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ORIGINAL ARTICLE

Numerical solution of the linear two-dimensional Fredholm integral equations of the second kind via two-dimensional triangular orthogonal functions

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Abstract In this paper, we will give some results for developing the two-dimensional triangular orthogonal functions (2D-TFs) for numerical solution of the linear two-dimensional Fredholm integral equations of the second kind. The product of 2D-TFs and some formulas for calculating definite integral of them are derived and utilized to reduce the solution of two-dimensional Fredholm integral equation to the solution of algebraic equations. Also a theorem is proved for convergence analysis. Numerical examples are presented and results are compared with analytical solution to demonstrate the validity and applicability the method.

1. Introduction

Many problems in engineering and mechanics can be transformed into two-dimensional Fredholm integral equations of the second kind. For example, it is usually required to solve Fredholm integral equations in the calculation of plasma physics (Farengo et al., 1983). There are many works on developing and analyzing numerical methods for solving Fredholm integral equations of the second kind (see Alipanah and Esmaeili, 2009; Alpert et al., 1993; Anselon, 1971; Atkinson, 1976, 1997; Baker, 1977; Delves and Mohamed, 1985). The subject of the presented paper is applying the 2D-TFs method for solving two-dimensional linear Fredholm integral equations. For this purpose we consider the two-dimensional Fredholm integral equations of the form

$$u_1(\tau,\mu) = f_1(\tau,\mu) + \int_a^b \int_c^d K_1(\tau,\mu,\lambda,\eta)u(\lambda,\eta)\,d\lambda\,d\eta; \quad (\tau,\mu)\in D, \tag{1}$$

where $f_1(\tau,\mu)$ and $K_1(\tau,\mu,\lambda,\eta)$ are given continuous functions defined, respectively, on $D = L^2([a,b]\times[c,d])$, $E = D\times D$ and $u_1(\tau,\mu)$ is unknown on D.

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2. Review of two-dimensional block pulse functions (2D-BPFs)

As shown in Maleknejad et al. (2010), a set of 2D-BPFs $\phi_{i,j}(x,y)$ $(i=0,1,\ldots,m_1-1; j=0,1,\ldots,m_2-1)$ is defined in the region of $x \in [0,1)$ and $y \in [0,1)$ as:

$$\phi_{i,j}(x,y) = \begin{cases} 1, & ih_1 \leqslant x < (i+1)h_1 \text{ and } jh_2 \leqslant y < (j+1)h_2, \\ 0, & \text{Otherwise}, \end{cases} \tag{2}$$

where m_1, m_2 are arbitrary positive integers, and $h_1 = \frac{1}{m_1}, h_2 = \frac{1}{m_2}$.

According to (2), the interval $[0,1)$ and $[0,1)$ are, respectively, divided into m_1 and m_2 subintervals.

One of the important properties of the 2D-BPFs is the disjointness of them,

$$\phi_{i_1,j_1}(x,y)\phi_{i_2,j_2}(x,y) = \begin{cases} \phi_{i_1,j_1}(x,y), & i_1 = i_2 \text{ and } j_1 = j_2, \\ 0, & \text{Otherwise}, \end{cases} \tag{3}$$

where $i_1, i_2 = 0,1,\ldots,m_1-1;\ j_1,j_2 = 0,1,\ldots,m_2-1$.

The orthogonality of 2D-BPFs is derived immediately from

$$\int_0^1 \int_0^1 \phi_{i_1,j_1}(x,y)\phi_{i_2,j_2}(x,y)\,dy\,dx = \begin{cases} h_1h_2, & i_1 = i_2 \text{ and } j_1 = j_2, \\ 0, & \text{Otherwise}. \end{cases} \tag{4}$$

The other property is completeness. For every $f(x,y) \in L^2([0,1)\times[0,1))$ when m_1 and m_2 approach to the infinity, Parseval's identity holds:

$$\int_0^1 \int_0^1 f^2(x,y)\,dy\,dx = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} f_{i,j}^2 \|\phi_{i,j}(x,y)\|^2, \tag{5}$$

where

$$f_{i,j} = \frac{1}{h_1h_2}\int_0^1 \int_0^1 f(x,y)\phi_{i,j}(x,y)\,dy\,dx. \tag{6}$$

In vector form, an arbitrary function can be expanded in the form $f(x,y) = F^T\phi(x,y) = \phi^T(x,y)F$, where

$$F = [f_{0,0},\ldots,f_{0,m_2-1},\ldots,f_{m_1-1,0},\ldots,f_{m_1-1,m_2-1}]^T,$$

with $f_{i,j}$'s as defined in (6) (see Maleknejad et al., 2010).

3. Two-dimensional triangular orthogonal functions

We usually call the triangular orthogonal functions containing one variable as one-dimensional (1D) triangular orthogonal functions (1D-TFs) and those containing two variables as two-dimensional (2D) triangular orthogonal functions (2D-TFs). 1D-TFs have been widely used for solving different problems (Babolian et al., 2008; Deb et al., 2006). A complete detail for 1D-TFs is given in Babolian et al. (2008, 2006). These discussions can also be extended to the 2D-TFs.

3.1. Definition

In the following, we have dissected a 2D-BPFs into two 2D-TFs as shown in Fig. 1. Thus, we have

$$\phi_{0,0}(x,y) = T1_{0,0}(x,y) + T2_{0,0}(x,y). \tag{7}$$

Now, we demonstrate the construction of 2D-TFs according to

$$\phi_{i,j}(x,y) = T1_{i,j}(x,y) + T2_{i,j}(x,y), \tag{8}$$

where $T1_{i,j}(x,y)$ is defined as

$$T1_{i,j}(x,y) = \begin{cases} 1 - \frac{y-jh_2}{h_2}, & ih_1 \leqslant x < (i+1)h_1 \\ & \text{and } jh_2 \leqslant y < (j+1)h_2, \\ 0, & \text{Otherwise}, \end{cases} \tag{9}$$

and $T2_{i,j}(x,y)$ is defined as

$$T2_{i,j}(x,y) = \begin{cases} \frac{y-jh_2}{h_2}, & ih_1 \leqslant x < (i+1)h_1 \text{ and } jh_2 \leqslant y < (j+1)h_2, \\ 0, & \text{Otherwise}, \end{cases} \tag{10}$$

where $h_1 = \frac{1}{m_1}, h_2 = \frac{1}{m_2}$, $i = 0,1,\ldots,m_1-1, j = 0,1,\ldots, m_2-1$. Therefore, for each $\phi_{i,j}(x,y), i = 0,1,\ldots,m_1-1$, $j = 0,\ 1,\ldots,m_2-1$, $T1_{i,j}(x,y)$ and $T2_{i,j}(x,y)$ can be constructed as above.

3.2. Vector forms

We can generate two vectors of orthogonal 2D-TFs, namely $T1(x,y)$ and $T2(x,y)$, such that

$$\phi(x,y) = T1(x,y) + T2(x,y); \quad (x,y) \in [0,1)\times[0,1). \tag{11}$$

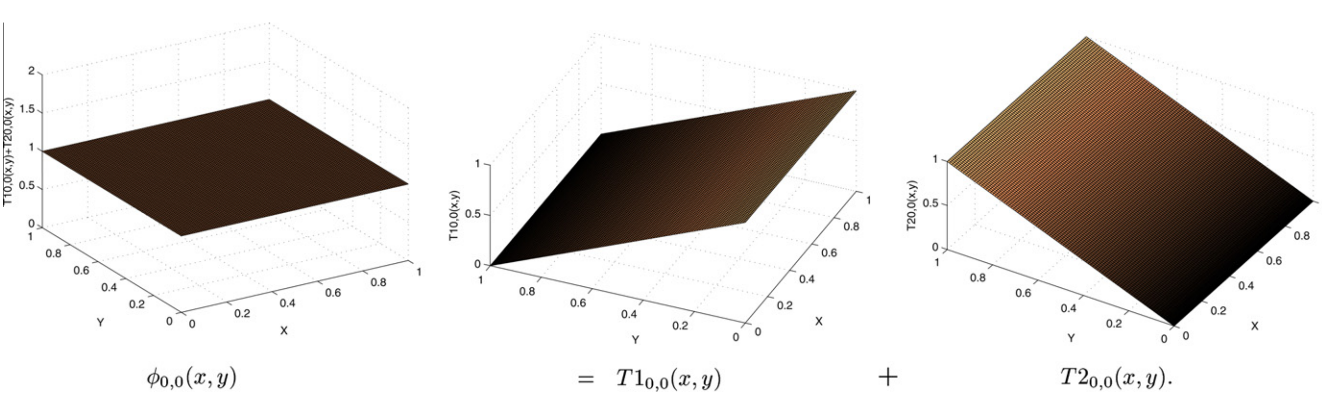

Figure 1 Dissection of $\phi_{0,0}(x,y)$ into two 2D-TFs.

It could be said that these two vectors are complementary to each other as far as 2D-BPFs are considered. We call $T1(x,y)$ and $T2(x,y)$ the left-handed two-dimensional triangular functions (LH2D-TFs) and the right-handed two-dimensional triangular functions (RH2D-TFs), respectively. Now, if we divide the interval $[0,1)\times[0,1)$ to m_1m_2 equal parts, we have

$$T1(x,y)=[T1_{0,0}(x,y),\ldots,T1_{0,m_2-1}(x,y),\ldots,T1_{m_1-1,0}(x,y),\ldots,T1_{m_1-1,m_2-1}(x,y)]^T, \quad (12)$$

$$T2(x,y)=[T2_{0,0}(x,y),\ldots,T2_{0,m_2-1}(x,y),\ldots,T2_{m_1-1,0}(x,y),\ldots,T2_{m_1-1,m_2-1}(x,y)]^T. \quad (13)$$

The orthogonality of LH2D-TFs set (similarly RH2D-TFs set) resulted from mutual disjointness of LH2D-TFs (and RH2D-TFs), i.e. for $i_1,i_2=0,1,\ldots,m_1-1, j_1,j_2=0,1,\ldots,m_2-1$,

$$\int_0^1\int_0^1 T1_{i_1,j_1}(x,y).T2_{i_2,j_2}(x,y)=\begin{cases}\frac{h_1h_2}{3}, & i_1=i_2 \text{ and } j_1=j_2,\\ 0, & \text{Otherwise.}\end{cases} \quad (14)$$

The following properties of the product of two 2D-TFs vectors will be used:

$$T2(x,y)T2^T(x,y)\simeq\begin{pmatrix} T2_{0,0}(x,y) & 0 & \cdots & 0\\ \ddots & \vdots & \vdots & \vdots\\ 0 & T2_{0,m_2-1}(x,y) & \cdots & 0\\ \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & T2_{m_1-1,0}(x,y) & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & T2_{m_1-1,m_2-1}(x,y)\end{pmatrix}, \quad (15)$$

$$T1(x,y)T1^T(x,y)\simeq\begin{pmatrix} T1_{0,0}(x,y) & 0 & \cdots & 0\\ \ddots & \vdots & \vdots & \vdots\\ 0 & T1_{0,m_2-1}(x,y) & \cdots & 0\\ \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & T1_{m_1-1,0}(x,y) & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & T1_{m_1-1,m_2-1}(x,y)\end{pmatrix} \quad (16)$$

and

$$T1(x,y)T2^T(x,y)\simeq\mathbf{0}, \quad (17)$$

$$T2(x,y)T1^T(x,y)\simeq\mathbf{0}, \quad (18)$$

where $\mathbf{0}$ is the zero $m_1m_2\times m_1m_2$ matrix. To prove the above properties notice that, functions $T1_{i,j}(x,y)$ and $T2_{i,j}(x,y)$ also can be presented by

$$T1_{i,j}(x,y)=u_{ih_1,jh_2}(x,y)-\frac{y-jh_2}{h_2}u_{ih_1,jh_2}(x,y)+\frac{y-(j+1)h_2}{h_2}u_{ih_1,(j+1)h_2}(x,y), \quad (19)$$

$$T2_{i,j}(x,y)=\frac{y-jh_2}{h_2}u_{ih_1,jh_2}(x,y)-\frac{y-(j+1)h_2}{h_2}u_{ih_1,(j+1)h_2}(x,y)-u_{ih_1,(j+1)h_2}(x,y), \quad (20)$$

for $i=0,1,\ldots,m_1-1, j=0,1,\ldots,m_2-1$, where $u_{a,b}(x,y)$ is the unit step function defined as

$$u_{a,b}(x,y)=\begin{cases}1, & y\geqslant b,\\ 0, & y<b.\end{cases} \quad (21)$$

Now, we show that (15) holds. For it, consider the product $T2(x,y)T2^T(x,y)$. It is sufficient to show that for each $i=0,1,\ldots,m_1-1, j=0,1,\ldots,m_2-1$, $T2_{i,j}(x,y)T2_{i,j}(x,y)\simeq T2_{i,j}(x,y)$. By disjointness of $T2_{i,j}(x,y)$ and $T2_{k,t}(x,y)$ for $i\neq k$ and $j\neq t$ the desired will be obtained:

$$\begin{aligned}T2_{i,j}(x,y)T2_{i,j}(x,y)&=\left[\frac{y-jh_2}{h_2}u_{ih_1,jh_2}(x,y)-\frac{y-(j+1)h_2}{h_2}u_{ih_1,(j+1)h_2}(x,y)\right]^2-u_{ih_1,(j+1)h_2}(x,y)\\&=\left[\frac{y-jh_2}{h_2}\left(u_{ih_1,jh_2}(x,y)-u_{ih_1,(j+1)h_2}(x,y)\right)+u_{ih_1,(j+1)h_2}(x,y)\right]^2\\&\quad-u_{ih_1,(j+1)h_2}(x,y)=\left(\frac{y-jh_2}{h_2}\right)^2\left(u_{ih_1,jh_2}(x,y)-u_{ih_1,(j+1)h_2}(x,y)\right)\\&\simeq T2_{i,j}(x,y).\end{aligned} \quad (22)$$

Eq. (16) can be obtained similarly. Notice that, we can approximate the result by $T1_{i,j}(x,y)$ and $T2_{i,j}(x,y)$,

$$\left(\frac{y-jh_2}{h_2}\right)^2\left(u_{ih_1,jh_2}(x,y)-u_{ih_1,(j+1)h_2}(x,y)\right)$$

is possibly nonzero only in $[jh_2,(j+1)h_2)$. Thus, the last relation in (22) is followed.

We now show that Eq. (17) holds. For it

$$T1_{i,j}(x,y)T2_{i,j}(x,y) = T2_{i,j}(x,y) - \left[\frac{y-jh_2}{h_2}u_{ih_1,jh_2}(x,y) - \frac{y-(j+1)h_2}{h_2}u_{ih_1,(j+1)h_2}(x,y)\right]^2 + u_{ih_1,(j+1)h_2}(x,y). \quad (23)$$

Similar argument discussed for (22) results that

$$T1_{i,j}(x,y)T2_{i,j}(x,y) \simeq T2_{i,j}(x,y) - (T2_{i,j}(x,y) + u_{ih_1,(j+1)h_2}(x,y)) + u_{ih_1,(j+1)h_2}(x,y) \simeq 0, \quad (24)$$

thus,

$$T1_{i,j}(x,y)T2_{i,j}(x,y) = T2_{i,j}(x,y)T1_{i,j}(x,y) \simeq 0, \quad (25)$$

where $0 \in R$.

3.3. 2D-TFs expansion

We can approximate the function $f(x,y) \in L^2([0,1)\times[0,1))$ by 2D-TFs as follows:

$$\begin{aligned} f(x,y) &\simeq [c_{0,0}T1_{0,0} + c_{0,1}T2_{0,0}] + \cdots + [c_{0,m_2-1}T1_{0,m_2-1} \\ &\quad + c_{0,m_2}T2_{0,m_2-1}] + \cdots + [c_{m_1-1,0}T1_{m_1-1,0} + c_{m_1-1,1}T2_{m_1-1,0}] \\ &\quad + \cdots + [c_{m_1-1,m_2-1}T1_{m_1-1,m_2-1} + c_{m_1-1,m_2}T2_{m_1-1,m_2-1}] \\ &= [c_{0,0},\cdots,c_{0,m_2-1},\cdots,c_{m_1-1,0},\cdots,c_{m_1-1,m_2-1}]T1(x,y) \\ &\quad + [c_{0,1},\cdots,c_{0,m_2},\cdots,c_{m_1-1,1},\cdots,c_{m_1-1,m_2}]T2(x,y), \end{aligned}$$

thus

$$f(x,y) \simeq \sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2-1} c_{i,j}T1_{i,j}(x,y) + \sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2-1} c_{i,j+1}T2_{i,j}(x,y), \quad (26)$$

where

$$\begin{cases} c_{i,j} = f(ih_1, jh_2), \\ c_{i,j+1} = f(ih_1,(j+1)h_2). \end{cases} \quad (27)$$

Preference (*(Maleknejad et al., 2010)*). It is apparent from (6) and (27) that unlike 2D-BPFs the representation by 2D-TFs does not need any integration to evaluate the coefficients, there by reducing a lot of computational efforts.

3.3.1. Expanding four variables function by 2D-TFs

We can expand $K(x,y,t,s) \in L^2([0,1)\times[0,1)\times[0,1)\times[0,1))$ by 2D-TFs vectors, with m_1m_2 and m_3m_4 components, respectively. For convenience, consider $m = m_1 = m_2 = m_3 = m_4$. For obtaining desired results, we first fix the independent variables t, s. Then, expand $K(x,y,t,s)$ by 2D-TFs with respect to independent variables x, y as follows:

$$K(x,y,t,s) \simeq T1^T(x,y)\begin{pmatrix} K(0,0,t,s) \\ \vdots \\ K(0,(m-1)h,t,s) \\ \vdots \\ K((m-1)h,0,t,s) \\ \vdots \\ K((m-1)h,(m-1)h,t,s) \end{pmatrix} + T2^T(x,y)\begin{pmatrix} K(0,h,t,s) \\ \vdots \\ K(0,mh,t,s) \\ \vdots \\ K((m-1)h,h,t,s) \\ \vdots \\ K((m-1)h,mh,t,s) \end{pmatrix}.$$

Now each of $K(ih,jh,t,s)s$ for $i,j = 0,1,\ldots,m-1$ can be expanded by 2D-TFs with respect to independent variables t, s. Hence, the expansion of $K(x,y,t,s)$ can be written as

$$\begin{aligned} K(x,y,t,s) &\simeq T1^T(x,y)\begin{pmatrix} K11_1^T T1(t,s) + K12_1^T T2(t,s) \\ \vdots \\ K11_m^T T1(t,s) + K12_m^T T2(t,s) \\ \vdots \\ K11_{m(m-1)}^T T1(t,s) + K12_{m(m-1)}^T T2(t,s) \\ \vdots \\ K11_{m^2}^T T1(t,s) + K12_{m^2}^T T2(t,s) \end{pmatrix} \\ &\quad + T2^T(x,y)\begin{pmatrix} K21_1^T T1(t,s) + K22_1^T T2(t,s) \\ \vdots \\ K21_m^T T1(t,s) + K22_m^T T2(t,s) \\ \vdots \\ K21_{m(m-1)}^T T1(t,s) + K22_{m(m-1)}^T T2(t,s) \\ \vdots \\ K21_{m^2}^T T1(t,s) + K22_{m^2}^T T2(t,s) \end{pmatrix} \\ &= T1^T(x,y)K11T1(t,s) + T1^T(x,y)K12T2(t,s) \\ &\quad + T2^T(x,y)K21T1(t,s) + T2^T(x,y)K22T2(t,s), \end{aligned}$$

which in equation above

$$K11 = \begin{pmatrix} \begin{pmatrix} K(0,0,0,0) \\ \vdots \\ K(0,0,0,(m-1)h) \\ \vdots \\ K(0,0,(m-1)h,0) \\ \vdots \\ K(0,0,(m-1)h,(m-1)h) \end{pmatrix}^T \\ \vdots \\ \begin{pmatrix} K(0,(m-1)h,0,0) \\ \vdots \\ K(0,(m-1)h,0,(m-1)h) \\ \vdots \\ K(0,(m-1)h,(m-1)h,0) \\ \vdots \\ K(0,(m-1)h,(m-1)h,(m-1)h) \end{pmatrix}^T \\ \vdots \\ \begin{pmatrix} K((m-1)h,0,0,0) \\ \vdots \\ K((m-1)h,0,0,(m-1)h) \\ \vdots \\ K((m-1)h,0,(m-1)h,0) \\ \vdots \\ K((m-1)h,0,(m-1)h,(m-1)h) \end{pmatrix}^T \\ \vdots \\ \begin{pmatrix} K((m-1)h,(m-1)h,0,0) \\ \vdots \\ K((m-1)h,(m-1)h,0,(m-1)h) \\ \vdots \\ K((m-1)h,(m-1)h,(m-1)h,0) \\ \vdots \\ K((m-1)h,(m-1)h,(m-1)h,(m-1)h) \end{pmatrix}^T \end{pmatrix}, K12 = \begin{pmatrix} \begin{pmatrix} K(0,0,0,h) \\ \vdots \\ K(0,0,0,mh) \\ \vdots \\ K(0,0,(m-1)h,h) \\ \vdots \\ K(0,0,(m-1)h,mh) \end{pmatrix}^T \\ \vdots \\ \begin{pmatrix} K(0,(m-1)h,0,h) \\ \vdots \\ K(0,(m-1)h,0,mh) \\ \vdots \\ K(0,(m-1)h,(m-1)h,h) \\ \vdots \\ K(0,(m-1)h,(m-1)h,mh) \end{pmatrix}^T \\ \vdots \\ \begin{pmatrix} K((m-1)h,0,0,h) \\ \vdots \\ K((m-1)h,0,0,mh) \\ \vdots \\ K((m-1)h,0,(m-1)h,h) \\ \vdots \\ K((m-1)h,0,(m-1)h,mh) \end{pmatrix}^T \\ \vdots \\ \begin{pmatrix} K((m-1)h,(m-1)h,0,h) \\ \vdots \\ K((m-1)h,(m-1)h,0,mh) \\ \vdots \\ K((m-1)h,(m-1)h,(m-1)h,h) \\ \vdots \\ K((m-1)h,(m-1)h,(m-1)h,mh) \end{pmatrix}^T \end{pmatrix}, \qquad (28)$$

$$
K21 = \begin{pmatrix}
\begin{pmatrix} K(0,h,0,0) \\ \vdots \\ K(0,h,0,(m-1)h) \\ \vdots \\ K(0,h,(m-1)h,0) \\ \vdots \\ K(0,h,(m-1)h,(m-1)h) \end{pmatrix}^{T} \\
\vdots \\
\begin{pmatrix} K(0,mh,0,0) \\ \vdots \\ K(0,mh,0,(m-1)h) \\ \vdots \\ K(0,mh,(m-1)h,0) \\ \vdots \\ K(0,mh,(m-1)h,(m-1)h) \end{pmatrix}^{T} \\
\vdots \\
\begin{pmatrix} K((m-1)h,h,0,0) \\ \vdots \\ K((m-1)h,h,0,(m-1)h) \\ \vdots \\ K((m-1)h,h,(m-1)h,0) \\ \vdots \\ k((m-1)h,h,(m-1)h,(m-1)h) \end{pmatrix}^{T} \\
\vdots \\
\begin{pmatrix} K((m-1)h,mh,0,0) \\ \vdots \\ K((m-1)h,mh,0,(m-1)h) \\ \vdots \\ K((m-1)h,mh,(m-1)h,0) \\ \vdots \\ K((m-1)h,mh,(m-1)h,(m-1)h) \end{pmatrix}^{T}
\end{pmatrix},
K22 = \begin{pmatrix}
\begin{pmatrix} K(0,h,0,h) \\ \vdots \\ K(0,h,0,mh) \\ \vdots \\ K(0,h,(m-1)h,h) \\ \vdots \\ K(0,h,(m-1)h,mh) \end{pmatrix}^{T} \\
\vdots \\
\begin{pmatrix} K(0,mh,0,h) \\ \vdots \\ K(0,mh,0,mh) \\ \vdots \\ K(0,mh,(m-1)h,h) \\ \vdots \\ K(0,mh,(m-1)h,mh) \end{pmatrix}^{T} \\
\vdots \\
\begin{pmatrix} K((m-1)h,h,0,h) \\ \vdots \\ K((m-1)h,h,0,mh) \\ \vdots \\ K((m-1)h,h,(m-1)h,h) \\ \vdots \\ K((m-1)h,h,(m-1)h,mh) \end{pmatrix}^{T} \\
\vdots \\
\begin{pmatrix} K((m-1)h,mh,0,h) \\ \vdots \\ K((m-1)h,mh,0,mh) \\ \vdots \\ K((m-1)h,mh,(m-1)h,h) \\ \vdots \\ K((m-1)h,mh,(m-1)h,mh) \end{pmatrix}^{T}
\end{pmatrix}, \tag{29}
$$

3.4. Other properties

Other properties that we will need are

$$\int_0^1 \int_0^1 T1(x,y).T1^T(x,y)\,dy\,dx = \int_0^1 \int_0^1 T2(x,y).T2^T(x,y)\,dy\,dx = \frac{h_1 h_2}{3} I, \quad (30)$$

$$\int_0^1 \int_0^1 T1(x,y).T2^T(x,y)\,dy\,dx = \int_0^1 \int_0^1 T2(x,y).T1^T(x,y)\,dy\,dx = \frac{h_1 h_2}{6} I, \quad (31)$$

where I is $m_1 m_2 \times m_1 m_2$ identity matrix. To show (30), consider (22) again

$$\begin{aligned}\int_0^1 \int_0^1 T2(x,y).T2^T(x,y)dydx &= \int_{ih_1}^{(i+1)h_1} \int_{jh_2}^{(j+1)h_2} \left(\frac{y-jh_2}{h_2}\right)^2 (u_{ih_1,jh_2}(x,y) - u_{ih_1,(j+1)h_2}(x,y))\,dy\,dx \\ &= \int_{ih_1}^{(i+1)h_1} \int_{jh_2}^{(j+1)h_2} \left(\frac{y-jh_2}{h_2}\right)^2 dy\,dx \\ &= \int_{ih_1}^{(i+1)h_1} \frac{h_2}{3} dx = \frac{h_1 h_2}{3}. \end{aligned} \quad (32)$$

Similarly,

$$\int_0^1 \int_0^1 T1(x,y).T1^T(x,y)\,dy\,dx = \frac{h_1 h_2}{3}.$$

Now

$$\begin{aligned}\int_0^1 \int_0^1 T1(x,y).T2^T(x,y)dydx &= \int_{ih_1}^{(i+1)h_1} \int_{jh_2}^{(j+1)h_2} \left(1-\frac{y-jh_2}{h_2}\right)\left(\frac{y-jh_2}{h_2}\right) dy\,dx \\ &= \int_{ih_1}^{(i+1)h_1} \int_{jh_2}^{(j+1)h_2} \frac{y-jh_2}{h_2}\,dy\,dx - \int_{ih_1}^{(i+1)h_1} \int_{jh_2}^{(j+1)h_2} \left(\frac{y-jh_2}{h_2}\right)^2 dy\,dx \\ &= \frac{h_1 h_2}{2} - \frac{h_1 h_2}{3} = \frac{h_1 h_2}{6}. \end{aligned} \quad (33)$$

Similarly,

$$\int_0^1 \int_0^1 T2(x,y).T1^T(x,y)\,dy\,dx = \frac{h_1 h_2}{6}.$$

4. Two-dimensional Fredholm integral equation of the second kind

In this section, we present a 2D-TFs method for solving Eq. (1). Changing the variables

$\tau = (b-a)x + a,\ \mu = (d-c)y + c,\ \lambda = (b-a)t + a$

and

$\eta = (d-c)s + c,$

Eq. (1) can be written as

$$u(x,y) = f(x,y) + (b-a)(d-c) \times \int_0^1 \int_0^1 K(x,y,t,s)u(t,s)\,dt\,ds; \quad (x,y) \in D, \quad (34)$$

where

$$\begin{aligned} f(x,y) &= f_1((b-a)x + a, (d-c)y + c), \\ K(x,y,t,s) &= K_1((b-a)x + a, (d-c)y + c, (b-a)t + a, (d-c)s + c), \\ u(x,y) &= u_1((b-a)x + a, (d-c)y + c), \end{aligned}$$

and $D = L^2([0,1) \times [0,1))$.

Let us expand $f(x,y)$ and $u(x,y)$ by 2D-TFs (LH2D-TF and RH2D-TF) as follows:

$$f(x,y) \simeq F_1^T T1(x,y) + F_2^T T2(x,y), \quad (35)$$

$$u(x,y) \simeq U_1^T T1(x,y) + U_2^T T2(x,y). \quad (36)$$

As described in Section 3.3.1, we can expand $K(x,y,t,s)$ in the interval $L^2([0,1) \times [0,1) \times [0,1) \times [0,1))$ by 2D-TF. Suppose that this approximation be as follows:

$$\begin{aligned} K(x,y,t,s) \simeq\ & T1^T(x,y)K11T1(t,s) \\ &+ T1^T(x,y)K12T2(t,s) \\ &+ T2^T(x,y)K21T1(t,s) \\ &+ T2^T(x,y)K22T2(t,s), \end{aligned} \quad (37)$$

where $K11, K12, K21$ and $K22$ are obtained from Eqs. (28) and (29). Then, we have

$$\begin{aligned} U_1^T T1(x,y) + U_2^T T2(x,y) = F_1^T T1(x,y) + F_2^T T2(x,y) &+ \rho \int_0^{m_1 h_1} \int_0^{m_2 h_2} \left(U_1^T T1(t,s) + U_2^T T2(t,s)\right) \left[T1^T(x,y)K11T1(t,s) \right. \\ &+ T1^T(x,y)K12T2(t,s) \\ &+ T2^T(x,y)K21T1(t,s) \\ &\left. + T2^T(x,y)K22T2(t,s)\right] ds\,dt, \end{aligned}$$

where $\rho = (b-a)(d-c)$. Thus, we have

$$\begin{aligned} U_1^T T1(x,y) + U_2^T T2(x,y) &= F_1^T T1(x,y) + F_2^T T2(x,y) \\ &+ \rho\Big[U_1^T \int_0^{m_1 h_1} \int_0^{m_2 h_2} T1(t,s)T1^T(t,s)K11^T T1(x,y)\,dt\,ds \\ &+ U_1^T \int_0^{m_1 h_1} \int_0^{m_2 h_2} T1(t,s)T2^T(t,s)K12^T T1(x,y)\,dt\,ds \\ &+ U_1^T \int_0^{m_1 h_1} \int_0^{m_2 h_2} T1(t,s)T1^T(t,s)K21^T T2(x,y)\,dt\,ds \\ &+ U_1^T \int_0^{m_1 h_1} \int_0^{m_2 h_2} T1(t,s)T2^T(t,s)K22^T T2(x,y)\,dt\,ds \\ &+ U_2^T \int_0^{m_1 h_1} \int_0^{m_2 h_2} T2(t,s)T1^T(t,s)K11^T T1(x,y)\,dt\,ds \\ &+ U_2^T \int_0^{m_1 h_1} \int_0^{m_2 h_2} T2(t,s)T2^T(t,s)K12^T T1(x,y)\,dt\,ds \\ &+ U_2^T \int_0^{m_1 h_1} \int_0^{m_2 h_2} T2(t,s)T1^T(t,s)K21^T T2(x,y)\,dt\,ds \\ &+ U_2^T \int_0^{m_1 h_1} \int_0^{m_2 h_2} T2(t,s)T2^T(t,s)K22^T T2(x,y)\,dt\,ds\Big]. \end{aligned}$$

By using definite integral formula for 2D-TFs in (30) and (31), we have

$$U_1^T T1(x,y) + U_2^T T2(x,y) = F_1^T T1(x,y) + F_2^T T2(x,y) + \rho\bigg[U_1^T\Big(\frac{h_1h_2}{3}K11^T T1(x,y) + \frac{h_1h_2}{6}K12^T T1(x,y) + \frac{h_1h_2}{3}K21^T T2(x,y) + \frac{h_1h_2}{6}K22^T T2(x,y)\Big) + U_2^T\Big(\frac{h_1h_2}{6}K11^T T1(x,y) + \frac{h_1h_2}{3}K12^T T1(x,y) + \frac{h_1h_2}{6}K21^T T2(x,y) + \frac{h_1h_2}{3}K22^T T2(x,y)\Big)\bigg]. \quad (38)$$

The coefficients of $T1(x,y)$ and $T2(x,y)$ on both sides of (38) must be equal; hence, we have the following equations for the corresponding coefficients of 2D-TFs:

$$\begin{cases} U_1^T\big(I - \rho(\frac{h_1h_2}{3}K11^T + \frac{h_1h_2}{6}K12^T)\big) \\ \quad -\rho U_2^T\big(\frac{h_1h_2}{6}K11^T + \frac{h_1h_2}{3}K12^T\big) = F_1^T \\ -\rho U_1^T\big(\frac{h_1h_2}{3}K21^T + \frac{h_1h_2}{6}K22^T\big) \\ \quad +U_2^T\big(I - \rho(\frac{h_1h_2}{6}K21^T + \frac{h_1h_2}{3}K22^T)\big) = F_2^T \end{cases}.$$

Set

$$A_1 = I - \rho\Big(\frac{h_1h_2}{3}K11 + \frac{h_1h_2}{6}K12\Big), \quad (39)$$

$$A_2 = -\rho\Big(\frac{h_1h_2}{6}K11 + \frac{h_1h_2}{3}K12\Big), \quad (40)$$

$$B_1 = -\rho\Big(\frac{h_1h_2}{3}K21 + \frac{h_1h_2}{6}K22\Big), \quad (41)$$

$$B_2 = I - \rho\Big(\frac{h_1h_2}{6}K21 + \frac{h_1h_2}{3}K22\Big). \quad (42)$$

Then, we have the following linear system:

$$\begin{pmatrix} A_1 & A_2 \\ B_1 & B_2 \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}. \quad (43)$$

After solving the above linear system, we can find U_1 and U_2 and then

$$u(x,y) \simeq U_1^T T1(x,y) + U_2^T T2(x,y), \quad (44)$$

is the estimation of the solution of the two-dimensional Fredholm integral equation of the second kind.

5. Convergence analysis

Assume $(C[J], \|.\|)$ the Banach space of all continuous functions on $J = [0,1) \times [0,1)$ with norm $\|f(x,y)\| = max_{\forall(x,y)\in J}|f(x,y)|$.

Let $\forall x,y,t,s \in [0,1), |k(x,y,s,t)| \leqslant M$. We denote the error 2D-TFS by

$$e_{\text{2D-TFS}} = \|u_m(x,y) - u(x,y)\|,$$

where $u_m(x,y)$ and $u(x,y)$ show the approximate and exact solutions of the two-dimensional linear Fredholm integral equation, respectively. If we note to Eq. (27), we will see the coefficients $c_{i,j}$'s and $d_{i,j+1}$'s are not optimal. By using the optimal coefficients, the representational errors e_{2D-TFS} can be reduced.

Theorem 1. *The solution of the two-dimensional linear Fredholm integral equation by using 2D-TFs approximation converges if* $0 < \alpha < 1$.

Proof. Let

$$\begin{aligned} &\|u_m(x,y) - u(x,y)\| \\ &= max_{\forall(x,y)\in J}|u_m(x,y) - u(x,y)| \\ &= max\Big|f(x,y) + \rho\int_0^1\int_0^1 k(x,y,t,s)u_m(t,s)\,dt\,ds - f(x,y) \\ &\quad -\rho\int_0^1\int_0^1 k(x,y,t,s)u(t,s)\,dt\,ds\Big| \\ &\leqslant max|\rho|\int_0^1\int_0^1 |k(x,y,t,s)||u_m(t,s) - u(t,s)|\,dt\,ds \\ &\leqslant |\rho|M\int_0^1\int_0^1 max|u_m(t,s) - u(t,s)|\,dt\,ds \\ &= |\rho|M\|u_m(x,y) - u(x,y)\|, \end{aligned}$$

$$\Rightarrow \|u_m(x,y) - u(x,y)\| \leqslant \alpha\|u_m(x,y) - u(x,y)\|,$$

where $\alpha = |\rho|M$.

We get $(1-\alpha)\|u_m(x,y) - u(x,y)\| \leqslant 0$ and choose $0 < \alpha < 1$, by increasing m, it implies $\|u_m(x,y) - u(x,y)\| \to 0$ as $m \to \infty$. □

6. Numerical illustration

In this section, we present one example and their numerical results to show the high accuracy of the solution obtained by 2D-TFs.

Example 5.1. Consider the two-dimensional linear Fredholm integral equation

Table 1 Numerical results of example 1 with 2D-TFs.

Nodes (x,y)	2D-TFs method with $m = 32$	Error of 2D-TFs method with $m = 32$	Exact solution
(0,0)	1.0000e−000	0	1.0000e−000
(0.1,0.1)	7.020e−001	−7.5800e−003	6.9444e−001
(0.2,0.2)	5.1974e−001	−9.5400e−003	5.1020e−001
(0.3,0.3)	4.0022e−001	−9.6000e−003	3.9062e−001
(0.4,0.4)	3.1767e−001	−9.0300e−003	3.0864e−001
(0.5,0.5)	2.5030e−001	−3.0000e−004	2.5000e−001
(0.6,0.6)	2.0704e−001	−4.3000e−004	2.0661e−001
(0.7,0.7)	1.7585e−001	−2.2400e−003	1.7361e−001
(0.8,0.8)	1.5055e−001	−2.6200e−003	1.4793e−001
(0.9,0.9)	1.3036e−001	−2.8100e−003	1.2755e−001

$$u(x,y) = g(x,y) + \int_0^1 \int_0^1 \frac{x}{(8+y)(1+t+s)} u(t,s)\, dt\, ds, \quad (45)$$

where $(x,y) \in [0,1) \times [0,1)$ and

$$g(x,y) = \frac{1}{(1+x+y)^2} - \frac{x}{6(8+y)}.$$

It's exact solution is $u(x,y) = \frac{1}{(1+x+y)^2}$. The solution for $u(x,y)$ is obtained by 2D-TFs method described in Section 4, for $m = 32$ is collected as shown in Table 1.

7. Conclusion

Two-dimensional Fredholm integral equations are usually difficult to solve analytically. In many cases, it is required to obtain the approximate solutions, for this purpose the presented method can be proposed. We have investigated the application of orthogonal functions by 2D-TFs for solving the linear two-dimensional Fredholm integral equations. This technique is very simple and involves less computation. Also we can expand this method to higher dimensional problems and other classes of integral equations such as nonlinear two-dimensional Fredholm integral equations, linear and nonlinear two-dimensional Volterra integral equations.

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