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A new one-parameter discrete exponential distribution: Properties, inference, and applications to COVID-19 data

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ABSTRACT

A new one-parameter discrete length-biased exponential distribution called the discrete moment exponential (DMEx) distribution is introduced using the survival discretizing approach. We derive the reliability measures including survival function, hazard function, residual reliability function, and the second rate of failure function. Further, the mathematical properties of the DMEx distribution are derived. The parameters of the DMEx distribution are estimated using seven estimation methods. A simulation study is carried out to explore the behavior of the proposed estimators. It is observed that the maximum likelihood approach provides efficient estimates. Finally, the DMEx is adopted for fitting the number of COVID-19 deaths in China and Europe countries. It is shown that the DMEx distribution fits the data better than other competing discrete distributions.

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1. Introduction

In the last decades, several discrete distributions have been introduced as analogs of the continuous distributions to have a better alternative model for modeling count data sets having complicated behavior. Although classical models such as binomial, Poisson, Geometric, and negative binomial distributions are used to model count data sets, in some situations, these probability models do not provide the best fit. Hence there is a need to develop more flexible distributions.

Several discretization methods have been adopted extensively in the literature to define new discrete models. Chakraborty (Chakraborty, 2015) conducted a survey on several discretization methods available to derive discrete analogs of continuous distributions, such as discretization methods based on survival function (SF), probability density function (PDF), cumulative distribution function (CDF), hazard rate function (HRF), reversed-HRF, the difference equation analog of Persian differential equation and a two-stage composite method.

Let a random variable (rv) X follows a continuous probability distribution with SF $S(x)$. Using the SF discretization method which is introduced by Kemp (2004), the probability mass function (PMF) of a discrete rv is specified by.

$$P(Y = k) = S(k) - S(k+1), \quad k = 0, 1, 2, 3, \tag{1}$$

Using this method of discretization several discrete distribution have been introduced, such as the discrete Weibull (Nakagawa and Osaki, 1975), discrete Rayleigh (Roy, 2004), discrete half-normal (Kemp, 2008), discrete Burr and discrete Pareto (Krishna and Singh Pundir, 2009), discrete inverse-Weibull (Jazi et al., 2010), generalized geometric (Gómez-Déniz, 2010), discrete Lindley (Gomez-Deniz and Calderin-Ojeda, 2011), generalized exponential type II (Nekoukhou et al., 2013), two-parameter discrete Lindley (Hussain et al., 2016), new discrete extended-Weibull (Jia et al.,

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2019), natural discrete-Lindley (Al-Babtain et al., 2020); discrete MO-Weibull (Opone et al., 2020), uniform Poisson–Ailamujia (Aljohani et al., 2021), discrete inverted Topp-Leone (Eldeeb et al., 2021), discrete power-Ailamujia (Alghamdi et al., 2022), Binomial-exponential 2 (Bakouch et al., 2017), Poisson-exponential (Rodrigues et al., 2018), and discrete Ramos-Louzada (Ramos and Louzada, 2019; Afify et al., 2021).

Dara and Ahmad (2012) introduced the moment-exponential (MEx) distribution and showed that it is more flexible than the exponential distribution. The PDF of MEx distribution is specified by.

$$f\ (x) = \frac{x}{\beta^2}\exp\left(-\frac{x}{\beta}\right), \quad x > 0, \quad \beta > 0 \tag{2}$$

The corresponding SF has the form

$$S\ (x) = \left(1 + \frac{x}{\beta}\right)\exp\left(-\frac{x}{\beta}\right) \tag{3}$$

Using the discretization method based on the SF, a discrete analog of continuous MEx distribution is introduced in this paper.

In this article, we propose the asymmetric discrete distribution called the discrete moment exponential (DMEx) distribution using the discretization method based on SF. The DMEx distribution is a competitor to some well-known discrete models such as the discrete Burr, discrete Burr-Hatke, discrete Rayleigh, discrete inverted Topp-Leon, discrete Pareto, discrete inverse Rayleigh, and Poisson distributions. The parameter β of the DMEx model is estimated using seven classical estimation approaches. We present the detailed simulation results to address the behavior of the estimators.

The rest of this paper is outlined six sections. In Section 2, we define the new DMEx distribution and provide some of its basic properties. The estimation approaches are presented in Section 3. The efficiency of the introduced estimators is assessed via simulation results in Section 4. Section 5 provides two real applications to COVID-19 data of the DMEx distribution. Section 6 presents some final conclusions.

2. The DMEx distribution and its properties

The DMEx distribution is obtained using Eqs. (1) and (3). The PMF of the DMEx distribution has the form (for $x = 0, 1, 2, \ldots$)

$$p(x;\beta) = \exp\left(-\frac{x}{\beta}\right)\left[\left(1 + \frac{x}{\beta}\right) - \left(1 + \frac{1+x}{\beta}\right)\exp\left(-\frac{1}{\beta}\right)\right], \quad \beta > 0 \tag{4}$$

The corresponding CDF of the DMEx model reduces to

$$F(x;\beta) = 1 - \left(1 + \frac{1+x}{\beta}\right)\exp\left(-\frac{1+x}{\beta}\right), \quad x = 0, 1, 2, \ldots, \quad \beta > 0 \tag{5}$$

Fig. 1 shows the PMF behavior of the DMEx distribution for different values of β. It is observed that the DMEs PMF has unimodal behavior and positively skewed for $\beta \geqslant 1$. The mode of the DMEx distribution moves towards the left for large values of β.

2.1. Survival, hazard rate, and quantile functions

The SF of the DMEx distribution reduces to

$$S(x) = \left(1 + \frac{1+x}{\beta}\right)\exp\left(-\frac{1+x}{\beta}\right).$$

The HRF of the DMEx distribution takes the form

$$h(x) = \frac{\left(1 + \frac{x}{\beta}\right) - \left(1 + \frac{1+x}{\beta}\right)\exp\left(-\frac{1}{\beta}\right)}{\left(1 + \frac{1+x}{\beta}\right)\exp\left(-\frac{1}{\beta}\right)}.$$

Fig. 2 shows that the behavior of the HRF of the DMEx model is increasing for different values ofβ.

The second rate of failure, say$\varphi(x)$, of the DMEs model is defined as.

$$\varphi(x) = \log\left[\frac{S(x)}{S(x+1)}\right] = \log\left[\frac{\left(1 + \frac{1+x}{\beta}\right)\exp\left(\frac{1}{\beta}\right)}{\left(1 + \frac{2+x}{\beta}\right)}\right].$$

The reversed HRF of the DMEx distribution is

$$r^*(x) = \frac{p(x)}{F(x)} = \frac{\exp\left(-\frac{x}{\beta}\right)\left[\left(1 + \frac{x}{\beta}\right) - \left(1 + \frac{1+x}{\beta}\right)\exp\left(-\frac{1}{\beta}\right)\right]}{1 - \left(1 + \frac{1+x}{\beta}\right)\exp\left(-\frac{1+x}{\beta}\right)}.$$

The quantile function (QF) of the DMEx model takes the form

$$Q(p) = -1 - \beta - \beta\, W\left(\frac{(p-1)}{\exp(1)}\right), \ 0 < p < 1,$$

where x is the integer part of x and $W\ (.)$ is the negative branch of the Lambert W function.

2.2. Probability generating function and moments

The probability generating function of the DMEx distribution follows as

$$G_x(Z) = \sum_{x=0}^{\infty} Z^x p(X) = 1 + (Z-1)\sum_{x=1}^{\infty} Z^{x-1}\left(1 + \frac{x}{\beta}\right)\exp\left(-\frac{x}{\beta}\right).$$

Differentiating $G_x(Z)$ with respect to Z and setting $Z = 1$, we can obtain the first four factorial moments of the DMEx distribution as follows

$$G'x\ (1) = \sum_{x=1}^{\infty}\left(1 + \frac{x}{\beta}\right)\exp\left(-\frac{x}{\beta}\right).$$

$$G_x''(1) = 2\sum_{x=1}^{\infty}(x-1)\left(1 + \frac{x}{\beta}\right)\exp\left(-\frac{x}{\beta}\right).$$

$$G_x'''(1) = 3\sum_{x=1}^{\infty}(x-1)(x-2)\left(1 + \frac{x}{\beta}\right)\exp\left(-\frac{x}{\beta}\right).$$

$$G_x''''(1) = 4\sum_{x=1}^{\infty}(x-1)(x-2)(x-3)\left(1 + \frac{x}{\beta}\right)\exp\left(-\frac{x}{\beta}\right).$$

The first four ordinary moments of the DMEx distribution can be calculated using the factorial moments as follows

$$\mu = G'x\ (1) = \sum_{x=1}^{\infty}\left(1 + \frac{x}{\beta}\right)\exp\left(-\frac{x}{\beta}\right).$$

$$\mu_2' = G''x\ (1) + G'x\ (1) = \sum_{x=1}^{\infty}(2x-1)\left(1 + \frac{x}{\beta}\right)\exp\left(-\frac{x}{\beta}\right).$$

$$\mu_3' = G'''x\ (1) + 3\,G'x\ (1) + G'x\ (1)\sum_{x=1}^{\infty}(3x^2 - 3x + 1)\left(1 + \frac{x}{\beta}\right)\exp\left(-\frac{x}{\beta}\right).$$

$$\mu_4' = G''''x\ (1) + 6\,G'''x\ (1) + 7\,G''x\ (1) + G'x\ (1) = \sum_{x=1}^{\infty}(4x^3 - 6x^2 + 4x - 1)\left(1 + \frac{x}{\beta}\right)\exp\left(-\frac{x}{\beta}\right).$$

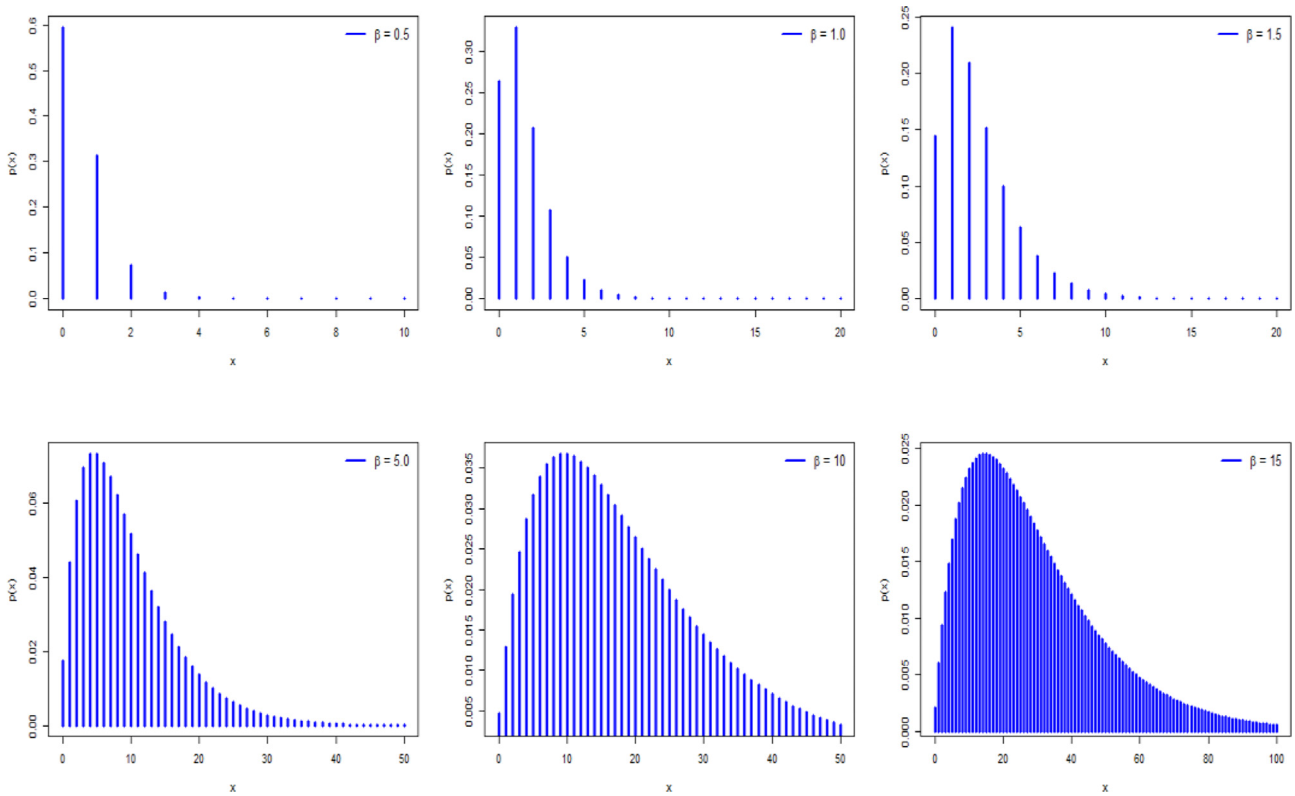


Fig. 1. The PMF plots of the DMEx distribution.

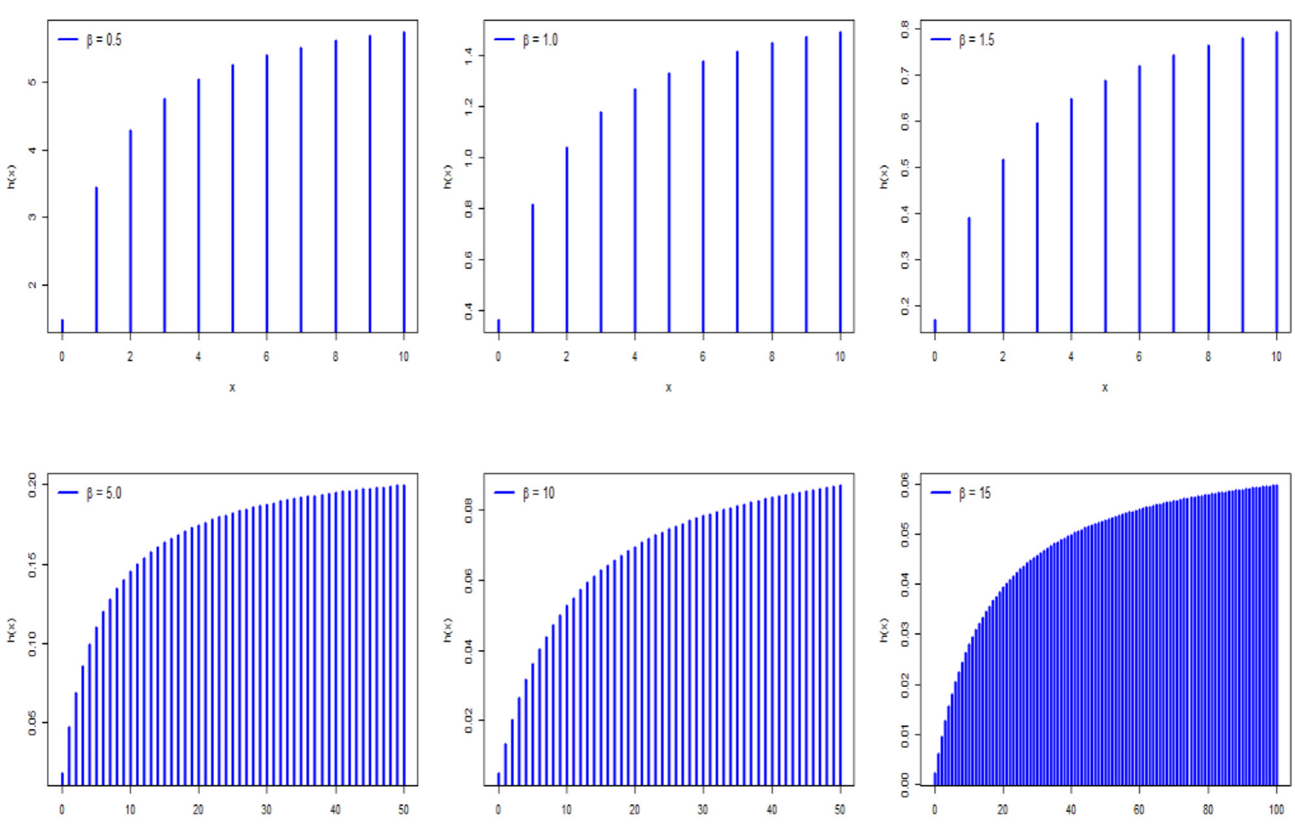


Fig. 2. The HRF plots of the DMEx distribution.

The variance of the DMEx distribution has the form

$$\sigma^2 = \sum_{x=1}^{\infty}(2x-1)\left(1+\frac{x}{\beta}\right)\exp\left(-\frac{x}{\beta}\right) - \left[\sum_{x=1}^{\infty}\left(1+\frac{x}{\beta}\right)\exp\left(-\frac{x}{\beta}\right)\right]^2.$$

Further, the coefficient of skewness and kurtosis of the DMEx distribution can be calculated by the formulae

$$CS = \frac{\mu_3' - 3\mu_2'\mu + 2\mu^3}{(\sigma^2)^{\frac{3}{2}}}$$

and

$$CK = \frac{\mu_4' - 4\mu_3'\mu + 6\mu_2'\mu^2 - 3\mu^4}{(\sigma^2)^2}.$$

2.3. Dispersion index

The index of dispersion (DI) is defined as

$$DI = \frac{Variance}{Mean}.$$

The DI shows that whether a distribution is suitable for modeling under-dispersed data or over-dispersed data. The distribution is over-dispersed for $DI > 1$ and for $DI < 1$ it is under-dispersed. Table 1 reports some numerical values for the descriptive measures of the DMEx distribution. Table 1 shows that the DMEs model is useful for both over-dispersed and under-dispersed data sets. The DMEx also exhibits positively skewed which supported by the plots of its PMF in Fig. 1.

3. Parameter estimation

In this section, we estimate the parameter β of the DMEx distribution using seven methods of estimation.

3.1. Maximum likelihood (ML)

Suppose $x_1, \ldots, x_n$ be a random sample of size n from the DMEx distribution. Then, the log-likelihood function reduces to

$$\log L = -\frac{1}{\beta}\sum_{i=1}^{n}x_i + \sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\beta}\right) - \left(1+\frac{x_i+1}{\beta}\right)\exp\left(-\frac{1}{\beta}\right)\right].$$

The ML estimator (MLE) of β follows by solving $\frac{\partial L}{\partial \beta} = 0$, that is

$$\frac{dL}{d\beta} = \frac{1}{\beta^2}\sum_{i=1}^{n}x_i + \sum_{i=1}^{n}\frac{\left(-\frac{x_i}{\beta^2}\right) - \frac{1}{\beta^2}\left(1+\frac{x_i+1}{\beta}\right)\exp\left(-\frac{1}{\beta}\right) + \left(\frac{x_i+1}{\beta^2}\right)\exp\left(-\frac{1}{\beta}\right)}{\left(1+\frac{x_i}{\beta}\right) - \left(1+\frac{x_i+1}{\beta}\right)\exp\left(-\frac{1}{\beta}\right)} = 0.$$

The MLE of β cannot be calculated explicitly. Hence, numerical methods can be adopted to obtain it.

3.2. Methods of Anderson-Darling and right-tail Anderson-Darling

Let $X_{i:n}$ be the i th order statistic in a sample of size$n$. The Anderson-Darling (AD) test is proposed by (Anderson and Darling, 1952). Let $\hat{\beta}$ be the AD estimator (ADE) which is obtained by minimizing the following equation

$$A(\beta) = -\sum_{i=1}^{n}\frac{2_i-1}{n}\left[\log\left(1-\left(1+\frac{1+x_{i:n}}{\beta}\right)e^{-\left(\frac{1+x_{i:n}}{\beta}\right)}\right) + \log\left(1-\left(1+\frac{1+x_{n+1-i:n}}{\beta}\right)e^{-\left(\frac{1+x_{n+1-i:n}}{\beta}\right)}\right)\right] - n.$$

This estimator of β can also be derived by solving

$$\sum_{i=1}^{n}2(i-1)\left[\frac{\varphi(x_{i:n}|\beta)}{F(x_{i:n}|\beta)} - \frac{\varphi(x_{n+1-i:n}|\beta)}{\bar{F}(x_{n+1-i:n}|\beta)}\right] = 0,$$

where $\varphi(x_{i:n}|\beta) = \frac{d}{d\beta}F(x_{i:n}|\beta)$ and it reduces to

$$\varphi(x_{i:n}|\beta) = -\frac{(1+x_{i:n})^2}{\beta^3}e^{-\left(\frac{1+x_{i:n}}{\beta}\right)} \tag{6}$$

The right-tail Anderson-Darling (RADE) of the parameter β follows by minimizing

$$RT(\beta) = \frac{n}{2} - 2\sum_{i=1}^{n}\left[1-\left(1+\frac{1+x_{i:n}}{\beta}\right)e^{-\left(\frac{1+x_{i:n}}{\beta}\right)}\right] - \frac{1}{n} \times \sum_{i=1}^{n}2(i-1)\log\left[\left(1+\frac{1+x_{n+1-i:n}}{\beta}\right)e^{-\left(\frac{1+x_{n+1-i:n}}{\beta}\right)}\right]$$

The RADE is also calculated by solving

$$-2\sum_{i=1}^{n}\phi(x_{i:n}|\beta) + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\varphi(x_{n+1-1:n}|\beta)}{1-F(x_{n+1-i:n}|\beta)} = 0$$

3.3. Method of Cramèr-von-Misses

Macdonald (Macdonald, 1971) proved that the Cramèr-von-Mises estimator (CVME) has a smaller bias as compared to other minimum distance type estimators. The CVME of β follows by minimizing

$$C(\beta) = \sum_{i=1}^{n}\left[1-\frac{2_i-1}{2\,n} - \left(1+\frac{1+x_{i:n}}{\beta}\right)e^{-\left(\frac{1+x_{i:n}}{\beta}\right)}\right]^2 + \frac{1}{12\,n}.$$

The CVME of β is also obtained by solving

$$\sum_{i=1}^{n}\left[1-\frac{2i-1}{2\,n} - \left(1+\frac{1+x_{i:n}}{\beta}\right)e^{-\left(\frac{1+x_{i:n}}{\beta}\right)}\right]\varphi(x_{i:n}|\beta) = 0$$

Table 1
The values of descriptive measures for the DMEx distribution.

Parameter	Descriptive Measures				
β	Mean	Variance	Skewness	Kurtosis	DI
0.2	0.0409	0.0403	4.8256	25.763	0.9839
0.5	0.5185	0.5253	1.5011	11.266	1.0130
0.8	1.1050	1.3364	1.3678	18.932	1.2093
1.0	1.5027	2.0653	1.3668	23.323	1.3745
1.5	2.5008	4.5750	1.3838	31.074	1.8294
2.0	3.5003	8.0786	1.3950	35.852	2.3079
2.5	4.5002	12.580	1.4013	39.022	2.7955
3.0	5.5001	18.081	1.4049	41.263	3.2874
4.0	7.5000	32.082	1.4089	44.209	4.2776
5.0	9.5000	50.083	1.4108	46.052	5.2718
9.0	17.500	162.08	1.4131	49.473	9.2618

3.4. Methods of least-squares and weighted least-squares

The least-squares estimator (LSE) of β follows by minimizing

$$LS(\beta)=\sum_{i=1}^{n}\left[1-\frac{i}{1+n}-\left(1+\frac{1+x_{i:n}}{\beta}\right)e^{-\left(\frac{1+x_{i:n}}{\beta}\right)}\right]^2,$$

with respect toβ. Moreover, the LSE of β is also obtained by solving

$$\sum_{i=1}^{m}\left[1-\frac{i}{1+n}-\left(1+\frac{1+x_{i:n}}{\beta}\right)e^{-\left(\frac{1+x_{i:n}}{\beta}\right)}\right]\varphi(x_{i:n}|\beta)=0,$$

The weighted least-squares estimator (WLSE) of β follows by minimizing

$$W(\beta)=\sum_{i=1}^{n}1\frac{(1+n)^2(2+n)}{i(n-i+1)}\left[1-\frac{i}{1+n}-\left(1+\frac{1+x_{i:n}}{\beta}\right)e^{-\left(\frac{1+x_{i:n}}{\beta}\right)}\right]^2$$

The WLSE of β can also be obtained by solving

$$\sum_{i=1}^{n}1\frac{(1+n)^2(2+n)}{i(n-i+1)}\left[-\frac{i}{1+n}-\left(1+\frac{1+x_{i:n}}{\beta}\right)e^{-\left(\frac{1+x_{i:n}}{\beta}\right)}\right]\varphi(x_{i:n}|\beta)=0,$$

In which $\varphi(x_{i:n}|\beta)$ is presented in (6).

3.5. Methods of percentiles

Suppose that $u_i=i/(m+1)$ is an unbiased estimator of$F(x_{i:n}|\beta)$. Hence, the percentiles estimator (PCE) (Kao, 1959) of the parameter β can be derived by minimizing the following equation

$$P(\theta)=\sum_{i=1}^{m}\left\{x_{i:n}+1+\beta+\beta\,W\left(\frac{(u_i-1)}{\exp(1)}\right)\right\}^2,$$

where W (.) is the negative branch of the Lambert W function.

4. Simulation study

Here we conduct a Monte Carlo simulation study to compare the performance of different methods of estimation. The performance of different estimators is evaluated based on mean square errors (MSEs), mean relative errors (MREs) and average absolute biases (ABBs) which are given by.

$$MSEs\left(\hat{\beta}\right)=\frac{1}{N}\sum_{i=1}^{N}\left(\hat{\beta}_i-\beta\right)^2,$$

$$MREs\left(\hat{\beta}\right)=\frac{1}{N}\sum_{i=1}^{N}\left|\hat{\beta}_i-\beta\right|/\beta$$

and

$$MSEs\left(\hat{\beta}\right)=\frac{1}{N}\sum_{i=1}^{N}\left|\hat{\beta}_i-\beta\right|,$$

Table 2
Simulation results of the DMEx parameter for $\beta=0.25$ and $\beta=0.5$.

n	Measure	MLE	LSE	WLE	PCE	CVME	ADE	RADE
$\beta=0.25$								
10	AABs	0.0588	0.3887	0.4145	0.2315	0.3888	0.3826	0.3710
25		0.0169	0.3879	0.4427	0.2022	0.3878	0.3792	0.3678
50		0.0032	0.3874	0.4650	0.1888	0.3874	0.3797	0.3665
100		0.0012	0.3878	0.4838	0.1808	0.3877	0.3791	0.3662
150		0.0007	0.3870	0.4945	0.1777	0.3876	0.3792	0.3656
200		0.0009	0.3872	0.5022	0.1752	0.3875	0.3790	0.3656
10	MREs	0.2354	1.5546	1.6581	0.9259	1.5551	1.5305	1.4840
25		0.0677	1.5518	1.7708	0.8087	1.5511	1.5168	1.4712
50		0.0128	1.5497	1.8601	0.7551	1.5497	1.5189	1.4659
100		0.0047	1.5511	1.9350	0.7232	1.5507	1.5166	1.4649
150		0.0029	1.5481	1.9780	0.7106	1.5503	1.5166	1.4624
200		0.0037	1.5488	2.0088	0.7008	1.5501	1.5159	1.4624
10	MSEs	0.0198	0.1530	0.1756	0.0586	0.1531	0.1480	0.1386
25		0.0058	0.1512	0.1980	0.0430	0.1511	0.1443	0.1356
50		0.0014	0.1505	0.2172	0.0366	0.1505	0.1445	0.1344
100		0.0006	0.1506	0.2345	0.0332	0.1505	0.1439	0.1342
150		0.0004	0.1499	0.2449	0.0319	0.1503	0.1439	0.1337
200		0.0003	0.1500	0.2526	0.0309	0.1503	0.1437	0.1337
$\beta=0.5$								
10	AABs	0.0051	0.3368	0.3715	0.2471	0.3371	0.3290	0.2983
25		0.0004	0.3306	0.3995	0.2117	0.3313	0.3224	0.2888
50		0.0011	0.3311	0.4241	0.1940	0.3294	0.3197	0.2864
100		0.0010	0.3290	0.4484	0.1835	0.3295	0.3189	0.2848
150		0.0001	0.3289	0.4633	0.1783	0.3289	0.3180	0.2841
200		0.0000	0.3285	0.4745	0.1758	0.3290	0.3181	0.2834
10	MREs	0.0103	0.6735	0.7429	0.4941	0.6741	0.6579	0.5966
25		0.0007	0.6611	0.7989	0.4234	0.6627	0.6448	0.5777
50		0.0022	0.6622	0.8482	0.3881	0.6587	0.6395	0.5729
100		0.0019	0.6580	0.8967	0.3670	0.6590	0.6377	0.5696
150		0.0002	0.6578	0.9266	0.3567	0.6578	0.6361	0.5682
200		0.0001	0.6570	0.9490	0.3517	0.6580	0.6362	0.5667
10	MSEs	0.0161	0.1271	0.1492	0.0795	0.1268	0.1222	0.1014
25		0.0058	0.1145	0.1639	0.0515	0.1149	0.1094	0.0885
50		0.0030	0.1123	0.1821	0.0409	0.1111	0.1048	0.0844
100		0.0015	0.1095	0.2022	0.0353	0.1098	0.1030	0.0823
150		0.0010	0.1090	0.2155	0.0329	0.1090	0.1020	0.0815
200		0.0007	0.1086	0.2258	0.0317	0.1089	0.1019	0.0809

Table 3
Simulation results of the DMEx parameter $\beta=1.0$ and $\beta=2.0$.

n	Measure	MLE	LSE	WLE	PCE	CVME	ADE	RADE
$\beta=1.0$								
10	AABs	0.0088	0.3362	0.3488	0.2921	0.3359	0.3348	0.2916
25		0.0011	0.3259	0.3496	0.2346	0.3230	0.3182	0.2750
50		0.0016	0.3197	0.3672	0.2140	0.3159	0.3099	0.2673
100		0.0009	0.3171	0.3847	0.1951	0.3185	0.3116	0.2643
150		0.0008	0.3154	0.3986	0.1884	0.3145	0.3074	0.2645
200		0.0001	0.3152	0.4066	0.1848	0.3146	0.3074	0.2620
10	MREs	0.0088	0.3362	0.3488	0.2921	0.3359	0.3348	0.2916
25		0.0011	0.3259	0.3496	0.2346	0.3230	0.3182	0.2750
50		0.0016	0.3197	0.3672	0.2140	0.3159	0.3099	0.2673
100		0.0009	0.3171	0.3847	0.1951	0.3185	0.3116	0.2643
150		0.0008	0.3154	0.3986	0.1884	0.3145	0.3074	0.2645
200		0.0001	0.3152	0.4066	0.1848	0.3146	0.3074	0.2620
10	MSEs	0.0502	0.1755	0.1768	0.1498	0.1740	0.1708	0.1388
25		0.0201	0.1305	0.1429	0.0799	0.1274	0.1233	0.0959
50		0.0105	0.1143	0.1442	0.0583	0.1116	0.1074	0.0819
100		0.0052	0.1065	0.1521	0.0439	0.1073	0.1028	0.0751
150		0.0035	0.1034	0.1617	0.0395	0.1028	0.0983	0.0734
200		0.0026	0.1023	0.1674	0.0372	0.1021	0.0975	0.0711
$\beta=2.0$								
10	AABs	0.0038	0.3618	0.3741	0.3855	0.3511	0.3566	0.3139
25		0.0047	0.3243	0.3399	0.2932	0.3221	0.3216	0.2730
50		0.0011	0.3231	0.3428	0.2427	0.3222	0.3187	0.2674
100		0.0002	0.3194	0.3410	0.2201	0.3149	0.3103	0.2583
150		0.0002	0.3167	0.3449	0.2017	0.3136	0.3092	0.2558
200		0.0019	0.3115	0.3472	0.1960	0.3129	0.3081	0.2524
10	MREs	0.0019	0.1809	0.1870	0.1928	0.1756	0.1783	0.1570
25		0.0023	0.1621	0.1700	0.1466	0.1611	0.1608	0.1365
50		0.0005	0.1616	0.1714	0.1213	0.1611	0.1594	0.1337
100		0.0001	0.1597	0.1705	0.1100	0.1575	0.1552	0.1292
150		0.0001	0.1583	0.1724	0.1009	0.1568	0.1546	0.1279
200		0.0010	0.1557	0.1736	0.0980	0.1565	0.1541	0.1262
10	MSEs	0.1982	0.3722	0.3850	0.4017	0.3602	0.3516	0.3187
25		0.0782	0.1980	0.2090	0.1805	0.1976	0.1920	0.1568
50		0.0411	0.1524	0.1615	0.1069	0.1519	0.1470	0.1141
100		0.0201	0.1253	0.1374	0.0720	0.1223	0.1181	0.0876
150		0.0134	0.1160	0.1330	0.0560	0.1142	0.1106	0.0793
200		0.0099	0.1083	0.1306	0.0506	0.1097	0.1061	0.0741

Table 4
Simulation results of the DMEx parameter $\beta = 3.0$ and $\beta = 5.0$.

n	Measure	MLE	LSE	WLE	PCE	CVME	ADE	RADE
$\beta = 3.0$								
10	AABs	0.0130	0.3975	0.4179	0.4705	0.3643	0.3769	0.3574
25		0.0014	0.3498	0.3519	0.3383	0.3386	0.3420	0.2850
50		0.0097	0.3335	0.3295	0.2823	0.3243	0.3245	0.2559
100		0.0003	0.3206	0.3306	0.2340	0.3168	0.3147	0.2509
150		0.0060	0.3175	0.3288	0.2218	0.3119	0.3100	0.2496
200		0.0037	0.3126	0.3289	0.2095	0.3114	0.3088	0.2447
10	MREs	0.0043	0.1325	0.1393	0.1568	0.1214	0.1256	0.1191
25		0.0005	0.1166	0.1173	0.1128	0.1129	0.1140	0.0950
50		0.0032	0.1112	0.1098	0.0941	0.1081	0.1082	0.0853
100		0.0001	0.1069	0.1102	0.0780	0.1056	0.1049	0.0836
150		0.0020	0.1058	0.1096	0.0739	0.1040	0.1033	0.0832
200		0.0012	0.1042	0.1096	0.0698	0.1038	0.1029	0.0816
10	MSEs	0.4482	0.7108	0.7379	0.7991	0.6734	0.6494	0.6324
25		0.1827	0.3402	0.3251	0.3305	0.3274	0.3155	0.2765
50		0.0859	0.2208	0.2068	0.1883	0.2129	0.2063	0.1631
100		0.0451	0.1558	0.1590	0.1079	0.1534	0.1489	0.1090
150		0.0305	0.1364	0.1401	0.0846	0.1327	0.1292	0.0942
200		0.0224	0.1234	0.1315	0.0704	0.1239	0.1207	0.0843
$\beta = 5.0$								
10	AABs	0.0021	0.4191	0.4318	0.6349	0.4216	0.4438	0.4216
25		0.0078	0.3448	0.3654	0.4577	0.3491	0.3573	0.3168
50		0.0023	0.3268	0.3433	0.3487	0.3228	0.3296	0.2746
100		0.0055	0.3230	0.3229	0.2768	0.3258	0.3271	0.2530
150		0.0065	0.3100	0.3224	0.2542	0.3174	0.3185	0.2508
200		0.0034	0.3117	0.3252	0.2326	0.3125	0.3136	0.2456
10	MREs	0.0004	0.0838	0.0864	0.1270	0.0843	0.0888	0.0843
25		0.0016	0.0690	0.0731	0.0915	0.0698	0.0715	0.0634
50		0.0005	0.0654	0.0687	0.0697	0.0646	0.0659	0.0549
100		0.0011	0.0646	0.0646	0.0554	0.0652	0.0654	0.0506
150		0.0013	0.0620	0.0645	0.0508	0.0635	0.0637	0.0502
200		0.0007	0.0623	0.0650	0.0465	0.0625	0.0627	0.0491
10	MSEs	1.2552	1.7171	1.6830	1.9589	1.6687	1.5782	1.5649
25		0.5255	0.7323	0.6971	0.8212	0.7176	0.6809	0.6329
50		0.2565	0.3988	0.4034	0.4206	0.4080	0.3907	0.3376
100		0.1213	0.2504	0.2401	0.2277	0.2486	0.2397	0.1982
150		0.0823	0.1969	0.1925	0.1614	0.1990	0.1928	0.1522
200		0.0632	0.1707	0.1734	0.1287	0.1757	0.1708	0.1238

The introduced methods are compared for sample sizes, n=5, 10, 30, 50, and 100. To this end, we generate 10,000 independent samples of size n from the DMEx distribution with different values of β = 0.5, 1.0, 2.0, 3.0, 5.0 and 9.0. It is observed that 10,000 repetitions are sufficiently large enough to have stable results. The results of the simulations are reported in Tables 2–4.

The following conclusions are drawn from Tables 2–4.

- All the considered estimation methods have consistency property, i.e., the MSEs and MREs decrease with the increasing of the sample size n.
- It is observed that the ML method performs better based on the MSEs as compared to other estimation methods.

5. Applications to COVID-19 data

The flexibility of the DMEx distribution is illustrated using two real-life COVID-19 data sets.

The first data refers to the number of COVID-19 daily deaths in China from 23 January to 28 March (https://www.worldometers.info/coronavirus/country/china/). The observations are listed below in ascending order.

3	3	4	5	5	6	6	7	7	7	8	8	9	10	11	11	13
3	14	15	16	17	22	22	24	26	26	27	28	29	30	31	31	35
38	38	42	43	44	45	46	47	52	57	64	65	71	73	73	86	89
97	97	97	98	105	108	109	114	118	121	136	142	143	146	150		

The second data represents the number of COVID-19 daily deaths in Europe from 1 March to 31 March (https://covid19.who.int/). The observations are given below

6	18	28	29	44	47	55	116	118	129	150
184	219	236	237	336	421	434	612	648	706	838
1129	1393	1540	1941	2175	2278	2667	2803	2824		

Some descriptive measures of both data sets are reported in Table 5.

Table 5
The descriptive measures of the two data sets.

n	Min.	Mean	Median	Var	Skewness	Kurtosis	DI	Q_3	Max.
Data I	3	49.74	33	1924.8	0.8365	2.4502	38.696	83	150
Data II	6	818.0	336	868739.6	1.0167	2.6308	1062.1	1407	2824

Table 6
The ML estimates, SE, and goodness of fit measures for number of deaths in China and Europe.

Model	Estimate (SE)	logL	AIC	BIC
Data 1: Number of deaths in China				
DMEx	$\beta = 25.1237(2.18696)$	−330.515	663.031	665.220
DBur	$\theta = 0.95087(0.10742) \alpha = 5.77181(12.9360)$	−374.500	753.001	757.380
DBH	$\theta = 0.99978(0.00185)$	−461.020	924.039	926.229
DR	$\theta = 47.0101(2.89338)$	−347.227	696.455	698.644
DITL	$\theta = 0.35393(0.04357)$	−366.907	735.815	738.005
DPr	$\theta = 0.28633(0.03524)$	−379.070	760.140	762.330
DIR	$\theta = 142.479(17.9843)$	−376.380	754.760	756.950
Poisson	$\theta = 49.7424(0.86814)$	−1409.78	2821.57	2823.75
Data 2: Number of deaths in Europe				
DMEx	$\beta = 409.362(52.0921)$	−257.122	516.243	517.677
DBur	$\theta = 0.96110(0.17072)$	−62.902	529.804	532.672
	$\alpha = 4.39613(19.6877)$			
DBH	$\theta = 0.99999(0.12196)$	−56.308	714.616	716.050
DR	$\theta = 869.110(78.4324)$	−72.816	547.634	549.068
DITL	$\theta = 0.19763(0.03549)$	−59.617	521.234	522.668
DPr	$\theta = 0.17401(0.03125)$	−63.363	528.727	530.161
DIR	$\theta = 1077.00(34.9945)$	−26.987	655.975	657.409
Poisson	$\theta = 785.710(14.2215)$	−5361.1	30724.2	30725.7

The fitting performance of the DMEx distribution is compared with the following discrete distributions:

- Discrete Burr (DBurr) distribution. Its PMF is

$$P(X=x)=\frac{\theta^2(2+x)(1-\theta)^x}{(1+\theta)},\quad \theta>0,\ x=0,1,2,\ldots$$

- Discrete Burr-Hatke (DBH) distribution. Its PMF is

$$P(X=x)=\left(\frac{1}{1+x}-\frac{\theta}{2+x}\right)\theta^x,\quad \theta>0,\ x=0,1,2,\ldots$$

- Discrete Rayleigh (DR) distribution. Its PMF is

$$P(X=x)=\exp\left(-\frac{x^2}{2\theta^2}\right)-\exp\left(-\frac{(x+1)^2}{2\theta^2}\right),\quad \theta>0,\ x=0,1,2,\ldots$$

- Discrete inverted Topp-Leon (DITL) distribution. Its PMF is

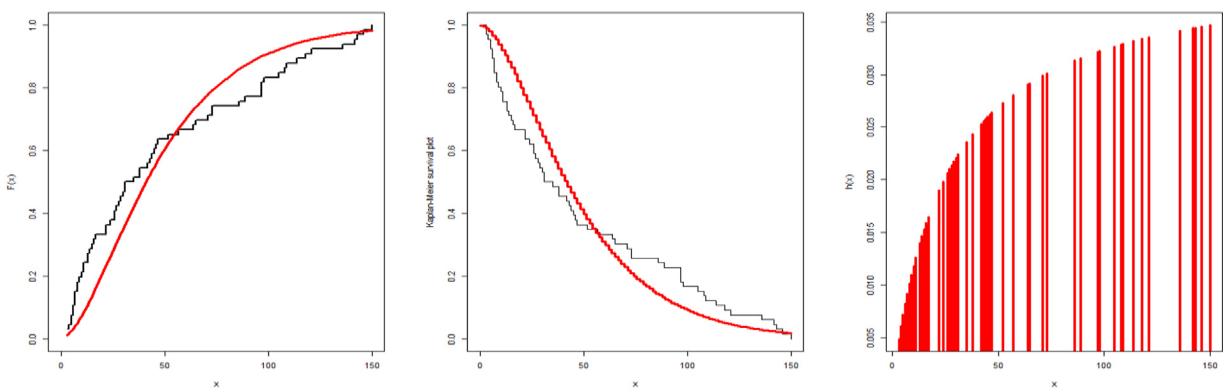
$$P(X=x)=\frac{(1+2x)^\theta}{(1+x)^{2\theta}}-\frac{(3+2x)^\theta}{(2+x)^{2\theta}},\quad \theta>0,\ x=0,1,2,\ldots$$

- Discrete Pareto (DPr) distribution. Its PMF is

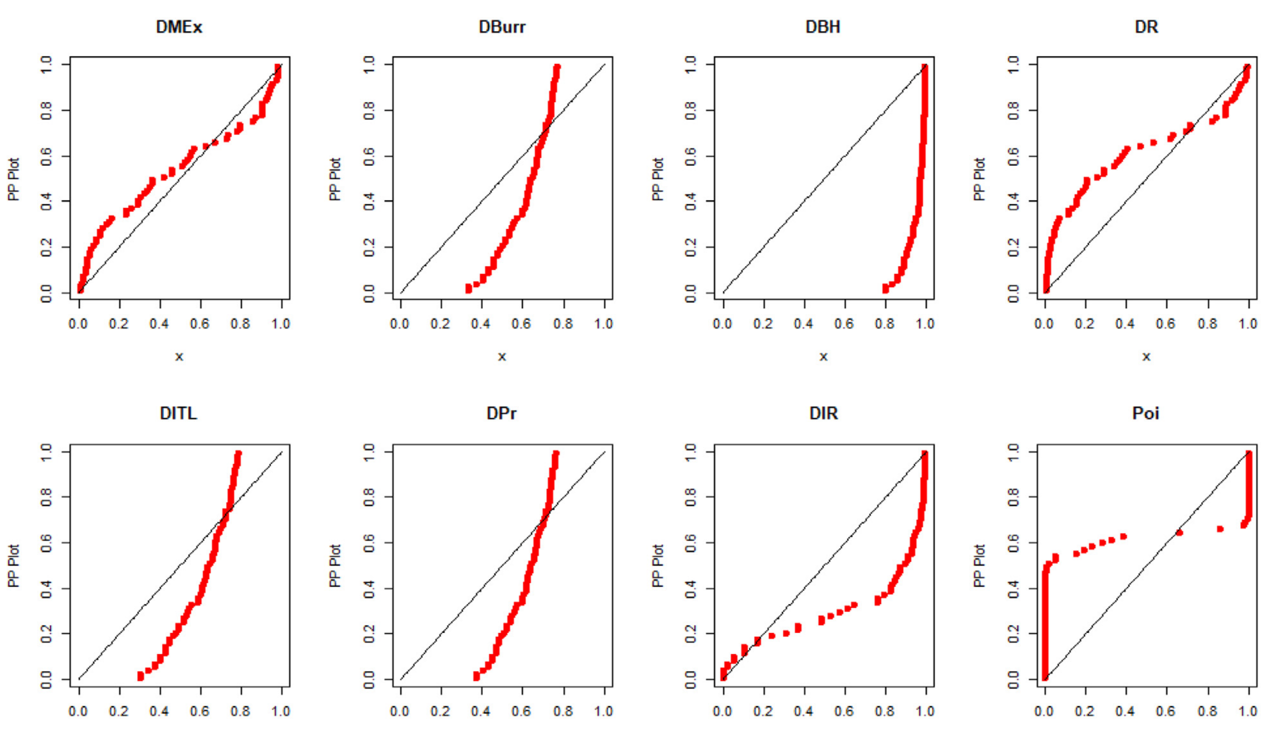
$$P(X=x)=\exp(-\theta\log(1+x))-\exp(-\theta\log(2+x)),\ \theta>0,\ x=0,1,2,\ldots$$

- Discrete inverse Rayleigh (DIR) distribution. Its PMF is

$$P(X=x)=\exp\left(-\frac{\theta}{(1+x)^2}\right)-\exp\left(-\frac{\theta}{(x)^2}\right),\quad \theta>0,\ x=0,1,2,\ldots$$



(a)



(b)

Fig. 3. (a) The fitted CDF, SF, and HRF plots of the DMEx model and (b) the P-P plots of the DMEx model and its competing discrete distributions for number of deaths in China.

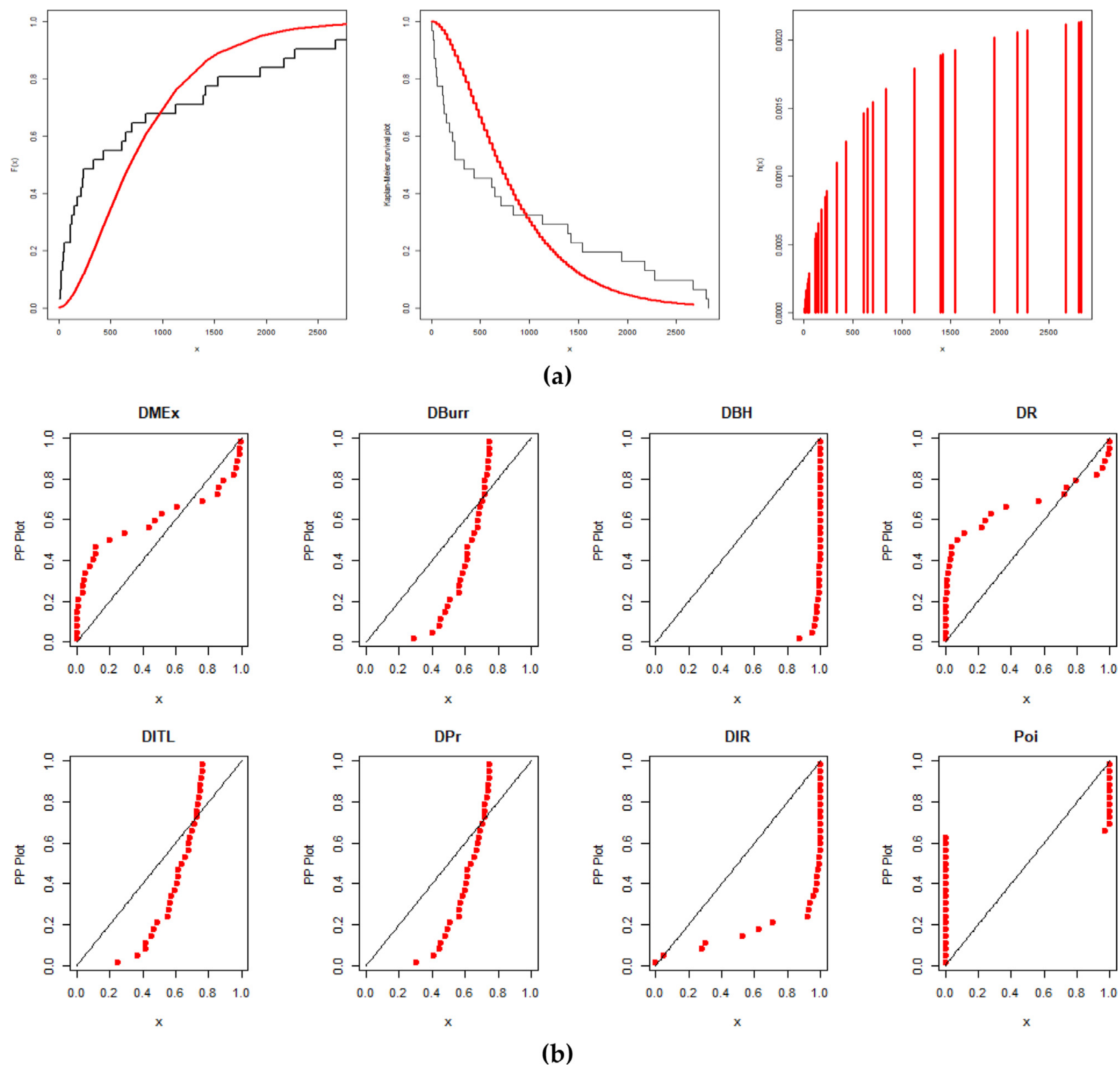


Fig. 4. (a) The fitted CDF, SF, and HRF plots of the DMEx model and (b) the P-P plots of the DMEx model and its competing discrete distributions for number of deaths in Europe.

- Poisson distribution. Its PMF is

$$P(X = x) = \frac{\theta^x \exp(-\theta)}{x!}, \quad \theta > 0, \; x = 0, 1, 2, \ldots$$

The R package called "fitdistrplus" is used to obtain the results in this section. The parameters of the four distributions are estimated using the ML approach. For model comparison, we have considered the log-likelihood (LogL), Akaike's information criterion (AIC), and Bayesian information criterion (BIC).

The estimates of the parameters and their standard errors (SE) along with goodness of fit measures are presented in Table 6 for both data sets. The values in these tables illustrate that the DMEx provides an adequate fit to the data as compared with other models.

Furthermore, the estimated CDF, SF, and HRF of the DMEx distribution are depicted in Figs. 3(a) and 4(a) for number of deaths in China and Europe, respectively. The probability–probability (P-P) plots of the fitted discrete distributions are displayed in Fisg. 3(b) and 4(b) for the two datasets, respectively.

In the two applications, we can indicate that the best model representing the daily deaths by COVID-19 is the DMEx distribution. Based on the fitted model, we can answer some questions such as the probability of having more than X deaths per day, whereas a prediction based on the mean to the deaths per day by COVID-19.

6. Conclusion

In this article, a new discrete probability distribution called the discrete moment-exponential (DMEx) distribution is proposed. It can be used as an alternative to some well-known discrete distributions. Its mathematical properties of the DMEx distribution are presented. The model parameters are estimated using seven different estimation methods. Comprehensive simulation results are carried out to compare these methods. Based on our study, the maximum likelihood is recommended to estimate the DMEx parameter. The usefulness of the DMEx distribution is illustrated empirically using two applications to the number of deaths due to COVID-19 in China and Europe. The DMEx distribution is quite competitive to the discrete Burr, discrete Burr-Hatke, discrete Rayleigh, discrete inverted Topp-Leon, discrete-Pareto, discrete inverse-Rayleigh, and Poisson distributions. We hope that the DMEx distribution can be applied to traumatic brain injury data following the paper of Ramos et al. (2019).

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

Afify, A.Z., Elmorshedy, M., Eliwa, M.S., 2021. A new skewed discrete model: properties, inference, and applications. Pakistan J. Stat. Oper. Res. 17, 799–816.
Al-Babtain, A.A., Ahmed, A.H.N., Afify, A.Z., 2020. A new discrete analog of the continuous Lindley distribution, with reliability applications. Entropy 22, 603.
Alghamdi, A.S., Ahsan-ul-Haq, M., Babar, A., Aljohani, H.M., Afify, A.Z., 2022. The discrete power-Ailamujia distribution: properties, inference, and applications. AIMS Math. 7, 8344–8360.
Aljohani, H.M., Akdoğan, Y., Cordeiro, G.M., Afify, A.Z., 2021. The uniform Poisson-Ailamujia distribution: actuarial measures and applications in biological science. Symmetry 13, 1258.
Anderson, T.W., Darling, D.A., 1952. Asymptotic theory of certain" goodness of fit" criteria based on stochastic processes. Ann. Math. Stat. 23, 193–212.
Bakouch, H.S., Dey, S., Ramos, P.L., Louzada, F., 2017. Binomial-exponential 2 distribution: different estimation methods with weather applications. TEMA (São Carlos) 2017 (18), 233–251.
Chakraborty, S., 2015. Generating discrete analogues of continuous probability distributions- a survey of methods and constructions. J. Stat. Distrib. Appl. 2, 1–30.
Dara, S.T., Ahmad, M., 2012. Recent advances in Moment distribution and their hazard rates. Lap Lambert academic Publishing GmbH KG.
Eldeeb, A.S., Ahsan-Ul-Haq, M., Babar, A., 2021. A discrete analog of inverted Topp-Leone distribution: properties, estimation and applications. Int. J. Anal. Appl. 19, 695–708.
Gómez-Déniz, E., 2010. Another generalization of the geometric distribution. Test 19 (2), 399–415.
Gomez-Deniz, E., Calderin-Ojeda, E., 2011. The discrete Lindley distribution: properties and application. J. Stat. Comput. Simul. 81, 1405–1416.
Hussain, T., Aslam, M., Ahmad, M., 2016. A two parameter discrete Lindley distribution. Revista Colom.Estadistica 39, 45–61.
Jazi, M.A., Lai, C., Alamastsaz, M.H., 2010. A discrete inverse Weibull distribution and estimation of its parameters. Stat. Methodol. 7, 121–132.
Jia, J.-M., Yan, Z.-Z., Peng, X.-Y., 2019. A new discrete extended Weibull distribution. IEEE Access 7, 175474–175486.
Kao, J., 1959. A graphical estimation of mixed Weibull parameters in life testing electron tube. Technometrics 1, 389–407.
Kemp, A.W., 2004. A classes of discrete lifetime distributions. Commun. Stat. Theory Methods 33 (12), 3069–3093.
Kemp, A., 2008. The discrete half normal distribution. Adv. Math. Stat. Model., 353–365
Krishna, H., Singh Pundir, P., 2009. Discrete Burr and discrete Pareto distributions. Stat. Methodol. 6 (2), 177–188.
Macdonald, P., 1971. Comment on "an estimation procedure for mixtures of distributions" by Choi and Bulgren. J. R. Stat. Soc. Ser. B Methodol. 33, 326–329.
Nakagawa, T., Osaki, S., 1975. Discrete Weibull distribution. . *IEEF Trans. Reliab.*, 300–301.
Nekoukhou, V., Alamatsaz, M.H., Bidram, H., 2013. Discrete generalized exponential distribution of a second type. Statistics 47, 876–887.
Opone, F.C., Izekor, E.K., Akata, I.U., Osagiede, F.E.U., 2020. A Discrete analogue of the continuous Marshall-Olkin Weibull distribution with application to count data. Earthline J. Math. Sci. 5, 415–428.
Ramos, P.L., Louzada, F., 2019. A Distribution for Instantaneous Failures. *Stats* 2 (2), 247–258.
Ramos, P.L., Nascimento, D.C., Ferreira, P.H., Weber, K.T., Santos, T.E.G., Louzada, F., Chen, F., 2019. Modeling traumatic brain injury lifetime data: improved estimators for the generalized gamma distribution under small samples. PLoS ONE 14 (8), e0221332.
Rodrigues, G.C., Louzada, F., Ramos, P.L., 2018. Poisson–exponential distribution: different methods of estimation. J. Appl. Stat. 45 (1), 128–144.
Roy, D., 2004. Discrete Rayleigh distribution. IEEF Trans. Reliab. 53 (2), 255–260.