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Introducing novel Θ-fractional operators: Advances in fractional calculus

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ABSTRACT

This study explores the foundational iterative processes of fractional calculus, focusing on Θ-conformable fractional derivatives (Θ-CFD). We introduce novel fractional operators and define their associated function spaces. Additionally, we establish a series of theorems that enhance our understanding of these operators within the context of fractional calculus.

1. Introduction

In the domain of fractional calculus (FC), which finds widespread applications in diverse scientific and engineering disciplines, a plethora of fractional derivatives have been extensively employed. Notably, the Caputo derivative (CD) and Riemann–Liouville derivative (RLD) have emerged as the most commonly used ones. These derivatives have proven to be highly effective in capturing intricate dynamics observed in biology, physics, engineering, and various other fields (Kilbas et al., 2006; Magin, 2006; Redhwan and Shaikh, 2021). Real-world phenomena often exhibit memory effects, necessitating the selection of an appropriate nonlocal model when dealing with different types of data. Consequently, researchers have dedicated substantial efforts to exploring novel fractional operators encompassing diverse characteristics such as singular, non-singular, local, and nonlocal kernels (see Atangana and Baleanu, 2016; Caputo and Fabrizio, 2015; Abdeljawad and Baleanu, 2017; Sadek, 2023a; Sadek and Lazar, 2023 and related references).

Recent advancements in FC have led to significant progress in solving nonlinear functional integral equations using various fractional operators. Pathak et al. (2023) explored the solvability of the Erdélyi-Kober fractional operator, offering new insights into the mathematical methods applicable in these contexts. Paul et al. (2023b) presented an effective method for solving nonlinear fractional integral equations. Additionally, Paul et al. (2023a) analyzed mixed-type nonlinear Volterra-Fredholm Erdélyi-Kober fractional integral equations, highlighting the operator's versatility in handling diverse integral equations. Bhat et al. (2024) focused on the precision and efficiency of an interpolation approach to weakly singular integral equations, showing improvements in computational methods for heat and fluid flow problems. Farid et al. (2023) and Farid et al. (2022) contributed to the field by developing fractional Hadamard and Fejér–Hadamard inequalities associated with exp. $(\alpha, h-m)$-convexity and Riemann–Liouville fractional versions of Hadamard inequality for strongly m-convex functions, respectively, providing new tools for mathematical analysis. Finally, Rathour et al. (2023) introduced k-fractional integral inequalities of Hadamard type for strongly exponentially $(\alpha, h-m)$-convex functions, further expanding the applications of fractional calculus in the study of convex functions. These studies collectively underscore the growing significance of fractional calculus in solving complex mathematical problems and its broad applicability across various scientific fields, as well as contributions from L. Sadek and collaborators. Sadek (2022) explored the application of FBDF method for solving fractional differential matrix equations, presenting a novel approach to these complex problems. In a subsequent study, Sadek et al. (2023) introduced a conformable finite element method tailored for conformable fractional partial differential equations, significantly enhancing the computational tools available for these equations. Sadek's (Sadek, 2023b) work on the stability of conformable linear infinite-dimensional systems provided crucial insights into the behavior and control of such systems. Furthermore, Sadek (2024b) investigated methods to solve two-term fractional differential

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Sylvester matrix equations using fractional BDF formulas, expanding the repertoire of techniques for handling fractional differential equations. In collaboration with Sadek and Sami Bataineh (2024) applied the general Bernstein function to Θ-fractional differential equations, showcasing innovative uses of mathematical functions in fractional calculus. Lastly, Sadek et al. (2024) developed the Galerkin-Bell method to address fractional optimal control problems, highlighting the method's efficiency and applicability in optimization problems. These studies collectively advance the understanding and application of fractional calculus in solving diverse and complex mathematical and control problems.

The conventional approach of FC may not furnish the necessary framework for extracting crucial insights from such systems. Consequently, a fundamental question arises: Can we extend the fractional Riemann–Liouville integral (RLI) in a manner that unifies RLD, Hadamard derivative (HD), and other fractional derivatives (Katugampola, 2011, 2014)? The crux of this endeavor lies in determining the appropriate differentiation operator to act as the foundation for the iterative process. In FC, we iterate the customary integral of a function and utilize the Cauchy formula to derive integrals of higher integer orders, thereby facilitating the substitution of the integer value with any $\omega \in \mathbb{C}$. In Abdeljawad (2015), it was suggested that the CFI should be appropriately fractionalized. Notably, a similar form of integral, as presented in (2), has previously emerged in El-Nabulsi and Torres (2008), finding applications in mathematical economics to depict discounting dynamics in economic contexts (El-Nabulsi and Torres, 2008). Moreover, this integral is employed to capture the behavior of nonlinear dissipative systems (El-Nabulsi and Torres, 2008). Let $0 < \omega \leq 1$, at this juncture, it is important to mention that the left and right CFDs, as defined in Abdeljawad (2015), are given by:

$$ {}_aT^{\omega} f(\ell) = \frac{\frac{df(\ell)}{d\ell}}{(\ell-a)^{\omega-1}} \text{ and } T_b^{\omega} f(\ell) = \frac{\frac{df(\ell)}{d\ell}}{(b-\ell)^{\omega-1}}. \tag{1} $$

The corresponding integrals, both left and right, can be expressed as follows:

$$ \left({}_aI^{\omega} f\right)(\ell) = \int_a^{\ell} f(s)(s-a)^{\omega-1} ds, \tag{2} $$

and

$$ \left(I_b^{\omega} f\right)(\ell) = \int_{\ell}^{b} f(s)(b-s)^{\omega-1} ds. \tag{3} $$

Let $r \in \mathbb{R}_+^*$ and $m = [r]+1$, the left RLI of order r is:

$$ \left({}_a\mathbf{I}^{r} f\right)(\ell) = \frac{1}{\Gamma(r)} \int_a^{\ell} (\ell-\tau)^{r-1} f(\tau) d\tau. \tag{4} $$

Meanwhile, the right RLI of order r is:

$$ \left(\mathbf{I}_b^{r} f\right)(\ell) = \frac{1}{\Gamma(r)} \int_{\ell}^{b} (s-\ell)^{r-1} f(s) ds. \tag{5} $$

Below is the expression for the left RLD of order r:

$$ \left({}_aD^{r} f\right)(\ell) = \left(\frac{d}{d\ell}\right)^m \left({}_a\mathbf{I}^{m-r} f\right)(\ell). \tag{6} $$

The right RLD of order r is:

$$ \left(D_b^{r} f\right)(\ell) = \left(-\frac{d}{d\ell}\right)^m \left(\mathrm{I}_b^{m-r} f\right)(\ell). \tag{7} $$

The form of the left CD of order r is:

$$ \left({}_a^C D^{r} f\right)(\ell) = \left({}_a\mathbf{I}^{m-r} f^{(m)}\right)(\ell). \tag{8} $$

The right CD is:

$$ \left({}^C D_b^{r} f\right)(\ell) = \left(\mathbf{I}_b^{m-r} (-1)^m f^{(m)}\right)(\ell). \tag{9} $$

Consider an increasing function $\Theta \in C^1([a,b])$ such that $\Theta(\ell) \neq 0$ for all $\ell \in [a,b]$. In accordance with Almeida (2017), the left Θ-RLI of order r starting at a is:

$$ \left({}_a\mathbf{I}^{r,\Theta} f\right)(\ell) = \frac{1}{\Gamma(r)} \int_a^{\ell} (\Theta(\ell)-\Theta(y))^{r-1} \Theta'(y) f(y) dy, \tag{10} $$

and the right Θ-RLI of order r is:

$$ \left(\mathbf{I}_b^{r,\Theta} f\right)(\ell) = \frac{1}{\Gamma(r)} \int_{\ell}^{b} (\Theta(y)-\Theta(\ell))^{r-1} \Theta'(y) f(y) dy. \tag{11} $$

Below is the expression for the left Θ-RLD of order r starting at a:

$$ \left({}_aD^{r,\Theta} f\right)(\ell) = \left(\frac{1}{\Theta'(\ell)} \frac{d}{d\ell}\right)^m \left({}_a\mathbf{I}^{m-r,\Theta} f\right)(\ell). \tag{12} $$

The right Θ-RLD of order r, ending at b is:

$$ \left(D_b^{r,\Theta} f\right)(\ell) = \left(-\frac{1}{\Theta'(\ell)} \frac{d}{d\ell}\right)^m \left(\mathrm{I}_b^{m-r,\Theta} f\right)(\ell). \tag{13} $$

The left Θ-CD is:

$$ \left({}_a^C D^{r,\Theta} f\right)(\ell) = {}_a\mathbf{I}^{m-r}\left(\left(\frac{1}{\Theta'(\ell)} \frac{d}{d\ell}\right)^m f(\ell)\right). \tag{14} $$

The right Θ-CD of order r, ending at b is:

$$ \left({}^C D_b^{r,\Theta} f\right)(\ell) = \left(\mathbf{I}_b^{m-r}\left(\left(\frac{-1}{\Theta'(\ell)} \frac{d}{d\ell}\right)^m f\right)\right)(\ell). \tag{15} $$

The left Hadamard fractional integral (Anatoly, 2001) (HI) is

$$ \left({}_a\mathfrak{J}^{r} f\right)(\ell) = \frac{1}{\Gamma(r)} \int_a^{\ell} (\ln(\ell)-\ln(y))^{r-1} f(y) \frac{dy}{y}, \tag{16} $$

and the right HI is:

$$ \left(\mathfrak{J}_b^{r} f\right)(\ell) = \frac{1}{\Gamma(r)} \int_{\ell}^{b} (\ln s - \ln \ell)^{r-1} f(s) \frac{ds}{s}. \tag{17} $$

The left HD of order r is:

$$ \left({}_a\mathfrak{D}^{r} f\right)(\ell) = \left(\ell \frac{d}{d\ell}\right)^m \left({}_aI^{m-r} f\right)(\ell), \tag{18} $$

the right HD of order r is

$$ \left(\mathfrak{D}_b^{r} f\right)(\ell) = \left(-\ell \frac{d}{d\ell}\right)^m \left(I_b^{m-r} f\right)(\ell). \tag{19} $$

The left and right Caputo-HD (Gambo et al., 2014; Jarad et al., 2012; Adjabi et al., 2016), respectively, as

$$ \left({}_a^C\mathfrak{D}^{r} f\right)(\ell) = {}_a\mathfrak{D}^{r}\left[f(t) - \sum_{l=0}^{m-1} \frac{(\ell \frac{d}{d\ell})^l f(a)}{l!} (\ln(t)-\ln(a))^l\right](\ell). \tag{20} $$

Let the space $C^m[a,b] = \left\{ f : [a,b] \to \mathbb{C} : (\ell \frac{d}{d\ell})^{m-1}[f(\ell)] \in C[a,b] \right\}$, The left and right Caputo-HD, respectively, are

$$ \left({}_a^C\mathfrak{D}^{r} f\right)(\ell) = \left({}_a\mathfrak{J}^{m-r}\left(\ell \frac{d}{d\ell}\right)^m f\right)(\ell), \tag{21} $$

and

$$ \left({}^C\mathfrak{D}_b^{r} f\right)(\ell) = \mathfrak{D}_b^{r}\left[f(t) - \sum_{l=0}^{m-1} \frac{(-1)^l (\ell \frac{d}{d\ell})^l f(b)}{l!} (\ln(b)-\ln(t))^k\right](\ell), \tag{22} $$

in $C^m[a,b]$ equivalently by

$$ \left({}^C\mathfrak{D}_b^{r} f\right)(\ell) = \left(\mathfrak{J}_b^{m-r}\left(-\ell \frac{d}{d\ell}\right)^m f\right)(\ell). \tag{23} $$

Let the function space

$$ X_c^q(a,b) = \left\{ f : [a,b] \to \mathbb{R} : \|f\|_{X_c^q} = \left(\int_a^b |\ell^c f(\ell)|^q \frac{d\ell}{\ell}\right)^{1/q} < \infty \right\}, $$

where $1 \leq q < \infty$, $a < b$, and $c \in \mathbb{R}$. In the context where $q = \infty$, the norm $|f|_{X_c^q}$ is defined as the essential supremum over $a \leq \ell \leq b$ of the expression $[\ell^c |f(\ell)|]$. Within the framework of the mentioned function space, the Katugampola FI (KFI) (Katugampola, 2011), can be expressed as follows:

$$ \left({}_aI^{r,\omega} f\right)(\ell) = \frac{1}{\Gamma(r)} \int_a^{\ell} \left(\frac{\ell^{\omega}-y^{\omega}}{\omega}\right)^{r-1} f(y) \frac{dy}{y^{1-\omega}}, \tag{24} $$

and

$$ \left(I_b^{r,\omega} f\right)(\ell) = \frac{1}{\Gamma(r)} \int_{\ell}^{b} \left(\frac{y^{\omega}-\ell^{\omega}}{\omega}\right)^{r-1} f(y) \frac{dy}{y^{1-\omega}}, \tag{25} $$

left and right, respectively. The left and right Katugampola fractional derivatives (Katugampola, 2014) are:

$$\left({}_aD^{r,\omega}f\right)(\ell) = \frac{(\ell^{1-\omega}\frac{d}{d\ell})^m}{\Gamma(m-r)}\int_a^{\ell}\left(\frac{x^{\omega}-y^{\omega}}{\omega}\right)^{m-r-1}f(y)\frac{dy}{y^{1-\omega}}, \tag{26}$$

and

$$\left(D_b^{r,\omega}f\right)(\ell) = \frac{(-\ell^{1-\omega}\frac{d}{d\ell})^m}{\Gamma(m-r)}\int_{\ell}^{b}\left(\frac{y^{\omega}-x^{\omega}}{\omega}\right)^{m-r-1}f(y)\frac{dy}{y^{1-\omega}}, \tag{27}$$

respectively. Based on the findings in Katugampola (2014), the authors of Jarad et al. (2017a) introduced the modified versions of the left and right generalized FD, following the Caputo type. These modifications are:

$$\left({}_a^CD^{r,\omega}f\right)(\ell) = \frac{1}{\Gamma(m-r)}\int_a^{\ell}\left(\frac{\ell^{\omega}-u^{\omega}}{\omega}\right)^{m-r-1}(\ell^{1-\omega}\frac{d}{d\ell})^m f(y)\frac{dy}{y^{1-\omega}}, \tag{28}$$

and

$$\left({}^CD_b^{r,\omega}f\right)(\ell) = \frac{1}{\Gamma(m-r)}\int_{\ell}^{b}\left(\frac{y^{\omega}-\ell^{\omega}}{\omega}\right)^{m-r-1}(-\ell^{1-\omega}\frac{d}{d\ell})^m f(y)\frac{dy}{y^{1-\omega}}. \tag{29}$$

Based on the information provided in Abdeljawad (2015), the authors of Jarad et al. (2017b) introduced the left and right FI. The expressions for these integrals are given as follows:

$${}_a^r\mathfrak{J}^{\omega}f(\ell) = \frac{1}{\Gamma(r)}\int_a^{\ell}\left(\frac{(\ell-a)^{\omega}-(\zeta-a)^{\omega}}{\omega}\right)^{r-1}\frac{f(\zeta)du}{(\zeta-a)^{1-\omega}}, \tag{30}$$

and

$${}^r\mathfrak{J}_b^{\omega}f(\ell) = \frac{1}{\Gamma(r)}\int_{\ell}^{b}\left(\frac{(b-\ell)^{\omega}-(b-\zeta)^{\omega}}{\omega}\right)^{r-1}\frac{f(\zeta)d\zeta}{(b-\zeta)^{1-\omega}}. \tag{31}$$

Drawing from the insights of Abdeljawad (2015), the authors in Jarad et al. (2017b) introduced the FD as follows:

$${}_a^r\mathfrak{D}^{\omega}f(\ell) = \frac{{}_a^mT^{\omega}}{\Gamma(m-r)}\int_a^{\ell}\left(\frac{(\ell-a)^{\omega}-(\zeta-a)^{\omega}}{\omega}\right)^{r-1}\frac{f(\zeta)d\zeta}{(\zeta-a)^{1-\omega}}, \tag{32}$$

and

$${}^r\mathfrak{D}_b^{\omega}f(\ell) = \frac{(-1)^m\,{}^mT_b^{\omega}}{\Gamma(m-r)}\int_{\ell}^{b}\left(\frac{(b-\ell)^{\omega}-(b-\xi)^{\omega}}{\omega}\right)^{r-1}\frac{f(\zeta)d\zeta}{(b-\zeta)^{1-\omega}}, \tag{33}$$

where

$${}_a^mT^{\omega} = \underbrace{{}_aT^{\omega}\,{}_aT^{\omega}\cdots{}_aT^{\omega}}_{m\text{ times}} \quad\text{and}\quad {}^mT_b^{\omega} = \underbrace{T_b^{\omega}T_b^{\omega}\cdots T_b^{\omega}}_{m\text{ times}}, \tag{34}$$

with ${}_aT^{\omega}$ and T_b^{ω} are the left and right CFD Eq. (1). The CFD in the Caputo type are

$${}_a^C\mathfrak{D}^{r,\omega}f(\ell) = \frac{1}{\Gamma(m-r)}\int_a^{\ell}\left(\frac{(\ell-a)^{\omega}-(\zeta-a)^{\omega}}{\omega}\right)^{m-r-1}\frac{{}_a^mT^{\omega}f(\zeta)}{(\zeta-a)^{1-\omega}}d\zeta, \tag{35}$$

and

$${}^C\mathfrak{D}_b^{r,\omega}f(\ell) = \frac{(-1)^m}{\Gamma(m-r)}\int_{\ell}^{b}\left(\frac{(b-\ell)^{\omega}-(b-\zeta)^{\omega}}{\omega}\right)^{m-r-1}\frac{{}^mT_b^{\omega}f(\zeta)}{(b-\zeta)^{1-\omega}}d\zeta, \tag{36}$$

left and right, respectively.

In recent years, there has been significant interest in the development and application of operators (32) and (33). Salah et al. (2024) utilized conformable fractional-order modeling to analyze the transmission dynamics of HIV/AIDS, providing new insights into disease spread and control strategies. Kiriş et al. (2024) introduced novel midpoint-type inequalities for coordinated convex functions using generalized CFI, expanding the mathematical toolkit for handling such functions. Ying et al. (2024) investigated conformable fractional Milne-type inequalities, contributing to the broader understanding of fractional inequalities and their applications. Hezenci and Budak (2024) developed Bullen-type inequalities for twice-differentiable functions using CFI, offering new perspectives on classical inequalities. Wang and Yuan (2024) explored the existence, uniqueness, and Ulam stability of solutions to the fractional conformable Langevin system on the ethane graph, further advancing the application of fractional calculus in complex systems analysis.

The structure of this article unfolds as follows:

I. Introduction:

- Background on fractional calculus and conformable derivatives.
- Objectives and contributions of the article.

II. Definitions and Notations:

- Left and right Θ-conformable fractional integrals (Θ-CFI).
- Left and right Θ-conformable fractional derivatives (Θ-CFD).
- Function spaces and properties.

III. Θ-CFD for Functions in Certain Spaces:

- Establishing Θ-CFD for functions in specified spaces.
- Proving properties of Θ-CFD in these spaces.

IV. Caputo-Type Θ-Conformable Derivatives:

- Definition and properties of Caputo-type Θ-CFD.
- Comparison with Riemann–Liouville and other types of fractional derivatives.

V. Conclusion

- Summary of main results and contributions.
- Implications and future research directions.

2. The Θ-CFD and Θ-CFI

Definition 1 (*Sadek (2024a)*)**.** Let Θ an increasing function and $\Theta \in C^1([a,b])$ such that $\Theta \neq 0$. The left Θ-CFD of order ω starting from a is:

$${}_aT^{\omega,\Theta}f(\ell) = \frac{\frac{df(\ell)}{d\ell}}{(\Theta(\ell)-\Theta(a))^{\omega-1}\frac{d\Theta(\ell)}{d\ell}}, \tag{37}$$

and the right Θ-CFD of order ω starting from a is:

$$T_b^{\omega,\Theta}f(\ell) = \frac{\frac{df(\ell)}{d\ell}}{(\Theta(b)-\Theta(\ell))^{\omega-1}\frac{d\Theta(\ell)}{d\ell}}. \tag{38}$$

If $\Theta(\ell) = \ell$, we get the left and right CFD Eq. (1) see Abdeljawad (2015).

Definition 2 (*Sadek (2024a)*)**.** Let $\Theta \in C^1([a,b])$ an increasing function with $\Theta(\ell) \neq 0$. The left Θ-CFI of order ω starting from a is:

$$\left({}_aI^{\omega,\Theta}f\right)(\ell) = \int_a^{\ell} f(u)(\Theta(u)-\Theta(a))^{\omega-1}\Theta'(u)du, \tag{39}$$

and the right Θ-CFI of order ω is:

$$\left(I_b^{\omega,\Theta}f\right)(\ell) = \int_{\ell}^{b} f(u)(\Theta(b)-\Theta(u))^{\omega-1}\Theta'(u)du. \tag{40}$$

If $\Theta(u) = u$, we get the left and right CFI Eqs. (2) and (3) see Abdeljawad (2015).

From Eq. (39), we have

$$
\begin{aligned}
{}_a^m I^{\omega,\Theta} f(\ell) = & \int_a^{\ell} (\Theta(\tau_1)-\Theta(a))^{\omega-1}\Theta'(\tau_1)d\tau_1 \\
& \times \int_a^{\tau_1} (\Theta(\tau_2)-\Theta(a))^{\omega-1}\Theta'(\tau_2)d\tau_2 \cdots \\
& \times \int_a^{\tau_{m-1}} (\Theta(\tau_m)-\Theta(a))^{\omega-1}\Theta'(\tau_m)f(\tau_m)d\tau_m \\
= & \frac{1}{\Gamma(m)}\int_a^{\ell}\left(\frac{(\Theta(\ell)-\Theta(a))^{\omega}-(\Theta(\tau)-\Theta(a))^{\omega}}{\omega}\right)^{m-1} \\
& \times \frac{f(\zeta)\Theta'(\zeta)d\zeta}{(\Theta(\zeta)-\Theta(a))^{1-\omega}}.
\end{aligned} \tag{41}
$$

Definition 3. Let $r \in \mathbb{C}, \mathrm{Rel}(r) > 0$. The left Θ-CFI by

$$
{}_a^r\mathfrak{J}^{\omega,\Theta} f(\ell) = \frac{1}{\Gamma(r)}\int_a^{\ell}\left(\frac{(\Theta(\ell)-\Theta(a))^{\omega}-(\Theta(\zeta)-\Theta(a))^{\omega}}{\omega}\right)^{r-1}\frac{f(\zeta)\Theta'(\zeta)d\zeta}{(\Theta(\zeta)-\Theta(a))^{1-\omega}}. \tag{42}
$$

Remark 1.

- If $\Theta(\tau)=\tau$ and $\omega=1$, we get the RL fractional integral (4).
- If $\omega=1$, we get the Θ-RLI of order r starting at a(10).
- If $\Theta(\tau)=\tau$, we get the RL conformable fractional integral (30).
- If $\Theta(\tau)=\tau$ and $\omega\to 0$, we get the Hadamard fractional integral (16).

Definition 4. Let $r \in \mathbb{C}, \mathrm{Rel}(r) > 0$. The right Θ-CFI is

$$
{}^r\mathfrak{J}_b^{\omega} f(\ell) = \frac{1}{\Gamma(r)}\int_{\ell}^{b}\left(\frac{(\Theta(b)-\Theta(\ell))^{\omega}-(\Theta(b)-\Theta(\zeta))^{\omega}}{\omega}\right)^{r-1}\frac{f(\zeta)\Theta'(\zeta)d\zeta}{(\Theta(b)-\Theta(\zeta))^{1-\omega}}. \tag{43}
$$

Remark 2. We have

- If $\Theta(t)=t$ and $\omega=1$, we get the RLI (5).
- If $\omega=1$, we get the right Θ-RLI of order r, ending at b (11).
- If $\Theta(t)=t$, we get the RL conformable fractional integral (31).
- If $\Theta(t)=t$ and $\omega\to 0$, we get the HI (25).

In Definition 5 present the Θ-CFD.

Definition 5. Let $r \in \mathbb{C}$ and $\mathrm{Rel}(r) \geq 0$. The left and right Θ-CFD, within the framework of Riemann-Liouville type. Specifically, they are defined as follows:

$$
\begin{aligned}
{}_a^r\mathfrak{D}^{\omega,\Theta} f(\ell) = & {}_a^m T^{\omega,\Theta}\left({}_a^{m-r}\mathfrak{J}^{\omega,\Theta}\right) f(\ell) \\
= & \frac{{}_a^m T^{\omega,\Theta}}{\Gamma(m-r)}\int_a^{\ell}\left(\frac{(\Theta(\ell)-\Theta(a))^{\omega}-(\Theta(\tau)-\Theta(a))^{\omega}}{\omega}\right)^{m-r-1} \\
& \times \frac{f(\zeta)\Theta'(\zeta)d\zeta}{(\Theta(\zeta)-\Theta(a))^{1-\omega}},
\end{aligned} \tag{44}
$$

and

$$
\begin{aligned}
{}^r\mathfrak{D}_b^{\omega,\Theta} f(\ell) = & {}^m T_b^{\omega,\Theta}\left({}^{m-r}\mathfrak{J}_b^{\omega}\right) f(\ell) \\
= & \frac{(-1)^{nn}T_b^{\omega,\Theta}}{\Gamma(m-r)}\int_{\ell}^{b}\left(\frac{(\Theta(b)-\Theta(\ell))^{\omega}-(\Theta(b)-\Theta(\zeta))^{\omega}}{\omega}\right)^{m-r-1} \\
& \times \frac{f(\zeta)\Theta'(\zeta)d\zeta}{(\Theta(b)-\Theta(\zeta))^{1-\omega}},
\end{aligned} \tag{45}
$$

where

$$
m=[\mathrm{Rel}(r)]+1, \quad {}_a^m T^{\omega,\Theta} = \underbrace{{}_aT^{\omega,\Theta}\,{}_aT^{\omega,\Theta}\cdots{}_aT^{\omega,\Theta}}_{m\ \text{times}}, \quad {}^m T_b^{\omega,\Theta} = \underbrace{T_b^{\omega,\Theta}T_b^{\omega,\Theta}\cdots T_b^{\omega,\Theta}}_{m\ \text{times}}, \tag{46}
$$

with ${}_aT^{\omega,\Theta}$ and $T_b^{\omega,\Theta}$ are presented in Eq. (37).

Remark 3. We have

- If $\Theta(t)=t$ and $\omega=1$, The Eq. (44) coincides with Eq. (12).
- If $\omega=1$, The Eq. (44) coincides with Eq. (32).
- If $\Theta(t)=t$ and $\omega\to 0$, The Eq. (44) coincides with Eq. (18).
- If $\omega=1$, The Eq. (44) coincides with the Eq. (11).
- If $\Theta(t)=t$ and $\omega=1$, we get Eq. (13).
- If $\omega=1$, we get Eq. (33).
- If $\Theta(t)=t$ and $\omega\to 0$, the Eq. (45) coincides with the Eq. (19).

Now we have some properties of the Θ-CFD and Θ-CFI.

Theorem 1. *Let $r,\gamma \in \mathbb{R}_+^*$. We have*

$$
{}_a^r\mathfrak{J}^{\omega,\Theta}\left({}^{\gamma}\mathfrak{J}^{\omega,\Theta}\right) f(\ell) = {}_a^{r+\gamma}\mathfrak{J}^{\omega,\Theta} f(\ell) \ \text{and}\ {}^r\mathfrak{J}_b^{\omega,\Theta}\left({}^{\gamma}\mathfrak{J}_b^{\omega,\Theta}\right) f(\ell) = {}^{r+\gamma}\mathfrak{J}_b^{\omega,\Theta} f(\ell). \tag{47}
$$

Proof.

$$
\begin{aligned}
{}_a^r\mathfrak{J}^{\omega,\Theta}\left({}_a^{\gamma}\mathfrak{J}^{\omega,\Theta}\right) f(\ell) = & \frac{1}{\Gamma(r)\Gamma(\gamma)} \\
& \times \int_a^{\ell}\int_a^{\mu}\left(\frac{(\Theta(\ell)-\Theta(a))^{\omega}-(\Theta(\mu)-\Theta(a))^{\omega}}{\omega}\right)^{r-1} \\
& \times \left(\frac{(\Theta(\mu)-\Theta(a))^{\omega}-(\Theta(\xi)-\Theta(a))^{\omega}}{\omega}\right)^{\gamma-1} f(\xi) \\
& \times \frac{\Theta'(\xi)d\xi}{(\Theta(\xi)-\Theta(a))^{1-\omega}}\frac{\Theta'(\mu)d\mu}{(\Theta(\mu)-\Theta(a))^{1-\omega}} \\
= & \frac{1}{\Gamma(r)\Gamma(\gamma)\omega^{r+\gamma-2}}\int_a^{\ell}\int_{\xi}^{\ell}((\Theta(\ell)-\Theta(a))^{\omega}-(\Theta(\mu)-\Theta(a))^{\omega})^{r-1} \\
& \times ((\Theta(\mu)-\Theta(a))^{\omega}-(\Theta(\xi)-\Theta(a))^{\omega})^{\gamma-1} f(\xi) \\
& \times \frac{\Theta'(\xi)d\xi}{(\Theta(\xi)-\Theta(a))^{1-\omega}}\frac{\Theta'(\mu)d\mu}{(\Theta(\mu)-\Theta(a))^{1-\omega}} \\
= & \frac{1}{\Gamma(r)\Gamma(\gamma)\omega^{r+\gamma-1}}\int_a^{\ell}((\Theta(\ell)-\Theta(a))^{\omega}-(\Theta(\xi)-\Theta(a)))^{r+\gamma-1} \\
& \times f(\xi)\frac{d\xi}{(\Theta(\xi)-\Theta(a))^{1-\omega}} \\
& \times \int_0^1 (1-z)^{r-1}z^{\gamma-1}dy \\
= & \frac{1}{\Gamma(r+\gamma)}\int_a^{\ell}\left(\frac{(\Theta(\ell)-\Theta(a))^{\omega}-(\Theta(u)-\Theta(a))}{\omega}\right)^{r+\gamma-1} \\
& \times f(\xi)\frac{\Theta'(\xi)d\xi}{(\Theta(\xi)-\Theta(a))^{1-\omega}} \\
= & {}_a^{r+\gamma}\mathfrak{J}^{\omega,\Theta} f(\ell).
\end{aligned}
$$

The second equation can be demonstrated using a similar method. □

Lemma 1. *We have*

$$
{}_a^r\mathfrak{J}^{\omega,\Theta}(\Theta(\ell)-\Theta(a))^{\omega v-\omega} = \frac{1}{\omega^r}\frac{\Gamma(v)}{\Gamma(r+v)}(\Theta(\ell)-\Theta(a))^{\omega(r+v-1)}, \tag{48}
$$

and

$$
{}^r\mathfrak{J}_b^{\omega,\Theta}(\Theta(b)-\Theta(\ell))^{\omega v-\omega} = \frac{1}{\omega^r}\frac{\Gamma(v)}{\Gamma(r+v)}(\Theta(b)-\Theta(\ell))^{\omega(r+v-1)}, \tag{49}
$$

with $v \in \mathbb{R}_+^*$.

Proof. Since

$$
\begin{aligned}
{}_a^r\mathfrak{J}^{\omega,\Theta}(\Theta(\ell)-\Theta(a))^{\omega v-\omega} = & \frac{1}{\Gamma(r)}\int_a^{\ell}\left(\frac{(\Theta(\ell)-\Theta(a))^{\omega}-(\Theta(t)-\Theta(a))^{\omega}}{\omega}\right)^{r-1} \\
& \times (\Theta(\xi)-\Theta(a))^{\omega v-\omega}\frac{\Theta'(\xi)d\xi}{(\Theta(\xi)-\Theta(a))^{1-\omega}}.
\end{aligned}
$$

Letting $u = \left(\frac{\Theta(t)-\Theta(a)}{\Theta(\ell)-\Theta(a)}\right)^{\omega}$, we obtain

$$
\begin{aligned}
{}^{r}_{a}\mathfrak{I}^{\omega,\Theta}(\Theta(\ell)-\Theta(a))^{\omega v-\omega} &= \frac{(\Theta(\ell)-\Theta(a))^{\omega(r+v-1)}}{\Gamma(r)\omega^{r}}\int_{0}^{1}(1-u)^{r-1}u^{v-1}du \\
&= \frac{\Gamma(v)}{\omega^{r}\Gamma(r+v)}(\Theta(\ell)-\Theta(a))^{\omega(r+v-1)}.
\end{aligned}
$$

Eq. (49) can be proved in a similar method. □

Lemma 2. *Let $m-\omega \in \mathbb{R}^{*}_{+}$, we get*

$$\left[{}^{r}_{a}\mathfrak{D}^{\omega,\Theta}(\Theta(t)-\Theta(a))^{\omega v-\omega}\right](\ell) = \omega^{r}\frac{\Gamma(v)}{\Gamma(v-r)}(\Theta(\ell)-\Theta(a))^{\omega(v-r-1)}, \quad (50)$$

and

$$\left[{}^{r}\mathfrak{D}^{\omega,\Theta}_{b}(\Theta(b)-\Theta(t))^{\omega v-\omega}\right](\ell) = \omega^{r}\frac{\Gamma(v)}{\Gamma(v-r)}(\Theta(b)-\Theta(\ell))^{\omega(v-r-1)}. \quad (51)$$

Proof. The demonstration can be derived through a simple and direct calculation. □

Remark 4. It can be shown that

$${}^{r}_{a}\mathfrak{D}^{\omega,\Theta}f = {}^{r}_{a}\mathfrak{I}^{-\omega,\Theta}f \quad \text{and} \quad {}^{r}\mathfrak{D}^{\omega,\Theta}_{b}f = {}^{r}\mathfrak{I}^{-\omega,\Theta}_{b}f. \quad (52)$$

3. Θ-CFD on the spaces $C^{m}_{\omega,\Theta,a}$ and $C^{m}_{\omega,\Theta,b}$

In this section, we explore the Θ-CFD of functions within the spaces specified by the following definitions.

Definition 6. Let $0<\omega\le 1$, $n\in\mathbb{N}^{*}$ and an interval $[a,b]$ define

$$C^{m}_{\omega,\Theta,a}([a,b]) = \left\{ f:[a,b]\to\mathbb{R} \Big/ {}^{m-1}_{a}T^{\omega,\Theta}f \in I_{\omega,\Theta}([a,b])\right\}, \quad (53)$$

and

$$C^{m}_{\omega,\Theta,b}([a,b]) = \left\{ f:[a,b]\to\mathbb{R} \Big/ {}^{m-1}T^{\omega,\Theta}_{b}f \in {}^{\omega,\Theta}I([a,b])\right\}, \quad (54)$$

with

$$
\begin{aligned}
I_{\omega,\Theta}([a,b]) = \Big\{ & f:[a,b]\to\mathbb{R} : f(\ell) = \left({}_{a}I^{\omega,\Theta}g\right)(\ell)+f(a), \\
& \text{for some } g\in L_{\omega,\Theta}(a) \Big\},
\end{aligned}
$$

and

$$
\begin{aligned}
{}^{\omega,\Theta}I([a,b]) = \Big\{ & f:[a,b]\to\mathbb{R} : f(\ell) = \left(I^{\omega,\Theta}_{b}g\right)(\ell)+f(b), \\
& \text{for some } g\in L_{\omega,\Theta}(b)\Big\},
\end{aligned}
$$

where

$$L_{\omega,\Theta}(a) = \left\{ f:[a,b]\to\mathbb{R} : \left({}_{a}I^{\omega,\Theta}f\right)(\ell) \text{ exists for all } f\in[a,b] \right\},$$

and

$$L_{\omega,\Theta}(b) = \left\{ f:[a,b]\to\mathbb{R} : \left(I^{\omega,\Theta}_{b}f\right)(\ell) \text{ exists for all } f\in[a,b] \right\}.$$

Lemma 3. *For $\omega>0$, $f\in C^{m}_{\omega,\Theta,a}([a,b])$ if and only if f can be expressed in the following manner:*

$$
\begin{aligned}
f(\ell) = & \frac{1}{(m-1)!}\int_{a}^{\ell}\left(\frac{(\Theta(\ell)-\Theta(a))^{\omega}-(\Theta(\xi)-\Theta(a))^{\omega}}{\omega}\right)^{m-1} \\
& \times \frac{g(\xi)\Theta'(\xi)d\xi}{(\Theta(\xi)-\Theta(a))^{1-\omega}} \\
& + \sum_{k=0}^{m-1}\frac{{}^{k}_{a}T^{\omega,\Theta}f(a)}{\omega^{k}k!}(\Theta(\ell)-\Theta(a))^{\omega k},
\end{aligned}
\quad (55)
$$

where $g(u) = {}^{m}_{a}T^{\omega,\Theta}f(u)$.

Proof. Since $f\in C^{m}_{\omega,\Theta,a}([a,b])$ we have ${}^{m-1}_{a}T^{\omega,\Theta}f\in I_{\omega,\Theta}([a,b])$ and thus

$${}^{m-1}_{a}T^{\omega,\Theta}f(\ell) = \int_{a}^{\ell}g(\tau)\frac{\Theta'(\tau)d\tau}{(\Theta(\tau)-\Theta(a))^{1-\omega}} + {}^{m-1}_{a}T^{\omega,\Theta}f(a), \quad (56)$$

where g is a continuous function. So

$$
\begin{aligned}
\frac{(\Theta(\ell)-\Theta(a))^{1-\omega}}{\Theta'(\ell)}\frac{d}{d\ell}\left({}^{m-2}_{a}T^{\omega,\Theta}f(\ell)\right) = & \int_{a}^{\ell}g(\zeta)\frac{\Theta'(\zeta)d\zeta}{(\Theta(\zeta)-\Theta(a))^{1-\omega}} \\
& + {}^{m-1}_{a}T^{\omega,\Theta}f(a),
\end{aligned}
\quad (57)
$$

and

$$
\begin{aligned}
\frac{d}{d\ell}\left({}^{m-2}_{a}T^{\omega,\Theta}f(\ell)\right) = & \frac{\Theta'(\ell)}{(\Theta(\ell)-\Theta(a))^{1-\omega}}\int_{a}^{\ell}g(\tau)\frac{\Theta'(\tau)d\tau}{(\Theta(\tau)-\Theta(a))^{1-\omega}} \\
& + \frac{\Theta'(\ell)\times{}^{m-1}_{a}T^{\omega,\Theta}f(a)}{(\Theta(\ell)-\Theta(a))^{1-\omega}}.
\end{aligned}
\quad (58)
$$

Integrating, we get

$$
\begin{aligned}
{}^{m-2}_{a}T^{\omega,\Theta}f(\ell) = & \int_{a}^{\ell}\frac{(\Theta(\ell)-\Theta(a))^{\omega}-(\Theta(\tau)-\Theta(a))^{\omega}}{\omega}\frac{g(\tau)\Theta'(\tau)d\tau}{(\Theta(\tau)-\Theta(a))^{1-\omega}} \\
& + {}^{m-1}_{a}T^{\omega,\Theta}f(a)\frac{(\Theta(\ell)-\Theta(a))^{\omega}}{\omega} + {}^{m-2}_{a}T^{\omega,\Theta}f(a).
\end{aligned}
\quad (59)
$$

Dividing by $(\Theta(\ell)-\Theta(a))^{1-\omega}$ and integrating once more we get

$$
\begin{aligned}
{}^{m-3}_{a}T^{\omega,\Theta}f(\ell) = & \frac{1}{2!}\int_{a}^{\ell}\left(\frac{(\Theta(\ell)-\Theta(a))^{\omega}-(\Theta(\tau)-\Theta(a))^{\omega}}{\omega}\right)^{2} \\
& \times\frac{g(\tau)\Theta'(\tau)d\tau}{(\Theta(\tau)-\Theta(a))^{1-\omega}} \\
& + {}^{m-2}_{a}T^{\omega,\Theta}f(a)\frac{(\Theta(\ell)-\Theta(a))^{2\omega}}{2\omega^{2}} \\
& + {}^{m-2}_{a}T^{\omega,\Theta}f(a)\frac{(\Theta(\ell)-\Theta(a))^{\omega}}{\omega} + {}^{m-3}_{a}T^{\omega,\Theta}f(a).
\end{aligned}
\quad (60)
$$

Iterating the described process an additional $m-3$ times yields:

$$
\begin{aligned}
f(\ell) = & \frac{1}{(m-1)!}\int_{a}^{\ell}\left(\frac{(\Theta(\ell)-\Theta(a))^{\omega}-(\Theta(\zeta)-\Theta(a))^{\omega}}{\omega}\right)^{m-1} \\
& \times\frac{g(\zeta)\Theta'(\zeta)d\zeta}{(\Theta(\zeta)-\Theta(a))^{1-\omega}} \\
& + \sum_{k=0}^{m-1}\frac{{}^{k}_{a}T^{\omega,\Theta}f(a)}{\omega^{k}k!}(\Theta(\ell)-\Theta(a))^{\omega k}.
\end{aligned}
\quad (61)
$$

It is clear from (56) that $g(\tau) = {}^{m}_{a}T^{\omega,\Theta}f(\tau)$.

The proof of sufficiency involves the application of the operator ${}^{m}_{a}T^{\omega,\Theta}$ to both sides of (65). □

Regarding the right-Θ-CFD, a parallel lemma can be formulated.

Lemma 4. *$f\in C^{m}_{\omega,\Theta,b}([a,b])$ if and only if*

$$
\begin{aligned}
f(\ell) = & \frac{1}{(m-1)!}\int_{\ell}^{b}\left(\frac{(\Theta(b)-\Theta(\ell))^{\omega}-(\Theta(b)-\Theta(u))^{\omega}}{\omega}\right)^{m-1} \\
& \times\frac{\Theta'(u)\left({}^{m}T^{\omega,\Theta}_{b}f\right)(u)}{(\Theta(b)-\Theta(u))^{1-\omega}}du \\
& + \sum_{k=0}^{m-1}\frac{(-1)^{k}{}^{k}T^{\omega,\Theta}_{b}f(b)}{k!}\frac{(\Theta(b)-\Theta(\ell))^{\omega k}}{\omega^{k}}.
\end{aligned}
\quad (62)
$$

Proof. The demonstration closely resembles the proof presented in Lemma 3. □

In Theorem 2, we state the Θ-CFD in $C^{m}_{\omega,\Theta,a}$ and $C^{m}_{\omega,\Theta,b}$

Theorem 2. *Let $r\in\mathbb{R}^{*}_{+}$, $m=[r]+1$. We have*

1. Let $f \in C^m_{\omega,\Theta,a}([a,b])$, the left Θ-CFD is

$$ {}^r_a\mathfrak{D}^{\omega,\Theta} f(\ell) = \frac{1}{\Gamma(m-r)} \int_a^\ell \left(\frac{(\Theta(\ell)-\Theta(a))^\omega - (\Theta(t)-\Theta(a))^\omega}{\omega} \right)^{m-r-1} \times \frac{{}^m_aT^{\omega,\Theta} f(t)\Theta'(t)dt}{(\Theta(t)-\Theta(a))^{1-\omega}} + \sum_{k=0}^{m-1} \frac{\left({}^k_aT^{\omega,\Theta} f(a)\right)(\Theta(\ell)-\Theta(a))^{\omega(k-r)}}{\omega^{k-r}\Gamma(k-r+1)}. \tag{63} $$

2. Let $f \in C^m_{\omega,\Theta,b}([a,b])$, the right Θ-CFD is

$$ {}^r\mathfrak{D}_b^{\omega,\Theta} f(\ell) = \frac{1}{\Gamma(m-r)} \int_\ell^b \left(\frac{(\Theta(b)-\Theta(\ell))^\omega - (\Theta(b)-\Theta(t))^\omega}{\omega} \right)^{m-r-1} \times \frac{{}^mT_b^{\omega,\Theta} f(t)}{(\Theta(b)-\Theta(t))^{1-\omega}}\Theta'(t)dt + \sum_{k=0}^{m-1} \frac{\left((-1)^{kk}T_b^{\omega,\Theta} f(b)\right)(\Theta(b)-\Theta(\ell))^{\omega(k-r)}}{\omega^{k-r}\Gamma(k-r+1)}. \tag{64} $$

Proof. Since $f \in C^m_{\omega,\Theta,a}[a,b]$, from Lemma 3, we have

$$ f(\ell) = \frac{1}{(m-1)!} \int_a^\ell \left(\frac{(\Theta(\ell)-\Theta(a))^\omega - (\Theta(t)-\Theta(a))^\omega}{\omega} \right)^{m-1} \times \frac{\Theta'(t)\left({}^m_aT^{\omega,\Theta} f\right)(t)}{(\Theta(t)-\Theta(a))^{1-\omega}} dt + \sum_{k=0}^{m-1} \frac{{}^m_aT^{\omega,\Theta} f(a)}{k!} \frac{(\Theta(\ell)-\Theta(a))^{\omega k}}{\omega^k}. \tag{65} $$

Therefore we have

$$ {}^r_a\mathfrak{D}^{\omega,\Theta} f(\ell) = \frac{{}^m_aT^{\omega,\Theta}}{(m-1)!\Gamma(m-r)} \int_a^\ell \int_a^t \left(\frac{(\Theta(\ell)-\Theta(a))^\omega - (\Theta(t)-\Theta(a))^\omega}{\omega} \right)^{m-r-1} \times \left(\frac{(\Theta(t)-\Theta(a))^\omega - (\Theta(u)-\Theta(a))^\omega}{\omega} \right)^m \times \left({}^m_aT^{\omega,\Theta} f(u)\right) \frac{\Theta'(u)du}{(\Theta(u)-\Theta(a))^{1-\omega}} \frac{\Theta'(t)dt}{(\Theta(t)-\Theta(a))^{1-\omega}} + \sum_{k=0}^{m-1} \frac{{}^m_aT^{\omega,\Theta} f(a)}{k!\omega^k} \omega^r \frac{\Gamma(k+1)}{\Gamma(k+1-r)} (\Theta(\ell)-\Theta(a))^{\omega(k-r)}. \tag{66} $$

From Lemma 2, and changing the order of integration $y = \frac{(\Theta(t)-\Theta(a))^\omega - (\Theta(u)-\Theta(a))^\omega}{(\Theta(\ell)-\Theta(a))^\omega - (\Theta(u)-\Theta(a))^\omega}$, we have

$$ {}^r_a\mathfrak{D}^{\omega,\Theta} f(\ell) = \left(\frac{\Gamma(m-r)\Gamma(m){}^m_aT^{\omega,\Theta}}{(m-1)!\Gamma(m-r)\Gamma(2m-r)} \times \int_a^\ell \left(\frac{(\Theta(\ell)-\Theta(a))^\omega - (\Theta(u)-\Theta(a))^\omega}{\omega} \right)^{2m-r-1} \times \frac{\left({}^m_aT^{\omega,\Theta} f\right)(u)\Theta'(u)du}{(\Theta(u)-\Theta(a))^{1-\omega}} \right) + + \sum_{k=0}^{m-1} \frac{{}^m_aT^{\omega,\Theta} f(a)}{\omega^{k-r}\Gamma(k+1-r)} (\Theta(\ell)-\Theta(a))^{\omega(k-r)}. \tag{67} $$

The result can be derived by applying the operator ${}^m_aT^{\omega,\Theta}$ to the integral in Eq. (67). The proof of Eq. (64) can be conducted similarly, following the same steps and logic. □

Theorem 3. *Let* $\mathrm{Rel}(r) > m > 0$ *with* $m \in \mathbb{N}$. *We have*

$$ {}^m_aT^{\omega,\Theta}\left({}^r_a\mathfrak{J}^{\omega,\Theta} f(\ell)\right) = {}^{r-m}_{a}\mathfrak{J}^{\omega,\Theta} f(\ell) \quad \text{and} \quad {}^mT_b^{\omega,\Theta}\left({}^r\mathfrak{J}_b^{\omega,\Theta} f(\ell)\right) = {}^{r-m}\mathfrak{J}_b^{\omega,\Theta} f(\ell). \tag{68} $$

Proof. We have

$$ \begin{aligned} &{}^m_aT^{\omega,\Theta}\left({}^r_a\mathfrak{J}^{\omega,\Theta} f(\ell)\right) = \\ &{}^m_aT^{\omega,\Theta}\left[\frac{1}{\Gamma(r)} \int_a^\ell \left(\frac{(\Theta(\ell)-\Theta(a))^\omega - (\Theta(\xi)-\Theta(a))^\omega}{\omega} \right)^{r-1} f(\xi) \times \frac{\Theta'(\xi)d\xi}{(\Theta(\xi)-\Theta(a))^{1-\omega}} \right] \\ &= {}^{m-1}_{a}T^{\omega,\Theta}\left[\frac{1}{\Gamma(r-1)} \int_a^\ell \left(\frac{(\Theta(\ell)-\Theta(a))^\omega - (\Theta(\xi)-\Theta(a))^\omega}{\omega} \right)^{r-2} f(\xi) \times \frac{\Theta'(\xi)d\xi}{(\Theta(\xi)-\Theta(a))^{1-\omega}} \right] \\ &= {}^{m-2}_{a}T^{\omega,\Theta}\left[\frac{1}{\Gamma(r-2)} \int_a^\ell \left(\frac{(\Theta(\ell)-\Theta(a))^\omega - (\Theta(\xi)-\Theta(a))^\omega}{\omega} \right)^{r-3} f(\xi) \times \frac{\Theta'(\xi)d\xi}{(\Theta(\xi)-\Theta(a))^{1-\omega}} \right] \\ &\vdots \\ &= \frac{1}{\Gamma(r-m)} \int_a^\ell \left(\frac{(\Theta(\ell)-\Theta(a))^\omega - (\Theta(\xi)-\Theta(a))^\omega}{\omega} \right)^{r-m-1} f(\xi) \times \frac{\Theta'(\xi)d\xi}{(\Theta(\xi)-\Theta(a))^{1-\omega}} \\ &= {}^{r-m}_{a}\mathfrak{J}^{\omega,\Theta} f(\ell). \end{aligned} $$ □

Corollary 1. *Let* $\mathrm{Rel}(\gamma) < \mathrm{Rel}(r)$, *we have*

$$ {}^\gamma_a\mathfrak{D}^{\omega,\Theta}\left({}^r_a\mathfrak{J}^{\omega,\Theta} f(\ell)\right) = {}^{r-\gamma}_{a}\mathfrak{J}^{\omega,\Theta} f(\ell) \quad \text{and} \quad {}^\gamma\mathfrak{D}_b^{\omega,\Theta}\left({}^r\mathfrak{J}_b^{\omega,\Theta} f(\ell)\right) = {}^{r-\gamma}\mathfrak{J}_b^{\omega,\Theta} f(\ell). \tag{69} $$

Proof. From Theorems 1 and 3, we have

$$ \begin{aligned} {}^\gamma_a\mathfrak{D}^{\omega,\Theta}\left({}^r_a\mathfrak{J}^{\omega} f(\ell)\right) &= {}^m_aT^{\omega,\Theta}\left({}^{m-\gamma}_{a}\mathfrak{J}^{\omega,\Theta}\left({}^r_a\mathfrak{J}^{\omega,\Theta} f(\ell)\right)\right) \\ &= {}^m_aT^{\omega}\left({}^{r+m-\gamma}_{a}\mathfrak{J}^{\omega} f(\ell)\right) \\ &= {}^{r-\gamma}_{a}\mathfrak{J}^{\omega,\Theta} f(\ell). \end{aligned} \tag{70} $$ □

Theorem 4. *Let* $r > 0$ *and* $f \in C^m_{\omega,\Theta,a}[a,b]\left(f \in C^m_{\omega,\Theta,b}[a,b]\right)$. *We get*

$$ {}^r_a\mathfrak{D}^{\omega,\Theta}\left({}^r_a\mathfrak{J}^{\omega,\Theta} f(\ell)\right) = f(\ell) \quad \text{and} \quad {}^r\mathfrak{D}_b^{\omega,\Theta}\left({}^r\mathfrak{J}_b^{\omega,\Theta} f(\ell)\right) = f(\ell). \tag{71} $$

Proof.

$$ \begin{aligned} &{}^r_a\mathfrak{D}^{\omega,\Theta}\left({}^r_a\mathfrak{J}^{\omega,\Theta} f(\ell)\right) \\ &= \frac{{}^m_aT^{\omega,\Theta}}{\Gamma(m-r)\Gamma(r)} \int_a^\ell \int_a^t \left(\frac{(\Theta(\ell)-\Theta(a))^\omega - (\Theta(t)-\Theta(a))^\omega}{\omega} \right)^{m-r} \times \left(\frac{(\Theta(t)-\Theta(a))^\omega - (\Theta(\xi)-\Theta(a))^\omega}{\omega} \right)^{r-1} f(\xi) \times \frac{\Theta'(\xi)d\xi}{(\Theta(\xi)-\Theta(a))^{1-\omega}} \frac{\Theta'(t)dt}{(\Theta(t)-\Theta(a))^{1-\omega}} \\ &= \frac{{}^m_aT^{\omega,\Theta}}{\Gamma(m-r)\Gamma(r)} \int_a^\ell \int_\xi^\ell \times \frac{((\Theta(\ell)-\Theta(a))^\omega - (\Theta(t)-\Theta(a))^\omega)^{m-r-1}((\Theta(t)-\Theta(a))^\omega - (\Theta(\xi)-\Theta(a))^\omega)^{r-1}}{\omega^m} \times \frac{\Theta'(t)dt}{(\Theta(t)-\Theta(a))^{1-\omega}} f(\xi) \frac{\Theta'(\xi)d\xi}{(\Theta(\xi)-\Theta(a))^{1-\omega}} \\ &= \frac{{}^mT^{\omega,\Theta}}{\omega^{m-1}\Gamma(m-r)\Gamma(r)} \int_a^\ell ((\Theta(\ell)-\Theta(a))^\omega - (\Theta(\xi)-\Theta(a))^\omega)^{m-1} f(\xi) \frac{\Theta'(\xi)d\xi}{(\Theta(\xi)-\Theta(a))^{\omega-1}} \times \int_0^1 (1-z)^{m-r-1} z^{r-1} dz \\ &= \frac{{}^m_aT^{\omega,\Theta}}{\Gamma(m)} \int_a^\ell \left(\frac{(\Theta(\ell)-\Theta(a))^\omega - (\Theta(\xi)-\Theta(a))^\omega}{\omega} \right)^{m-1} f(\xi) \frac{\Theta'(\xi)d\xi}{(\Theta(\xi)-\Theta(a))^{1-\omega}} \\ &= {}^m_aT^{\omega,\Theta}\left({}^m_aI^{\omega,\Theta} f(\ell)\right). \end{aligned} $$ □

Theorem 5. *Let* $\mathrm{Rel}(r) > 0$, $m = -[-\mathrm{Rel}(r)]$, $f \in L(a,b)$ *and* ${}^r_a\mathfrak{J}^{\omega,\Theta} f \in C^m_{\omega,\Theta,a}[a,b]$, ${}^r\mathfrak{J}_b^{\omega,\Theta} f \in C^m_{\omega,\Theta,b}[a,b]$. *Then*

$$ {}^r_a\mathfrak{J}^{\omega,\Theta}\left({}^r_a\mathfrak{D}^{\omega,\Theta} f(\ell)\right) = f(\ell) - \sum_{j=1}^{m} \frac{{}^{r-j}_{a}\mathfrak{D}^{\omega,\Theta} f(a)}{\omega^{r-j}\Gamma(r-j+1)} (\Theta(\ell)-\Theta(a))^{\omega r-\omega j}, \tag{72} $$

and

$$ {}^{r}\mathfrak{J}^{\omega,\Theta}_{b}\left({}^{r}\mathfrak{D}^{\omega,\Theta}_{b} f(\ell)\right) = f(\ell) - \sum_{j=1}^{m} \frac{(-1)^{m-jr-j}\mathfrak{D}^{\omega,\Theta}_{b} f(b)}{\omega^{r-j}\Gamma(r-j+1)}(\Theta(b)-\Theta(\ell))^{\omega r-\omega j}. \tag{73} $$

Proof. We have

$$ \begin{aligned} {}^{r}_{a}\mathfrak{J}^{\omega,\Theta}\left({}^{r}_{a}\mathfrak{D}^{\omega,\Theta} f(\ell)\right) &= \frac{1}{\Gamma(r)}\int_a^{\ell}\left(\frac{(\Theta(\ell)-\Theta(a))^{\omega}-(\Theta(t)-\Theta(a))^{\omega}}{\omega}\right)^{r-1} \\ &\quad \left({}^{m}_{a}T^{\omega,\Theta}\left(\begin{matrix} m-r \\ a \end{matrix}\ \mathfrak{J}^{\omega,\Theta} f(t)\right)\right)\frac{\Theta'(t)dt}{(\Theta(t)-\Theta(a))^{1-\omega}} \\ &= \frac{{}^{1}_{a}T^{\omega,\Theta}}{\Gamma(r+1)}\left[\int_a^{\ell}\left(\frac{(\Theta(\ell)-\Theta(a))^{\omega}-(\Theta(t)-\Theta(a))^{\omega}}{\omega}\right)^{r}\right. \\ &\quad \left.\left({}^{m}_{a}T^{\omega,\Theta}\left({}^{m-r}_{a}\mathfrak{J}^{\omega,\Theta} f(t)\right)\right)\frac{\Theta'(t)dt}{(\Theta(t)-\Theta(a))^{1-\omega}}\right]. \end{aligned} $$

Using the integration by parts m times, we get

$$ \begin{aligned} &{}^{r}_{a}\mathfrak{J}^{\omega,\Theta}\left({}^{r}_{a}\mathfrak{D}^{\omega,\Theta} f(\ell)\right) \\ &= {}^{1}_{a}T^{\omega,\Theta}\left[\frac{1}{\Gamma(r-m+1)}\int_a^{\ell}\left(\frac{(\Theta(\ell)-\Theta(a))^{\omega}-(\Theta(t)-\Theta(a))^{\omega}}{\omega}\right)^{r-m}\right. \\ &\times\left({}^{m-r}_{a}\mathfrak{J}^{\omega,\Theta} f(t)\right)\frac{\Theta'(t)dt}{(\Theta(t)-\Theta(a))^{1-\omega}} - \sum_{j=1}^{m}\frac{\left({}^{m-j}_{a}T^{\omega,\Theta}\left({}^{m-r}_{a}\mathfrak{J}^{\omega,\Theta} f(a)\right)\right)}{\Gamma(r+2-j)\omega^{r-j+1}} \\ &\left.\times(\Theta(\ell)-\Theta(a))^{\omega r-\omega j+\omega}\right] \\ &= {}^{1}_{a}T^{\omega,\Theta}\left[{}^{r-m+1}_{a}\mathfrak{J}^{\omega,\Theta}\left({}^{m-r}_{a}\mathfrak{J}^{\omega,\Theta} f(\ell)\right)\right. \\ &\left.-\sum_{j=1}^{m}\frac{\left({}^{m-j}_{a}T^{\omega,\Theta}\left({}^{m-r}_{a}\mathfrak{J}^{\omega,\Theta} f(a)\right)\right)}{\Gamma(r+2-j)\omega^{r-j+1}}(\Theta(\ell)-\Theta(a))^{\omega r-\omega j+\omega}\right]. \end{aligned} $$

Now by using Theorem 1, we get

$$ \begin{aligned} {}^{r}_{a}\mathfrak{J}^{\omega,\Theta}\left({}^{r}_{a}\mathfrak{D}^{\omega,\Theta} f(\ell)\right) &= {}^{1}_{a}T\left[{}^{1}_{a}\mathfrak{J}^{\omega,\Theta} f(\ell) - \sum_{j=1}^{m}\frac{\left({}^{m-j}_{a}T^{\omega,\Theta}\left({}^{m-r}_{a}\mathfrak{J}^{\omega,\Theta} f(a)\right)\right)}{\Gamma(r+2-j)\omega^{r-j+1}}\right. \\ &\quad\left.\times(\Theta(\ell)-\Theta(a))^{\omega r-\omega j+\omega}\right] \\ &= f(\ell) - \sum_{j=1}^{m}\frac{{}^{r-j}_{a}\mathfrak{D}^{\omega,\Theta} f(a)}{\omega^{r-j}\Gamma(r-j+1)}(\Theta(\ell)-\Theta(a))^{\omega r-\omega j}. \quad \square \end{aligned} $$

4. Θ-CFD with the Caputo type

Definition 7. Let $\omega > 0, \mathrm{Rel}(r) \geq 0$ and $m = [\mathrm{Rel}(r)] + 1$.

- If $f \in C^m_{\omega,\Theta,a}$. The left Caputo Θ-CFD of f is

$$ {}^{C}_{a}\mathfrak{D}^{r,\omega,\Theta} f(\ell) := {}^{r}_{a}\mathfrak{D}^{\omega,\Theta}\left[f(t) - \sum_{k=0}^{m-1}\frac{{}^{k}_{a}T^{\omega,\Theta} f(a)}{k!\omega^k}(\Theta(t)-\Theta(a))^{\omega k}\right](\ell). \tag{74} $$

- If $f \in C^m_{\omega,\Theta,b}$. The right Caputo Θ-CFD of f is

$$ {}^{C}\mathfrak{D}^{r,\omega,\Theta}_{b} f(\ell) := {}^{r}\mathfrak{D}^{\omega,\Theta}_{b}\left[f(t) - \sum_{k=0}^{m-1}\frac{(-1)^{kk}T^{\omega}_{b} f(b)}{k!\omega^k}(\Theta(b)-\Theta(t))^{\omega k}\right](\ell). \tag{75} $$

Theorem 6. *Let* $\mathrm{Rel}(r) \geq 0, m = [\mathrm{Rel}(r)] + 1, f \in C^m_{\omega,\Theta,a}([a,b])$ $\left(f \in C^m_{\omega,\Theta,b}([a,b])\right)$. *We have the left and right Θ-CFD in the Caputo type are*

$$ \begin{aligned} {}^{C}_{a}\mathfrak{D}^{r,\omega,\Theta} f(\ell) &= \frac{1}{\Gamma(m-r)} \\ &\quad\times\int_a^{\ell}\left(\frac{(\Theta(\ell)-\Theta(a))^{\omega}-(\Theta(u)-\Theta(a))^{\omega}}{\omega}\right)^{m-r-1} \\ &\quad\times\frac{{}^{m}_{a}T^{\omega,\Theta} f(u)\Theta'(u)du}{(\Theta(u)-\Theta(a))^{1-\omega}} \\ &= {}^{m-r}_{a}\mathfrak{J}^{\omega}\left({}^{m}_{a}T^{\omega,\Theta} f(\ell)\right), \end{aligned} \tag{76} $$

and

$$ \begin{aligned} {}^{C}\mathfrak{D}^{r,\omega,\Theta}_{b} f(\ell) &= \frac{(-1)^m}{\Gamma(m-r)}\int_{\ell}^{b}\left(\frac{(\Theta(b)-\Theta(\ell))^{\omega}-(\Theta(b)-\Theta(u))^{\omega}}{\omega}\right)^{m-r-1} \\ &\quad\times\frac{{}^{m}T^{\omega,\Theta}_{b} f(u)\Theta'(u)du}{(\Theta(b)-\Theta(u))^{1-\omega}} \\ &= {}^{m-r}\mathfrak{J}^{\omega}_{b}\left({}^{m}T^{\omega,\Theta}_{b} f(\ell)\right), \end{aligned} \tag{77} $$

respectively.

Proof. From Eq. (74), Lemma 2 and Theorem 2, we get

$$ \begin{aligned} {}^{C}\mathfrak{D}^{r,\omega,\Theta}_{b} f(\ell) &= {}^{r}_{a}\mathfrak{D}^{\omega,\Theta} f(\ell) - \sum_{k=0}^{m-1}\frac{{}^{k}T^{\omega,\Theta}_{a} f(a)}{\omega^{k-r}k!}\frac{\Gamma(k+1)}{\Gamma(k-r+1)} \\ &\quad\times(\Theta(\ell)-\Theta(a))^{k\omega-r\omega} \\ &= {}^{r}_{a}\mathfrak{D}^{\omega,\Theta} f(\ell) - \sum_{k=0}^{m-1}\frac{{}^{k}T^{\omega,\Theta}_{a} f(a)}{\omega^{k-r}\Gamma(k-r+1)}(\Theta(\ell)-\Theta(a))^{k\omega-r\omega} \\ &= \frac{1}{\Gamma(m-r)}\int_a^{\ell}\left(\frac{(\Theta(\ell)-\Theta(a))^{\omega}-(\Theta(u)-\Theta(a))^{\omega}}{\omega}\right)^{m-r-1} \\ &\quad\times\frac{{}^{m}_{a}T^{\omega,\Theta} f(u)}{(\Theta(u)-\Theta(a))^{1-\omega}}\Theta'(u)du \\ &= {}^{m-r}_{a}\mathfrak{J}^{\omega,\Theta}\left({}^{m}_{a}T^{\omega,\Theta} f(\ell)\right). \end{aligned} $$

The identity (77) is proved by using (75), Lemma 2 and Theorem 2 as well. $\square$

Remark 5. We have

- If $\Theta(t) = t$ and $\omega = 1$, the Eq. (76) coincides with the Eq. (8).
- If $\omega = 1$, the Eq. (76) coincides with the Eq. (14).
- If $\Theta(t) = t$ and $\omega \to 0$, the Eq. (76) coincides with the Eq. (21).
- If $\Theta(t) = t$, the Eq. (76) coincides with the Eq. (35).
- If $\Theta(t) = t$ and $\omega = 1$, the Eq. (77) coincides with the Eq. (9).
- If $\omega = 1$, the Eq. (77) coincides with the Eq. (15).
- If $\Theta(t) = t$ and $\omega \to 0$, the Eq. (77) coincides with the Eq. (22).
- If $\Theta(t) = t$, the Eq. (77) coincides with the Eq. (36).

Lemma 5. *Let* $\mathrm{Rel}(r) > 0$ *with* $m = [\mathrm{Rel}(r)]+1, \mathrm{Rel}(r) \notin \mathbb{N}$ *and* $f \in C[a,b]$. *We have*

$$ {}^{r-l}_{a}\mathfrak{J}^{\omega,\Theta} f(a) = 0, $$

and

$$ {}^{r-l}\mathfrak{J}^{\omega,\Theta}_{b} f(b) = 0, $$

for $l = 0, 1, \ldots, m-1$.

Proof. We have

$$ \left|{}^{r-l}_{a}\mathfrak{J}^{\omega,\Theta} f(\ell)\right| \leq \frac{\|f\|_C}{|\Gamma(r-k)|(\mathrm{Rel}(r)-l)}\frac{(\Theta(\ell)-\Theta(a))^{\omega(\mathrm{Rel}(r)-l)}}{\omega^{\mathrm{Rel}(r)-l}}. $$

The result is obtained by replacing ℓ by a. $\square$

Lemma 6. *Let $R(r) \geq 0, m = [\mathrm{Rel}(r)] + 1$ and ${}_a^m T^{\omega,\Theta} \in C[a,b]$ $\left({}^m T_b^{\omega,\Theta} \in C[a,b]\right)$. We have*

$${}_a^C \mathfrak{D}^{r,\omega,\Theta} f(a) = 0,$$

and

$${}^C \mathfrak{D}_b^{r,\omega,\Theta} f(b) = 0.$$

Proof. We have

$$\left|{}_a^C \mathfrak{D}^{r,\omega,\Theta} f(\ell)\right| \leq \frac{\left\|{}_a^m T^{\omega,\Theta} f\right\|_C}{|\Gamma(m-r)|(m-\mathrm{Rel}(r))} \frac{(\Theta(\ell)-\Theta(a))^{\omega(m-\mathrm{Rel}(r))}}{\omega^{m-\mathrm{Rel}(r)}},$$

and

$$\left|{}^C \mathfrak{D}_b^{r,\omega,\Theta} f(\ell)\right| \leq \frac{\left\|{}^m T_b^{\omega} f\right\|_C}{|\Gamma(m-r)|(m-\mathrm{Rel}(r))} \frac{(\Theta(b)-\Theta(\ell))^{\omega(m-\mathrm{Rel}(r))}}{\omega^{m-\mathrm{Rel}(r)}}. \quad \square$$

Theorem 7. *Let $\mathrm{Rel}(r) > 0, m = [\mathrm{Rel}(r)] + 1$, and $f \in C[a,b]$.*

1. *If $\mathrm{Rel}(r) \notin \mathbb{N}$ or $r \in \mathbb{N}$, we have*
$$ {}_a^C \mathfrak{D}^{r,\omega,\Theta} \left({}_a^r \mathfrak{J}^{\omega,\Theta} f(\ell)\right) = f(\ell); \quad {}^{Cr} \mathfrak{D}_b^{\omega,\Theta} \left({}^r \mathfrak{J}_b^{\omega,\Theta} f(\ell)\right) = f(\ell). \tag{78}$$
2. *If $\mathrm{Rel}(r) \neq 0$ and $\mathrm{Rel}(\omega) \in \mathbb{N}$, we have*

$${}_a^C \mathfrak{D}^{r,\omega,\Theta} \left({}_a^r \mathfrak{J}^{\omega,\Theta} f(\ell)\right) = f(\ell) - \frac{{}_a^{r+1-m} \mathfrak{J}^{\omega,\Theta} f(a)}{\omega^{m-r}\Gamma(m-r)} (\Theta(\ell)-\Theta(a))^{\omega m-\omega r}, \tag{79}$$

$${}^C \mathfrak{D}_b^{r,\omega,\Theta} \left({}^r \mathfrak{J}_b^{\omega,\Theta} f(\ell)\right) = f(\ell) - \frac{{}^{r+1-m} \mathfrak{J}_b^{\omega,\Theta} f(b)}{\omega^{m-r}\Gamma(m-r)} (\Theta(b)-\Theta(\ell))^{\omega m-\omega r}. \tag{80}$$

Proof. From the definition (74) we have

$${}_a^C \mathfrak{D}^{r,\omega,\Theta} \left({}_a^r \mathfrak{J}^{\omega,\Theta} f(\ell)\right) = {}_a^r \mathfrak{D}^{\omega,\Theta} \left({}_a^r \mathfrak{J}^{\omega,\Theta} f(\ell)\right) - \sum_{k=0}^{m-1} \frac{{}_a^k T^{\omega,\Theta}\left({}_a^r \mathfrak{J}^{\omega,\Theta} f(a)\right)(\Theta(\ell)-\Theta(a))}{\omega^{k-r}\Gamma(k-r+1)}.$$

Using Theorem 3 and Theorem 4, we have

$${}_a^C \mathfrak{D}^{r,\omega,\Theta} \left({}_a^r \mathfrak{J}^{\omega,\Theta} f(\ell)\right) = f(\ell) - \sum_{l=0}^{m-1} \frac{{}_a^{r-l} \mathfrak{J}^{\omega,\Theta} f(a)(\Theta(\ell)-\Theta(a))^{\omega-l-r\omega}}{\omega^{l-r}\Gamma(l-r+1)}.$$

If $\mathrm{Rel}(r) \notin \mathbb{N}$, by Lemma 5, we have ${}_a^{r-k} \mathfrak{J}^{\omega,\Theta} f(a) = 0$.

The case $r \in \mathbb{N}$ is trivial. Now if $\mathrm{Rel}(r) \in \mathbb{N}$, it can be proved that ${}_a^{r-l} \mathfrak{J}^{\omega,\Theta} f(a) = 0$ for $l = 0, 1, \ldots, m-2$ using the steps used in proving Lemma 5. $\square$

Theorem 8. *Let $f \in C^m_{\omega,\Theta,a}[a,b]$ $\left(f \in C^m_{\omega,\Theta,b}[a,b]\right)$ and $r \in \mathbb{C}$. We get*

$${}_a^r \mathfrak{J}^{\omega,\Theta} \left({}_a^C \mathfrak{D}^{r,\omega,\Theta} f(\ell)\right) = f(\ell) - \sum_{l=0}^{m-1} \frac{{}_a^l T^{\omega,\Theta} f(a)(\Theta(\ell)-\Theta(a))^{\omega l}}{\omega^l l!}, \tag{81}$$

and

$${}^r \mathfrak{J}_b^{\omega,\Theta} \left({}^C \mathfrak{D}_b^{r,\omega,\Theta} f(\ell)\right) = f(\ell) - \sum_{l=0}^{m-1} \frac{(-1)^{l}\,{}^l T_b^{\omega,\Theta} f(b)(\Theta(b)-\Theta(\ell))^{\omega l}}{\omega^l l!}. \tag{82}$$

Proof. We have

$$\begin{aligned}
{}_a^r \mathfrak{J}^{\omega,\Theta} \left({}_a^C \mathfrak{D}^{r,\omega,\Theta} f(\ell)\right) &= {}_a^r \mathfrak{J}^{\omega,\Theta} \left({}_a^{m-r} \mathfrak{J}^{\omega,\Theta} \left({}_a^m T^{\omega,\Theta} f(\ell)\right)\right) \\
&= {}_a^m \mathfrak{J}^{\omega,\Theta} \left({}_a^m T^{\omega,\Theta} f(\ell)\right) \\
&= f(\ell) - \sum_{j=1}^{m} \frac{{}_a^{m-j} \mathfrak{D}^{\omega,\Theta} f(a)}{\omega^{m-j}\Gamma(m-j+1)} (\Theta(\ell)-\Theta(a))^{(m-j)\omega} \\
&= f(\ell) - \sum_{k=0}^{m-1} \frac{a^m T^{\omega,\Theta} f(a)}{\omega^k k!} (\Theta(\ell)-\Theta(a))^{k\omega}. \quad \square
\end{aligned}$$

Theorem 9. *Let $\mathrm{Rel}(r) \geq 0, \mathrm{Rel}(\gamma) \geq 0, m-1 < \mathrm{Rel}(r) \leq n$ and $m-1 < \mathrm{Rel}(\gamma) \leq m$, $f \in C^{m+n}_{\omega,\Theta,a}[a,b]$ $\left(f \in C^{m+n}_{\omega,\Theta,b}[a,b]\right)$. We have*

$${}_a^C \mathfrak{D}^{r,\omega,\Theta} \left({}_a^C \mathfrak{D}^{\gamma,\omega,\Theta} f(\ell)\right) = {}_a^C \mathfrak{D}^{(r+\gamma),\omega,\Theta} f(\ell), \tag{83}$$

and

$${}^C \mathfrak{D}_b^{r,\omega,\Theta} \left({}^C \mathfrak{D}_b^{\gamma,\omega,\Theta} f(\ell)\right) = {}^C \mathfrak{D}_b^{(r+\gamma),\omega,\Theta} f(\ell). \tag{84}$$

Proof. The proof can be accomplished by utilizing Theorems 1, 4, 6, and Lemma 6. $\square$

5. Conclusion

This study introduced and analyzed novel fractional derivatives and integrals obtained through an iterative process involving conformable integrals with respect to another function. We successfully derived both left and right fractional Θ-conformable integrals, and based on the Riemann–Liouville and Caputo definitions, we established left and right fractional Θ-conformable derivatives. Rigorous mathematical proofs confirmed that these fractional operators exhibit properties analogous to their classical counterparts. Additionally, we defined fractional derivatives for functions within specific spaces, aiming to elucidate the relationship between these novel fractional differential operators.

While classical fractional calculus is widely recognized for its ability to uncover hidden dynamics in complex systems, each nonlocal system possesses unique characteristics that may not be adequately captured by existing fractional integrals and derivatives. Our proposed fractional operators, although reducible to established operators like Riemann–Liouville, Caputo, and Hadamard under specific conditions, introduce new generalized fractional operators that extend beyond the scope of these established ones.

We anticipate that these novel fractional operators may offer fresh insights into fractional variational problems, optimal control problems, and the modeling of intricate systems. The dependence on two parameters, including the Θ-conformable operator parameter that enhances the detection of memory effects, represents a significant advantage of these operators. Overall, this study provides a foundation for further exploration of these generalized fractional operators and their potential applications in uncovering the dynamics of complex systems. One direction is to expand these two papers (Hogeme et al., 2024; Negero et al., 2023) using this definition.

CRediT authorship contribution statement

Lakhlifa Sadek: Writing – original draft, Visualization, Validation, Supervision, Methodology, Conceptualization. **Dumitru Baleanu:** Validation, Supervision. **Mohammed S. Abdo:** Writing – original draft, Validation, Formal analysis, Conceptualization. **Wasfi Shatanawi:** Validation, Funding acquisition, Formal analysis.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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