to develop iterative methods for solving the quasi-variational inequality and its various variant forms.

Related to the variational inequalities, we have the problem of solving the Wiener–Hopf equations, which were introduced by Robinson (1992) and Shi (1991). Normal maps (Shi, 1991) were introduced by using linear transformation technique, whereas the Wiener–Hopf equations were considered by Robinson (1992) by using the projection operator theory. We would like to point out that normal maps problem and the Wiener–Hopf equations are exactly the same. Robinson (1992) and Shi (1991) have proved that the variational inequalities are equivalent to the Wiener–Hopf equations (normal maps) using quite different techniques. These alternative formulations have played a very significant role in the developments of numerical methods, sensitivity analysis, dynamical systems and other aspects of variational inequalities. Noor (1997a) has shown that the quasi-variational inequalities are equivalent to the Wiener–Hopf (normal maps) using the projection operator techniques. This alternative equivalent formulation to suggest numerical methods and other techniques for solving the quasi variational inequalities, see Borwein and Lewis (2006), Kravchuk and Neittaanmaki (2007), Noor (1975, 1985, 1988a,b, 1993, 1997a,b,c, 1998, 1999, 2000, 2004, 2007, 2008a, 2010a, 2009a,b,c, 2010b, 2008b), Noor et al. (1993, 2010) and the references therein.

It is well known that the variational inequality represents the optimality condition for the minimum of a convex function on the convex set. We would like to point out that all the works carried out in this direction assumed that the underlying set is a convex set. In many practical problems, a choice set may not be a convex so that the existing results may not be applicable. In recent years, the concept of convexity has been generalized and extended in several directions using some different techniques. Cristescu and Lupsa (2002) introduced the concept of g-convex set, which is non–convex set. Noor (2008b) studied a class of functions on the g-convex set, which is called the g-convex function. It is well known that every convex function is a g-convex function, but the converse is not true. Noor, 2008b has shown that the minimum of a differentiable g-convex function can be characterized by a class of variational inequalities, which is called the general variational inequality.

Inspired and motived by these research activities, we consider and study a new class of quasi-variational inequalities involving two operators, which is called the general quasi-variational inequality. This class is quite general and unifying ones. The general quasi-variational inequalities include the classical quasi-variational inequalities and related optimization problems as special cases. Using the projection method, we prove that the general quasi-variational inequality is equivalent to the fixed-point problem, which is Lemma 3.1. This equivalent is used to discuss the existence of a solution of the general quasi-variational inequality, which is the main motivation of Theorem 3.1. In section, we use Lemma 3.1 to suggest and analyze a number of iterative methods for solving the general quasi-variational inequalities, see, for example Algorithm 4.1. We also consider the convergence analysis of Algorithm 4.1 under some suitable conditions, which is the main result (Theorem 3.1). Several special cases are also discussed. In Section 5, we introduce a new class for solving the Wiener–Hopf equations(normal maps), which is called the general implicit Wiener–Hopf equation. Using Lemma 3.1, we show that the general implicit Wiener–Hopf equations are equivalent to the general quasi-variational inequalities. This equivalence is more flexible than the fixed-point problem. We use this alternative equivalent to suggest some new iterative methods for solving the general quasi-variational inequalities. We consider the convergence criteria of these new methods under the same conditions as in Section 4. Several special cases are also discussed. Since the general quasi-variational inequality includes quasi-variational and related optimization problems as special cases, the results proved in this paper continue to hold for these problems.

2. Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|.\|$, respectively. Let $K : H \to H$ be a point to set mapping, which is closed and convex valued, In other words, for every $u \in H$, the set $K(u)$ is closed and convex.

For given nonlinear operators $T,\ g : H \to H$, consider the problem of finding $u \in H : g(u) \in K(u)$ such that

$$\langle \rho T(u) + g(u) - u, v - g(u) \rangle \geqslant 0, \quad \forall v \in K(u), \tag{2.1}$$

where $\rho > 0$ is a constant. Inequality of type (2.1) is called the *general quasi-variational inequality involving two operators*. This class of quasi-variational inequalities is quite general and unified one.

If $K(u) \equiv K$, that is, the convex set $K(u)$ is independent of the solution u, then the general quasi-variational inequalities (2.1) are equivalent to finding $u \in H : g(u) \in K$ such that

$$\langle T(u) + g(u) - u, v - g(u) \rangle \geqslant 0, \quad \forall v \in K, \tag{2.2}$$

which is called the *general variational inequality involving two operators* and was introduced and studied by Noor, 2008b. For $u = g(u)$, the problem (2.2) is equivalent to finding $u \in H : g(u) \in K$ such that

$$\langle T(g(u)), v - g(u) \rangle \geqslant 0, \quad \forall v \in K. \tag{2.3}$$

Inequality of type (2.3) is also called the *general variational inequality involving two operators*, which was introduced and studied by Noor, 1988a in 2008. For the numerical analysis, applications and other aspects of these variational inequalities, see Glowinski et al. (1981), Noor (2008a, 2010b, 2008b), Noor et al. (1993) and the references therein.

We now show that the minimum of a differentiable g-convex function on a non-convex set K in H can be characterized by the general variational inequality of type (2.3).

For this purpose, we recall the following well-known concepts, see Noor (2008a, 2010a, 2009a).

Definition 2.1 (*Cristescu and Lupsa, 2002; Noor, 2008b*). Let K be any set in H. The set K is said to be g-convex, if there exists a function $g : H \to H$ such that

$$g(u) + t(v - g(u)) \in K, \quad \forall u, v \in H : g(u), v \in K, \quad t \in [0, 1].$$

Note that every convex set is g-convex, but the converse is not true, see Noor (2010a, 2009a,b,c).

Definition 2.2 Noor, 2008b. The function $F : K \to H$ is said to be g-convex, if there exists a function g such that

$$F(g(u)+t(v-g(u))) \leqslant (1-t)F(g(u))+tF(v),$$
$$\forall u,v \in H : g(u), v \in K, \quad t \in [0,1].$$

Clearly every convex function is g-convex, but the converse is not true, see Noor (2008a), Noor (2010a), Noor (2009a).

Lemma 2.1 Noor, 2008b. *Let $F: K \to H$ be a differentiable g-convex function. Then $u \in H : g(u) \in K$ is the minimum of g-convex function F on K, if and only if, $u \in H : g(u) \in K$ satisfies the inequality*

$$\langle F'(g(u)), v-g(u)\rangle \geqslant 0, \quad \forall v \in K, \tag{2.4}$$

where $F'(u)$ is the differential of F at $g(u) \in K$.

Proof. Let $u \in H : g(u) \in K$ be a minimum of g-convex function F on K. Then

$$F(g(u)) \leqslant F(v), \quad \forall v \in K. \tag{2.5}$$

Since K is a g-convex set, so, for all $u, v \in H : g(u), \ v \in K, \ t \in [0,1], \ g(v_t) = g(u)+t(v-g(u)) \in K$. Setting $v = g(v_t)$ in (2.5), we have

$$F(g(u)) \leqslant F(g(u)+t(v-g(u))).$$

Dividing the above inequality by t and taking $t \to 0$, we have

$$\langle F'(g(u)), v-g(u)\rangle \geqslant 0, \quad \forall v \in H : v \in K,$$

which is the required result (2.4).

Conversely, let $u \in H : g(u) \in K$ satisfy the inequality (2.4). Since F is a g-convex function, $\forall u,v \in H : g(u), \ v \in K, \ t \in [0,1], \ g(u)+t(v-g(u)) \in K$ and

$$F(g(u)+t(v-g(u))) \leqslant (1-t)F(g(u))+tF(v),$$

which implies that

$$F(g(v))-F(g(u)) \geqslant \frac{F(g(u)+t(v-g(u)))-F(g(u))}{t}.$$

Letting $t \to 0$, and using (2.4), we have

$$F(g(v))-F(g(u)) \geqslant \langle F'(h(u)), v-g(u)\rangle \geqslant 0,$$

which implies that

$$F(g(u)) \leqslant F(v), \quad \forall v \in K$$

showing that $u \in H : g(u) \in K$ is the minimum of F on K in H. □

For $g = I$, the identity operator, the general quasi-variational inequality (2.1) is equivalent to finding $u \in K(u)$ such that

$$\langle T(u), v-u\rangle \geqslant 0, \quad \forall v \in K(u), \tag{2.6}$$

which is known as the classical quasi-variational inequality, introduced and studied by Bensoussan and Lions (1978) in the study of impulse control theory. See also Borwein and Lewis (2006), Cristescu and Lupsa (2002), Giannessi et al. (2001), Glowinski et al. (1981), Noor (1975, 1988a, 1993, 1997a,b,c, 2000, 2004, 2007, 2010a, 2009a,b,c, 2010b, 2008b).

If $K(u) \equiv K$, and $g = I$, the identity operator, then the problem (2.1) is equivalent to finding $u \in K$ such that

$$\langle T(u), v-u\rangle \geqslant 0, \quad \forall v \in K, \tag{2.7}$$

which is known as the classical variational inequality which was introduced in 1964 by Stampacchia (1964). For the recent applications, numerical methods, sensitivity analysis, dynamical systems and formulation of variational inequalities, see Baiocchi and Capelo (1984), Bensoussan and Lions (1978), Borwein and Lewis (2006), Cristescu and Lupsa (2002), Giannessi et al. (2001), Gilbert et al. (2001), Glowinski et al. (1981), Kravchuk and Neittaanmaki (2007), Noor (1975, 1985, 1988a,b, 1993, 1997a,b,c, 1998, 1999, 2000, 2004, 2007, 2008a, 2010a, 2009a,b,c, 2010b, 2008b), Noor et al. (1993, 2010), Robinson (1992), Shi (1991), Stampacchia (1964) and the references therein. Thus, we conclude that the general quasi-variational inequality (2.1) is quite general and includes several classes of variational inequalities and related optimization problems as special cases.

We also need the following standard and classical result.

Lemma 2.2. *Let $K(u)$be a closed and convex set in H. Then, for a given $z \in H, u \in K(u)$ satisfies the inequality*

$$\langle u-z, v-u\rangle \geqslant 0, \quad \forall v \in K(u),$$

if and only if

$$u = P_{K(u)}z,$$

where $P_{K(u)}$ is the projection of Honto the closed convex set $K(u)$ in H.

We would like to point out that the implicit projection operator $P_{K(u)}$ is not non-expansive. We shall assume that the implicit projection operator $P_{K(u)}$ satisfies the Lipschitz type continuity, which plays an important and fundamental role in the existence theory and in developing numerical methods for solving the quasi- variational inequalities.

Assumption 2.1. For all $u, \ v, \ w \in H$, the implicit projection operator $P_{K(u)}$ satisfies the condition

$$\|P_{K(u)}w - P_{K(v)}w\|| \leqslant v\|u-v\|, \tag{2.8}$$

where $v > 0$ is a positive constant.

Assumption 2.1 has been used to prove the existence of a solution of the quasi-variational inequalities as well as in analyzing convergence of the iterative methods. One can easily show that the Assumption 2.1 holds for certain cases.

Remark 2.1. In many important applications (Glowinski et al., 1981; Noor, 1975, 2000, 2004) the convex-valued set $K(u)$ can be written as

$$K(u) = m(u) + K, \tag{2.9}$$

where $m(u)$ is a point-point mapping and K is a convex set. In this case, we have

$$P_{K(u)}w = P_{m(u)+K}(w) = m(u) + P_K[w-m(u)], \quad \forall u, v \in H. \tag{2.10}$$

We note that if $K(u)$ is defined by (2.9), and $m(u)$ is a Lipschitz continuous mapping with constant $\gamma > 0$, then

$$\begin{aligned}\|P_{K(u)}w - P_{K(v)}w\| &= \|m(u)-m(v)+P_K[w-m(u)] \\ &\quad - P_K[w-m(v)]\| \\ &\leqslant 2\|m(u)-m(v)\| \leqslant 2\gamma\|u-v\|, \\ &\quad \forall u,v,w \in H.\end{aligned}$$

which shows that Assumption 2.1 holds with $v = 2\gamma$.

Definition 2.3. For all $u, v \in H$, an operator $T : H \to H$ is said to be:

(i) *strongly monotone*, if there exists a constant $\alpha > 0$ such that

$$\langle T(u) - T(v), u - v\rangle \geqslant \alpha\|u - v\|^2$$

(ii) *Lipschitz continuous*, if there exists a constant $\beta > 0$ such that

$$\|T(u) - T(v)\| \leqslant \beta\|u - v\|.$$

If T verifies (i) and (ii), then $\alpha \leqslant \beta$.

3. Existence results

In this section, we consider the existence of a solution of the general quasi-variational inequality (2.1) under some conditions. First of all, we prove that the general quasi-variational inequality (2.1) is equivalent to the fixed-point problem using Lemma 2.2.

Lemma 3.1. $u \in H : g(u) \in K(u)$ is a solution of the general quasi-variational inequality *(2.1) if and only if* $u \in H : g(u) \in K(u)$ *satisfies the relation*

$$g(u) = P_{K(u)}[u - \rho T(u)], \tag{3.1}$$

where $P_{K(u)}$ *is the projection operator and* $\rho > 0$ *is a constant.*

Proof. Let $u \in H : g(u) \in K(u)$ be solution of (2.1). Then, from (2.1), we have

$$\langle g(u) - (u - \rho T(u)), v - g(u)\rangle \geqslant 0, \quad \forall v \in K(u),$$

which is equivalent to finding $u \in H : g(u) \in K(u)$ such that

$$g(u) = P_{K(u)}[u - \rho T(u)],$$

using Lemma 2.1, the required result (3.1).

Lemma 3.1 implies that the general quasi-variational inequality (2.1) is equivalent to the fixed-point problem (3.1). This alternative equivalent formulation is very useful from numerical and theoretical points of view. We use this alternative fixed-point formulation to discuss the existence of a solution of the general quasi-variational inequality (2.1) and this is the main motivation of our next result. □

Theorem 3.1. *Let the operators* $T, g : H \to H$ *be both strongly monotone with constants* $\alpha > 0, \sigma > 0$ *and Lipschitz continuous with constants with* $\beta > 0, \delta > 0$, *respectively. If Assumption* 2.1 *holds and there exists a constant* $\rho > 0$ *such that*

$$|\rho - \frac{\alpha}{\beta^2}| < \frac{\sqrt{\alpha^2 - \beta^2 k(2-k)}}{\beta^2}, \quad \alpha > \beta\sqrt{k(2-k)}, \quad k < 1, \tag{3.2}$$

where

$$k = \sqrt{1 - 2\sigma + \delta^2} + v, \tag{3.3}$$

then there exists a solution $u \in K(u)$ *satisfying the general quasi-variational inequality* (2.1).

Proof. Let $u \in K(u)$ be a solution of the general quasi-variational inequality (2.1). Then, using Lemma 3.1, we have

$$g(u) = P_{K(u)}[u - \rho T(u)].$$

Thus we can define the mapping $F(u)$ as:

$$F(u) = u - g(u) + P_{K(u)}[u - \rho T(u)]. \tag{3.4}$$

In order to prove the existence of a solution of (2.1), it is enough to show that the mapping $F(u)$, defined by (3.4), is a contraction mapping.

For $u_1 \neq u_2 \in H$, and using Assumption 2.1, we have

$$\begin{aligned}\|F(u_1) - F(u_2)\| &= \|u_1 - u_2 - (g(u_1) - g(u_2))\| \\ &\quad + \|P_{K(u_1)}[u_1 - \rho T(u_1)] - P_{K(u_2)}[u_2 - \rho T(u_2)]\| \\ &\leqslant \|u_1 - u_2 - (g(u_1) - g(u_2))\| \\ &\quad + \|P_{K(u_1)}[u_1 - \rho T(u_1)] - P_{K(u_2)}[u_1 - \rho T(u_1)]\| \\ &\quad + \|P_{K(u_2)}[u_1 - \rho T(u_1)] - P_{K(u_2)}[u_2 - \rho T(u_2)]\| \\ &= v\|u_1 - u_2\| + \|u_1 - u_2 - (g(u_1) - g(u_2))\| \\ &\quad + \|u_1 - u_2 - \rho(T(u_1) - T(u_2))\|. \end{aligned} \tag{3.5}$$

Since the operator T is strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$, it follows that:

$$\begin{aligned}&\|u_1 - u_2 - \rho(T(u_1) - T(u_2))\|^2 \\ &\quad \leqslant \|u_1 - u_2\|^2 - 2\rho\langle T(u_1) - T(u_2),\ u_1 - u_2\rangle \\ &\qquad + \rho^2\|T(u_1) - T(u_2)\|^2 \\ &\quad \leqslant (1 - 2\rho\alpha + \rho^2\beta^2)\|u_1 - u_2\|^2. \end{aligned} \tag{3.6}$$

In a similar way, we have

$$\|u_1 - u_2 - (g(u_1) - g(u_2))\|^2 \leqslant (1 - 2\sigma + \delta^2)\|u_1 - u_2\|^2, \tag{3.7}$$

where we have used the fact that g is strongly monotone with constant $\sigma > 0$ and Lipschitz continuous with constant $\delta > 0$.

From (3.3), (3.5)–(3.7), we have

$$\begin{aligned}\|F(u_1) - F(u_2)\| &\leqslant \left\{v + \sqrt{(1 - 2\sigma + \delta^2)} \right. \\ &\quad \left. + \sqrt{(1 - 2\rho\alpha + \rho^2\beta^2)}\right\}\|u_1 - u_2\| \\ &= \left\{k + \sqrt{(1 - 2\rho\alpha + \rho^2\beta^2)}\right\}\|u_1 - u_2\| \\ &= \theta\|u_1 - u_2\|, \end{aligned}$$

where

$$\theta = k + \sqrt{(1 - 2\rho\alpha + \rho^2\beta^2)}. \tag{3.8}$$

From (3.2), we see that $\theta < 1$. Thus it follows that the mapping $F(u)$, defined by (3.4), is a contraction mapping and consequently it has a fixed point, which belongs to $K(u)$ satisfying the general quasi-variational inequality (2.1), the required result. □

Remark 3.1. We would like to emphasize that the conditions that ensure the existence of the constant $\rho > 0$, which satisfies (3.2) in Theorem 3.1 have been verified in Noor (1988a, 2000, 2004, 2007, 2008a, 2010a, 2009a,b,c) for the quasi-variational inequalities and their related optimization problems. In several cases, these conditions have been used in the existing results and also in the studies of the convergence criteria of the iterative methods for solving the general quasi-variational inequalities. For the examples of the function g, see Noor (2000, 2009c) and the references therein.

4. Projection iterative method

In this section, we use the fixed-point formulation (3.1) to suggest and analyze the following projection iterative method for solving the general quasi-variational inequality (2.1).

Algorithm 4.1. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative schemes

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n\{u_n - g(u_n) + P_{K(u_n)}[u_n - \rho T(u_n)]\},\quad n = 0,\ 1,\ldots \tag{4.1}$$

which is known as the Mann iteration process for solving the general quasi-variational inequality (2.1).

Note that if $g = I$, then the Algorithm 4.1 reduces to the following iterative method for solving the quasi-variational inequality (2.10) and appears to be a new one.

Algorithm 4.2. For a given $u_0 \in H$, find the approximate solution $u_{n+1}$by the iterative schemes

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n P_{K(u_n)}[u_n - \rho T(u_n)],\quad n = 0,\ 1,\ldots$$

If $K(u) \equiv K$, that is, the convex set $K(u)$ is independent of the solution u, then Algorithm 4.1 reduces to the following algorithm for solving the general variational inequalities (2.2), which was suggested by Noor (2008a).

Algorithm 4.3. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative schemes

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n\{u_n - g(u_n) + P_K[g(u_n) - \rho T(u_n)]\},\quad n = 0,\ 1,\ldots$$

For the convergence analysis of Algorithm 4.3, see Noor (2008a).

We now consider the convergence analysis of Algorithm 4.1 and this is the main motivation of our next result.

Theorem 4.1. *Let the operators $T,\ g : H \to H$ be both strongly monotone with constants $\alpha > 0,\ \sigma > 0$ and Lipschitz continuous with constants with $\beta > 0,\ \delta > 0$, respectively. Let Assumption* 2.1 *hold and θ be as in the proof of Theorem* 3.1. *If (3.1) holds and $0 \leqslant \alpha_n \leqslant 1$, for all $n \geqslant 0$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$, then the approximate solution u_n obtained from Algorithm* 4.1 *converges to a solution $u \in K(u)$ satisfying the general quasi-variational inequality (2.1).*

Proof. From Theorem 3.1, it follows that there does exist a unique solution of the general quasi-variational inequality (2.1). Let $u \in K(u)$ be a solution of the general quasi-variational inequality (2.1). Then, from Lemma 3.1, we have

$$u = (1-\alpha_n)u + \alpha_n\{u - g(u) + P_{K(u)}[u - \rho T(u)]\}, \tag{4.2}$$

where $0 \leqslant \alpha_n \leqslant 1$ is a constant.

From (4.1) and (4.2), we have

$$\begin{aligned}\|u_{n+1} - u\| &= \|(1-\alpha_n)(u_n - u) + \alpha_n\|u_n - u - (g(u_n) \\ &\quad - g(u))\|\alpha_n\{P_{K(u)}[u_n - \rho T(u_n)] - P_{K(u)}[u - \rho T(u)]\}\| \\ &\leqslant (1-\alpha_n)\|u_n - u\| + \alpha_n\|u_n - u - (g(u_n) - g(u))\| \\ &\quad + \alpha_n\|P_{K(u_n)}[u_n - \rho T(u_n)] - P_{K(u_n)}[u - \rho T(u)]\| \\ &\quad + \alpha_n\|P_{K(u_n)}[u - \rho T(u)] - P_{K(u)}[u - \rho T(u)]\| \\ &\leqslant (1-\alpha_n)\|u_n - u\| + \alpha_n\|u_n - u - (g(u_n) - g(u))\| \\ &\quad + \alpha_n v\|u_n - u\| + \alpha_n\|u_n - u - \rho(T(u_n) - T(u))\|.\end{aligned} \tag{4.3}$$

From (3.6), (3.7) and (4.3), we have

$$\begin{aligned}\|u_{n+1} - u\| &\leqslant (1-\alpha_n)\|u_n - u\| + \alpha_n\{v + \sqrt{1 - 2\sigma + \delta^2} \\ &\quad + \sqrt{1 - 2\alpha\rho + \beta^2\rho^2}\}\|u_n - u\| \\ &= (1-\alpha_n)\|u_n - u\| + (k + t(\rho))\|u_n - u\|, \\ &\quad \text{using (3.3).} = (1-\alpha_n)\|u_n - u\| + \theta\|u_n - u\|,\end{aligned}$$

where

$$t(\rho) = \sqrt{1 - 2\alpha\rho + \rho^2\beta^2}. \tag{4.4}$$

From (3.2), it follows that $\theta < 1$.

Thus

$$\begin{aligned}\|u_{n+1} - u\| &\leqslant (1-\alpha_n)\|u_n - u\| + \alpha_n\theta\|u_n - u\| \\ &= [1 - (1-\theta)\alpha_n]\|u_n - u\| \\ &\leqslant \prod_{i=0}^{n}[1 - (1-\theta)\alpha_i]\|u_0 - u\|.\end{aligned}$$

Since $\sum_{n=0}^{\infty}\alpha_n$ diverges and $1 - \theta > 0$, we have $\lim_{n\to\infty}\prod_{i=0}^{n}[1-(1-\theta)\alpha_i] = 0$. Consequently the sequence $\{u_n\}$ convergences strongly to u. This completes the proof. □

5. Wiener–Hopf equations

In this section, we introduce a new class of Wiener–Hopf equations (normal maps), which is called the general implicit Wiener–Hopf equation. Using Lemma 3.1, we establish the equivalence between the Wiener–Hopf equations and the general quasi-variational inequalities. This alternative equivalent formulation is used to suggest and analyze some iterative methods for solving the general quasi-variational inequality (2.1).

To be more precise, let $Q_{K(u)} = I - g^{-1}P_{K(u)}$, where I is the identity operator and g^{-1} exists. For the given nonlinear operators $T,\ g$, we consider the problem of finding $z \in H,\ u \in K(u)$ such that

$$T(g^{-1}P_{K(u)}z) + \rho^{-1}Q_{K(u)}z = 0, \tag{5.1}$$

which is called the general implicit Wiener–Hopf equation. We note that, if $K(u) \equiv K$, then the general Wiener–Hopf equations (5.1) are due to Noor (2010a). If $g = I$ and $K(u) \equiv K$, then one can obtain the original Wiener–Hopf equations, which were introduced and studied by Robinson (1992) and Shi (1991) using quite different techniques. In this case, problem (5.1) is equivalent to finding $z \in H$ such that

$$T(P_K z) + \rho^{-1}Q_K z = 0,\quad \rho > 0.$$

It has been shown that the Wiener–Hopf equations have played an important and significant role in developing several numerical techniques for solving variational inequalities and related optimization problems, see Borwein and Lewis (2006); Noor (1975, 1985, 1988a, 1993, 1997a, 1998, 1999, 2000, 2004, 2007, 2008a, 2010a,2009a,b,c, 2010b, 2008b), Noor et al. (1993, 2010), Robinson (1992), Shi (1991) and the references therein.

Lemma 5.1. *The solution $u \in H : g(u) \in K(u)$ satisfies the general quasi-variational inequality (2.1) if and only if $z \in H, u \in K(u)$ is a solution of the general Wiener–Hopf equation (5.1), where*

$$g(u) = P_{K(u)}z \tag{5.2}$$

$$z = u - \rho T(u), \quad \rho > 0, \quad \text{a constant}. \tag{5.3}$$

Proof. Let $u \in H : g(u) \in K(u)$ be a solution of (2.1). Then, from Lemma 3.1, we have

$$g(u) = P_{K(u)}[u - \rho T(u)]. \tag{5.4}$$

Let

$$z = u - \rho T(u).$$

Then

$$g(u) = P_{K(u)}z. \tag{5.5}$$

Combining (5.3)–(5.5), we have

$$z = u - \rho T(u) = g^{-1}P_{K(u)}z - \rho T(g^{-1}P_{K(u)}z),$$

from which it follows that $z \in H$ is a solution of the general Wiener–Hopf equation (5.1), the required result. □

Lemma 5.1 implies that the general quasi-variational inequality (2.1) and the general implicit Wiener–Hopf equation (5.1) are equivalent. We use this equivalent formulation to suggest a number of iterative methods for solving the general quasi-variational inequalities.

(I) Using (5.2), the Wiener–Hopf equations (5.1) can be rewritten as:

$$Q_K(u)z = -\rho T(g^{-1}P_{K(u)}z),$$

which implies that

$$z = g^{-1}P_{K(u)}z - \rho T(g^{-1}P_{K(u)}z) = u - \rho T(u),$$

This fixed-point formulation enables to suggest the following iterative method for solving problem (2.1).

Algorithm 5.1. For a given $z_0 \in H$, compute the approximate solution z_{n+1} by the iterative schemes

$$g(u_n) = P_{K(u_n)}z_n, \tag{5.6}$$

$$z_{n+1} = (1 - \alpha_n)z_n + \alpha_n\{u_n - \rho T(u_n)\}, \quad n = 0, 1, \ldots, \tag{5.7}$$

where $0 \leqslant \alpha_n \leqslant 1$, for all $n \geqslant 0$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$.

(II) By an appropriate and suitable rearrangement of the terms and using (5.12), the Wiener–Hopf equations (5.1) can be written as:

$$z = g^{-1}P_{K(u)}z - \rho T(g^{-1}P_{K(u)}z) + (1 - \rho^{-1})Q_{K(u)}z$$
$$= u - \rho T(u) + (1 - \rho^{-1})Q_{K(u)}z,$$

which is another fixed-point formulation. Using this fixed-point formulation, we can suggest the following iterative method.

Algorithm 5.2. For a given $z_0 \in H$, compute the approximate solution z_{n+1} by the iterative schemes

$$g(u_n) = P_{K(u_n)}z_n$$
$$z_{n+1} = u_n - \rho T(u_n) + (1 - \rho^{-1})Q_{K(u_n)}z_n, \quad n = 0, 1, \ldots.$$

(III) If T is linear and T^{-1} exists, then the Wiener–Hopf equations (5.1) can be written as:

$$z = \left(I - \rho^{-1}gT^{-1}\right)Q_{K(u)}z.$$

This fixed-point formulation allows us to suggest the following iterative method for solving the general quasi-variational inequality (2.1).

Algorithm 5.3. For a given $z_0 \in H$, compute the approximate solution z_{n+1} by the iterative schemes

$$z_{n+1} = \left(I - \rho^{-1}gT^{-1}\right)Q_{K(u_n)}z_n, \quad n = 0, 1, \ldots.$$

If $K(u) \equiv K$, then Algorithms 5.1, 5.2, 5.3 reduce to the following iterative methods for solving the general variational inequality (2.2), which are due to Noor (2008a).

Algorithm 5.4. For a given $z_0 \in H$, compute the approximate solution z_{n+1} by the iterative schemes

$$g(u_n) = P_K z_n,$$
$$z_{n+1} = (1 - \alpha_n)z_n + \alpha_n\{u_n - \rho T(u_n)\}, \quad n = 0, 1, \ldots,$$

where $0 \leqslant \alpha_n \leqslant 1$, for all $n \geqslant 0$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$.

Algorithm 5.5. For a given $z_0 \in H$, compute the approximate solution z_{n+1} by the iterative schemes

$$g(u_n) = P_K z_n,$$
$$z_{n+1} = u_n - \rho T(u_n) + (1 - \rho^{-1})Q_K z_n, \quad n = 0, 1, \; n = 0, 1, \ldots,$$

where $0 \leqslant \alpha_n \leqslant 1$, for all $n \geqslant 0$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$.

Algorithm 5.6. For a given $z_0 \in H$, compute the approximate solution z_{n+1} by the iterative schemes

$$z_{n+1} = \left(I - \rho^{-1}gT^{-1}\right)Q_K z_n, \quad n = 0, 1, \ldots.$$

For $g = I$, the identity operator, Algorithms 5.1, 5.1, 5.3 are due to Noor (1997a,b, 1998) for solving the quasi-variational inequality(2.4). In brief, for appropriate and suitable rearrangements of the terms of the general Wiener–Hopf equations (5.1), one can suggest and analyze a number of iterative methods for solving the general quasi-variational inequality (2.1) and the related optimization problems. For the investigation of such type of projection iterative methods and the verification of their numerical efficiency, further research efforts are needed.

We now consider the convergence analysis of Algorithm 5.1. In a similar way, one can study the convergence analysis of Algorithms 5.2 and 5.3.

Theorem 5.1. *Let the operators T, g satisfy all the assumptions of Theorem 3.1. If the condition (3.2) holds, then the approximate solution $\{z_n\}$ obtained from Algorithm 5.1 converges to a solution $z \in H$ satisfying the Wiener–Hopf equation (5.1) strongly.*

Proof. Let $u \in H : g(u) \in K(u)$ be a solution of (2.1). Then, using Lemma 5.1, we have

$$g(u) = P_{K(u)}z \tag{5.8}$$

$$z = (1 - \alpha_n)z + \alpha_n\{u - \rho T(u)\}, \tag{5.9}$$

where $0 \leqslant \alpha_n \leqslant 1$, and $\sum_{n=0}^{\infty}\alpha_n = \infty$.

From (5.7), (5.9), (3.6) and (3.7), we have

$$\|z_{n+1} - z\| \leqslant (1-\alpha_n)\|z_n - z\| + \alpha_n\|u_n - u - \rho(T(u_n) - T(u))\| \leqslant (1-\alpha_n)\|z_n - z\| + \alpha_n\sqrt{1-2\rho\alpha+\beta^2}\|u_n - u\|. \tag{5.10}$$

Also, from (5.6), (5.8) and Assumption 2.1, we have

$$\begin{aligned}\|u_n - u\| &\leqslant \|u_n - u - (g(u_n) - g(u))\| + \|P_{K(u_n)}z_n - P_{K(u)}z\| \\ &\leqslant \|u_n - u - (g(u_n) - g(u))\| + \|P_{K(u_n)}z_n - P_{K(u_n)}z\| \\ &\quad + \|P_{K(u_n)}z - P_{K(u)}z\| \\ &\leqslant \{v + \sqrt{1-2\delta+\sigma^2}\}\|u_n - u\| + \|z_n - z\| \\ &= k\|u_n - u\| + \|z_n - z\|,\end{aligned}$$

from which it follows that:

$$\|u_n - u\| \leqslant \frac{1}{1-k}\|z_n - z\|. \tag{5.11}$$

Combining (5.10) and (5.11), we have

$$\|z_{n+1} - z\| \leqslant (1-\alpha_n)\|z_n - z\| + \alpha_n\theta_1\|z_n - z\|, \tag{5.12}$$

where

$$\theta_1 = \left\{\frac{\sqrt{1-2\alpha\rho+\beta^2\rho^2}}{1-k}\right\}. \tag{5.13}$$

From (3.2), we see that $\theta_1 < 1$ and consequently

$$\begin{aligned}\|z_{n+1} - z\| &\leqslant (1-\alpha_n)\|z_n - z\| + \alpha_n\theta_1\|z_n - z\| \\ &= [1-(1-\theta_1)\alpha_n]\|z_n - z\| \\ &\leqslant \prod_{i=0}^{n}[1-(1-\theta_1)\alpha_i]\|z_0 - z\|.\end{aligned}$$

Since $\sum_{n=0}^{\infty}\alpha_n$ diverges and $1-\theta_1 > 0$, we have $\lim_{n\to\infty}\prod_{i=0}^{n}[1-(1-\theta_1)\alpha_i] = 0$. Consequently the sequence $\{z_n\}$ convergences strongly to z in H, the required result. □

6. Conclusion

In this paper, we have introduced and considered a new class of quasi-variational inequalities involving two operators, which is called the general quasi-variational inequality. We have shown that the general quasi-variational inequalities are equivalent to the fixed point and Wiener–Hopf equations. These equivalent formulations have been used to suggest and analyse several iterative methods for solving the quasi-variational inequalities. We have also considered the convergence analysis of these new iterative methods under suitable conditions. We expect that the ideas and techniques of this paper will motivate and inspire interested readers to explore its applications in various fields of pure and applied mathematical sciences.

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