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Numerical solution of singular boundary value problems using Green's function and Sinc-Collocation method

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A R T I C L E I N F O

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A B S T R A C T

This study is the construction of the Green's function and Sinc function for a class of nonhomogeneous singular boundary value problems (SBVPs). The equivalent Volterra-Fredholm integral equations can be derived from SBVPs by applying Green's function. This can be approximated by Sinc-Collocation method. Convergence analysis is given. Our approach applied on three various examples. Errors in the solution are demonstrated in the tables. We conclude that our approach converge rapidly with the exponential order.

1. Introduction

We consider a class of nonhomogeneous SBVPs:

$$Ly(x) = y''(x) + P(x)y'(x) + Q(x)y = R(x), \quad x \in [0,1], \tag{1}$$

with boundary conditions:

$$\begin{cases} a_0 y(0) + a_1 y'(0) = a_2, \\ b_0 y(1) + b_1 y'(1) = b_2. \end{cases} \tag{2}$$

If $P(x)$ has singularity at $x = 0$ such as $P(x) = \frac{\beta}{x}$ with $\beta > 0$, then the problems (1)–(2) is called Lane-Emden type equations. We assume that constants a_0 and a_1, and likewise b_0 and b_1, both are not zero. The unique solution of problems (1) subjected to the boundary conditions (2) is depend on the following conditions on $P(x), Q(x)$, and $R(x)$:

E1. Let $P(x)$ is measurable on $[0,1]$ and continuous on $(0,1]$;
E2. $P(x) > 0$ on $(0,1]$;
E3. $\int_0^1 xP(x) < \infty$;
E4. Let $Q(x)$ is continuous on $[0,1]$.

The Lane-Emden equations is the model for several phenomena in physics and astrophysics Parand and Pirkhedri, 2010; Wazwaz, 2011; Yuzbası and Sezer, 2013.

Many researchers have tried to solve problems (1)–(2) numerically. Mohanty et al. Mohanty et al., 2004 using cubic spline method for solving SBVPs. Variational iteration scheme by Wazwaz Wazwaz, 2011. Pirabaharan et al. Pirabaharan and Chandrakumar,

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2016 discussed a new Bernstein polynomials method. The modified Bessel Collocation by Suayip Yuzbası, et al. Yuzbası and Sezer, 2013.

The Sinc function is widely used in various numerical methods which introduced by F. Stenger Stenger, 1993; Stenger, 2011. Rashidinia Rashidinia and Nabati, 2013; Rashidinia and Taheri, 2015 used Sinc methods based on Sinc function for SBVPs.

The Green' function is used for solving BVPs in one, two, and three dimensionalRahman, 2007. Z. Cen Cen, 2006 used Green's function to developed equivalent integral equation for SBVPs. Singh et al. Singh et al., 2013 using Adomian decomposition and Green's function.

In this paper, we apply a new direction for approximating the nonhomogeneous SBVPs (1)–(2) which can be reduced to a Volterra-Fredholm integral equation with the help of Green's function. In our approach, the convergence accuracy of the solution is $O\left(e^{(-\vartheta\sqrt{N})}\right)$, where $\vartheta > 0$, and also converges at an optimal rate, because the singularity on the boundary of approximation is ignoredRashidinia and Zarebnia, 2007; Rashidinia and Zarebnia, 2007.

Section 2 deals with representation of the solution of the SBVPs (1)–(2) by Green's function. In Section 3, after some preliminary definitions and theorems of Sinc function, the Sinc-Collocation method has been used to replace Volterra-Fredholm integral equation corresponding to problems (1)–(2). The converges of the methods are considered in Section 4. Finally, three test examples are presented in Section 5 and the conclusion is considered in the rest of the section.

2. Representation of the Green's function

The method of variation of parameter is used to construct Green's function for an integral representation of nonhomogeneous SBVPs (1)–(2). First of all, we convert the nonhomogeneous boundary conditions (2) to homogeneous Stenger, 1993, then we define an interpolating boundary function

$$\Gamma(x) = \frac{(a_2b_0 - a_0b_2)x + (a_1b_2 - (b_0 + b_1)a_2)}{-a_0b_0 + a_1b_0 - a_0b_1}, \quad (3)$$

and also using the linear shift

$$u(x) = y(x) - \Gamma(x). \quad (4)$$

Consequently, the problems (1)–(2) reduce to the following problem:

$$Lu(x) \equiv u''(x) + P(x)u'(x) + Q(x)u(x) = R(x) - L(\Gamma(x)),$$
$$x \in [0,1] \quad (5)$$

subjected to homogeneous boundary conditions:

$$\begin{cases} a_0u(0) + a_1u'(0) = 0, \\ b_0u(1) + b_1u'(1) = 0, \end{cases} \quad (6)$$

where

$$L(\Gamma(x)) = \Gamma''(x) + P(x)\Gamma'(x) + Q(x)\Gamma(x).$$

The homogeneous part of Eqs. (5) is:

$$u''(x) + P(x)u'(x) + Q(x)u(x) = 0 \quad (7)$$

Let $u_1(x)$ and $u_2(x)$ be two solutions of problem (7) which are linearly independent. By using variation of parameters (see Rahman, 2007), the Green's function of (5) and (6) can easily be constructed as:

$$G(x,t) = \begin{cases} \frac{u_1(x)u_2(t)}{W(t)}, & 0 \leqslant t \leqslant x \leqslant 1 \\ \frac{u_2(x)u_1(t)}{W(t)}, & 0 \leqslant x \leqslant t \leqslant 1, \end{cases} \quad (8)$$

where $W = u_1u_2' - u_2u_1'$ = Wronskian $\neq 0$.

It is obvious that $G(x,t)$ is matched with boundary conditions (6). The properties of the $G(x,t)$ are summarised in appendix A.

Now, by using (8), a particular solution of Eqs. 5,6 in integral form can be obtained as:

$$\begin{aligned} u(x) &= -\int_0^1 G(x,t)f(t)dt \\ &= -u_1(x)\int_0^x \frac{u_2(t)f(t)}{W(t)}dt - u_2(x)\int_x^1 \frac{u_1(t)f(t)}{W(t)}dt, \end{aligned} \quad (9)$$

where $f(t) = R(t) - L(\Gamma(t))$ is on the right hand side of problem (5). We apply variation of variable in second integral on the right hand side of (9), so that

$$\begin{aligned} u(x) = & -u_1(x)\int_0^x \frac{u_2(t)f(t)}{W(t)}dt \\ & + u_2(x)\int_0^x \frac{u_1(t+1)f(t+1)}{W(t+1)}dt - u_2(x)\int_0^1 \frac{u_1(t+x)f(t+x)}{W(t+x)}dt. \end{aligned} \quad (10)$$

The proof of Eq. (10) is in the appendix B.

3. Sinc function preliminaries and the Sinc-Collocation method

We first introduce some preliminaries of Sinc function, Sinc interpolation Stenger, 1993; Stenger, 2011 that are important here, and then we discussed the Sinc-Collocation method.

3.1. Some preliminary results using Sinc function

Definition. Stenger, 1993 We assume that $\mathscr{D} = \{z \in \mathbf{C} : |\mathscr{I}m(z)| < d\}$ is a simply connected domain in the complex plane($\mathbf{C}$). We consider two separate points 0 and 1 of $\partial\mathscr{D}$(boundary of $\mathscr{D}$). In $(0,1)$, we define a conformal map $\varphi(z) = \ln(\frac{z}{1-z})$, which has inverse $\psi(z) = \varphi^{-1}(z) = \frac{e^z}{1+e^z}$. For φ, ψ, and a step size $h > 0$, we consider $z_k = \psi(kh), k \in \mathbf{Z}$ which is the Sinc points.

Theorem 1. We assume $\mathbf{L}_\alpha(\mathscr{D})$ is the set of all analytic functions and $u \in \mathbf{L}_\alpha(\mathscr{D})$ for $\alpha > 0$. By considering $h = (\pi d/(\alpha N))^{1/2}$, there exist a constant $k_1 > 0$, so

$$\left| u(z) - \sum_{j=-N}^{N} u(z_j)\Theta_j(\varphi(z)) \right| \leqslant k_1 e^{(\pi d/(\alpha N))^{1/2}}. \quad (11)$$

We define the basis function $\Theta_j(\varphi(z))$ as follows:

$$\Theta_j(\varphi(z)) = \begin{cases} \frac{1}{1+e^{\varphi(z)}} - \sum_{k=-N+1}^{N} \frac{1}{1+e^{kh}} S(k,h)\circ\varphi(z), & j = -N \\ S(j,h)o(\varphi(z)), & j = -N+1,\ldots,N-1 \\ \frac{e^{\varphi(z)}}{1+e^{\varphi(z)}} - \sum_{k=-N}^{N-1} \frac{e^{kh}}{1+e^{kh}} S(k,h)\circ\varphi(z), & j = N \end{cases} \quad (12)$$

based on the Sinc function

$$S(j,h)\circ\varphi(z) = \frac{\sin([\varphi(z) - jh]/h)}{[\varphi(z) - jh]/h}, \quad j = -N,\ldots,N, \quad (13)$$

with the following property for points $z_k = \psi(kh), k \in \mathbf{Z}$:

$$\delta_{jk}^{(0)} = [S(j,h)\circ\varphi(z)]|_{z=z_k} = \begin{cases} 1, & j = k; \\ 0, & j \neq k. \end{cases} \quad (14)$$

The proof of this theorem is given in Stenger, 1993.

Theorem 2. Stenger, 1993 We cosider $\frac{u}{\varphi'} \in L_\alpha(\mathscr{D})$, $(\alpha > 0)$, and $0 < d \leqslant \pi$. By selecting $h = (\pi d/(\alpha N))^{1/2}$, there exist a constant, $k_2 > 0$, so

$$\left| \int_a^{z_j} u(t)dt - h\sum_{k=-N}^{N} \delta_{jk}^{(-1)} \frac{u(z_k)}{\varphi\prime(z_k)} \right| \leqslant k_2 e^{(\pi d/(\alpha N))1/2}, \quad (15)$$

with

$$\delta_{jk}^{(-1)} = \frac{1}{2} + \int_0^{j-k} \frac{\sin(\pi z)}{\pi z} dz. \quad (16)$$

Theorem 3. Stenger, 1993 We cosider $\frac{u}{\varphi'} \in L_\alpha(\mathscr{D}), \alpha > 0$, and $0 < d \leqslant \pi$. By selecting $h = (\pi d/(\alpha N))^{1/2}$, there exist a constant, $k_3 > 0$, there fore, for $\Lambda = [0, 1]$ the trapezoidal quadrature rule in the Sinc methods is:

$$\left| \int_\Lambda u(z)dz - h\sum_{k=-N}^{N} \frac{u(z_k)}{\varphi'(z_k)} \right| \leqslant k_3 e^{(\pi d/(\alpha N))^{1/2}}. \quad (17)$$

3.2. Sinc-Collocation method

The approximate solution of integral Eq. (10) by the Sinc basis function (12) is:

$$u_m(x) \equiv c_{-N-1}\varpi_0(x) + \sum_{j=-N} Nc_j\Theta_j(\varphi(x)) + c_{N+1}\varpi_1(x). \quad (18)$$

The boundary basis functions ϖ_0 and ϖ_1 are cubic Hermite functions given by:

$$\begin{aligned} \varpi_0(x) &= a_0(-x)(1-x)^2 + a_1(2x+1)(1-x)^2, \\ \varpi_1(x) &= b_0(-2x+3)x^2 + b_1(1-x)x^2. \end{aligned} \quad (19)$$

Upon replacing $u(x)$ in the Volterra-Fredholm integral Eq. (10) by $u_m(x)$, and Sinc function defined in 12, and applying Theorem 2, for the first and second terms (Volterra integral equation) on the right hand side of (10), and Theorem 3, for the third term (Fredholm integral equation) of (10), and setting points x_k, we get the following system:

$$\begin{aligned} &c_{-N-1}\varpi_0(x_k) + c_{-N}\Theta_{-N}(\varphi(x_k)) + \sum_{j=-N+1} N-1c_j\Theta_j(\varphi(x_k)) \\ &+c_N\Theta_N(\varphi(x_k)) + c_{N+1}\varpi_1(x_k) \\ &= h\sum_{j=-N}^{N} \frac{\delta_{kj}^{(-1)}}{\varphi'(t_j)} \left(-\frac{u_1(x_k)u_2(t_j)f(t_j)}{W(t_j)} + \frac{u_2(x_k)u_1(t_j+1)f(t_j+1)}{W(t_j+1)} \right) \\ &-h\sum_{j=-N}^{N} \frac{u_2(x_k)u_1(t_j+x_k)f(t_j+x_k)}{\varphi'(t_j)W(t_j+x_k)}; \quad k = -N-1, \ldots, N+1. \end{aligned} \quad (20)$$

Where

$$\varphi(x) = \ln\left(\frac{x_k}{1-x_k}\right), \quad x_k = \frac{e^{kh}}{1+e^{kh}}. \quad (21)$$

The matrix form of the system (20) is:

$$AX = \left[C_{n\times1}|D_{n\times1}|I_{n\times(n-4)}|E_{n\times1}|F_{n\times1}\right] = B; \quad n = 2N+3, \quad (22)$$

where

$$\begin{aligned} C &= [\varpi_0(x_{-N-1}), \varpi_0(x_{-N}), \ldots, \varpi_0(x_N), \varpi_0(x_{N+1})]^T, \\ D &= [\Theta_{-N}(\varphi(x_{-N-1})), \Theta_{-N}(\varphi(x_{-N})), \ldots, \Theta_{-N}(\varphi(x_N)), \Theta_{-N}(\varphi(x_{N+1}))]^T, \\ I &= \left[\delta_{jk}^{(0)}\right], \quad j = -N+1, \ldots, N-1, k = -N-1, \ldots, N+1, \\ E &= [\Theta_N(\varphi(x_{-N-1})), \Theta_N(\varphi(x_{-N})), \ldots, \Theta_N(\varphi(x_N)), \Theta_N(\varphi(x_{N+1}))]^T \\ F &= [\varpi_1(x_{-N-1}), \varpi_1(x_{-N}), \ldots, \varpi_1(x_N), \varpi_1(x_{N+1})]^T, \\ X &= [c_{-N-1}, c_N, \ldots, c_N, c_{N+1}]^T. \end{aligned}$$

We replace the right hand side of the system (20) by $g(x_k)$ as follows:

$$B = [g(x_{-N-1}), g(x_{-N}), \ldots, g(x_N), g(x_{N+1})]^T.$$

In solving system (22), we apply Newton's method with $X_{(0)} = \vec{0}$, which stop iteration whenever $\|X_{(k+1)} - X_{(k)}\| < \varepsilon$.

4. Convergence analysis

By the following theorem, we proof that the Volterra-Fredholm Eq. (9) has the unique solution.

Theorem 4. We consider the assumptions (E1)-(E4), and Green's function 8, so we have.

I. $m_1 := \max_{0\leqslant x\leqslant1} |\int_0^1 G(x,t)dt| < \infty$,
II. $m_2 := \max_{0\leqslant x\leqslant1} |\int_0^1 P(x)G_x(x,t)dt| < \infty$.

Proof. (i) The proof is clear. It follows from the Green's function (8) and the assumptions (E1)-(E4).(ii) For $G_x(x,t) = \frac{\partial G(x,t)}{\partial x}$, we have

$$G_x(x,t) = \begin{cases} \frac{u_1'(x)u_2(t)}{W(t)}, & t \leqslant x \leqslant 1, \\ \frac{u_2'(x)u_1(t)}{W(t)}, & 0 \leqslant x \leqslant t. \end{cases}$$

Hence,

$$\frac{\partial G}{\partial x}|_{t^{+0}} - \frac{\partial G}{\partial x}|_{t^{-0}} = \frac{u_1'(t)u_2(t) - u_2'(t)u_2(t)}{W(t)} = -1 \quad (23)$$

Because, $G(x,t)$ satisfies the differential Eq. (5) as follows:

$$G_{xx} + P(x)G_x + Q(x)G = -\delta(t-x), \quad (24)$$

where $\delta(t-x)$ is a Dirac delta function. We integrate from the relation (24) then we have

$$\begin{aligned} &\int_{t^{-0}}^{t^{+0}} G_{xx}dx + \int_{t^{-0}}^{t^{+0}} (P(x)G_x + Q(x)G)dx = -\int_{t^{-0}}^{t^{+0}} \delta(t-x)dx \\ &= \frac{\partial G}{\partial x}|_{t^{+0}} - \frac{\partial G}{\partial x}|_{t^{-0}} + P(x)\int_{t^{-0}}^{t^{+0}} \frac{\partial G}{\partial x}dx + Q(x)\int_{t^{-0}}^{t^{+0}} Gdx \\ &= -1 + 0 + 0 = -1 \end{aligned}$$

Hence, from (23) we have

$$P(x)G_x(x,t) = \begin{cases} 0, & 0 \leqslant x \leqslant t \leqslant 1, \\ -1, & 0 \leqslant t \leqslant x \leqslant 1. \end{cases}$$

Hence, we obtain $c = \max_{0\leqslant x,t\leqslant1} |P(x)G_x(x,t)dt| < \infty$, and then.
$m_2 = \max_{0\leqslant x,t\leqslant1} |\int_{t^{-0}}^{t^{+0}} P(x)G_x(x,t)| \leqslant c < \infty$.

Theorem 5. We assume that $u_m(x)$ and $u(x) \in L_\alpha(\mathscr{D})$ are the approximate and exact solutions of integral Eq. (10) respectively. Suppose that all conditions of Theorems 1, 2, and 3 are fulfilled. By considering $h = (\pi d/(\alpha N))^{1/2}$ and $\frac{u}{\varphi'} \in L_\alpha(\mathscr{D}), (\alpha > 0)$, there exist a constant $\zeta > 0$, so

$$\max_{0\leqslant x\leqslant1} |u(x) - u_m(x)| < \zeta e^{(\pi d/(\alpha N))^{1/2}}.$$

Proof. Using the relations (11), (15), and (17), we have

$$\begin{aligned} |e_m(x)| &= \max_{0\leqslant x\leqslant1} |u(x) - u_m(x)| \\ &\leqslant \max_{0\leqslant x\leqslant1} |u(x) - \left(c_{-N-1}\varpi_0(x) + \sum_{j=-N} Nc_j\Theta_j(\varphi(x)) + c_{N+1}\varpi_1(x) \right)| \\ &+| -u_1(x)\int_0^x \frac{u_2(t)f(t)}{W(t)}dt + h\sum_{j=-N}^{N} \frac{\delta_{kj}^{(-1)}}{\varphi'(t_j)} \frac{u_1(x_k)u_2(t_j)f(t_j)}{W(t_j)}| \\ &+|u_2(x)\int_0^x \frac{u_1(t+1)f(t+1)}{W(t+1)}dt - h\sum_{j=-N}^{N} \frac{\delta_{kj}^{(-1)}}{\varphi'(t_j)} \frac{u_2(x_k)u_1(t_j+1)f(t_j+1)}{W(t_j+1)}| \\ &+| -u_2(x)\int_0^1 \frac{u_1(t+x)f(t+x)}{W(t+x)}dt + h\sum_{j=-N}^{N} \frac{u_2(x_k)u_1(t_j+x_k)f(t_j+x_k)}{\varphi'(t_j)W(t_j+x_k)}| \\ &\leqslant k_1e^{(\pi d/(\alpha N))^{1/2}} + k_2e^{(\pi d/(\alpha N))^{1/2}} + k_3e^{(\pi d/(\alpha N))^{1/2}} \end{aligned}$$

By considering $\zeta = \max\{k_1, k_2, k_3\}$, we have

$$|e_m(x)| \leqslant \zeta e^{(\pi d/(\alpha N))^{1/2}}$$

then the proof is completed.

Theorem 5 demonstrates that the mentioned method converges at the rate of $O\left(e^{-\vartheta\sqrt{N}}\right)$, where $\vartheta > 0$.

5. Numerical results

In this section, three examples are presented based on the Green's function and the Sinc-Collocation method Parand and Pirkhedri, 2010; Yuzbası and Sezer, 2013 for illustrating the effectiveness and importance of the proposed method. All experiments were performed in Mathematica 11.0. Also, in order to show the errors and the accuracy of the approximation, on the set of sinc grid points

$$S = \{x_{-N-1}, x_{-N}, \ldots, x_N, x_{N+1}\},$$

$$x_k = \frac{e^{kh}}{1+e^{kh}}, \quad k = -N-1, -N, \ldots, N, N+1,$$

we apply the following criteria:

1. The relative error is defined by

$$E_{rel} = \left|\frac{u_m(x_k) - u(x_k)}{u(x_k)}\right|. \tag{25}$$

2. The maximum absolute error is defined by

$$E_{abs} = \max_{-N-1 \leqslant k \leqslant N+1} |u(x_k) - u_m(x_k)|. \tag{26}$$

3. The root mean square (RMS) error is defined for $M = 2N+3$ by

$$RMS = \sqrt{\frac{1}{M}\sum_{k=-N-1}^{N+1}(u(x_k) - u_m(x_k))^2}. \tag{27}$$

4. The L_2 error norm is defined by

$$||.||_2 = \sqrt{\sum_{k=-N-1}^{N+1}(u(x_k) - u_m(x_k))^2}. \tag{28}$$

Table 1
Relative errors in the solution of Example 1.

x	$N = 10$	$N = 20$	$N = 30$	$N = 40$	$N = 50$
0.1	3.91×10^{-3}	2.38×10^{-4}	1.95×10^{-5}	1.88×10^{-6}	2.19×10^{-8}
0.3	5.73×10^{-4}	1.11×10^{-5}	3.94×10^{-7}	1.23×10^{-7}	1.51×10^{-8}
0.5	5.16×10^{-5}	6.21×10^{-7}	3.30×10^{-8}	3.27×10^{-9}	4.52×10^{-10}
0.7	1.20×10^{-4}	3.25×10^{-6}	4.64×10^{-8}	2.41×10^{-8}	3.33×10^{-9}
0.9	1.19×10^{-4}	5.83×10^{-6}	1.51×10^{-7}	2.80×10^{-8}	2.80×10^{-10}

Table 2
Errors in the solution of Example 1.

N	$h = \frac{\pi}{\sqrt{2N}}$	E_{abs}	$\|\|.\|\|_2$	RMS
10	0.702481	3.39×10^{-6}	6.55×10^{-6}	1.43×10^{-7}
20	0.496729	3.90×10^{-8}	1.20×10^{-7}	1.88×10^{-8}
30	0.405578	2.06×10^{-9}	8.83×10^{-9}	1.13×10^{-9}
40	0.351241	2.05×10^{-10}	1.04×10^{-9}	1.16×10^{-10}
50	0.314159	2.86×10^{-11}	1.62×10^{-10}	1.61×10^{-11}

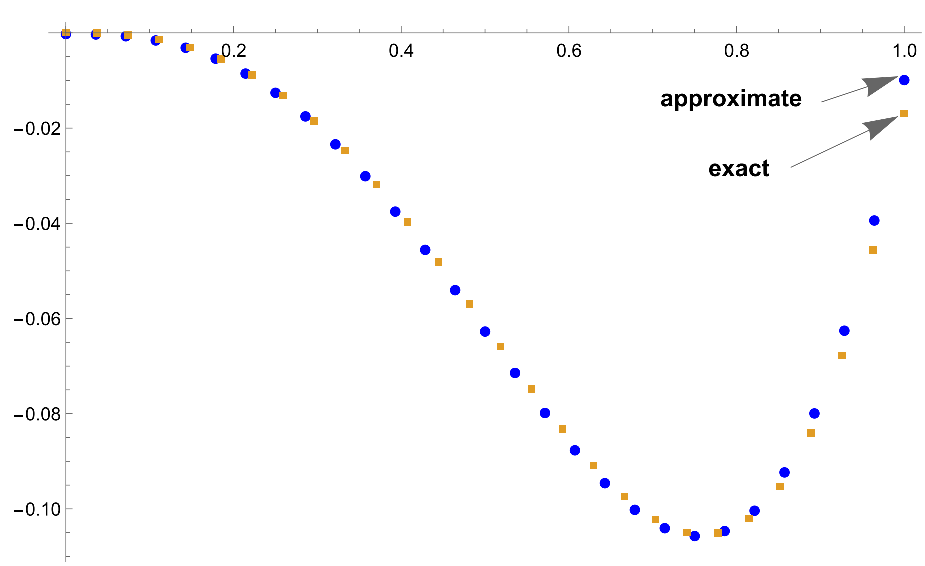

Fig. 1. Comparison between the exact and approximate solutions of Example 1 with $N = 10$.

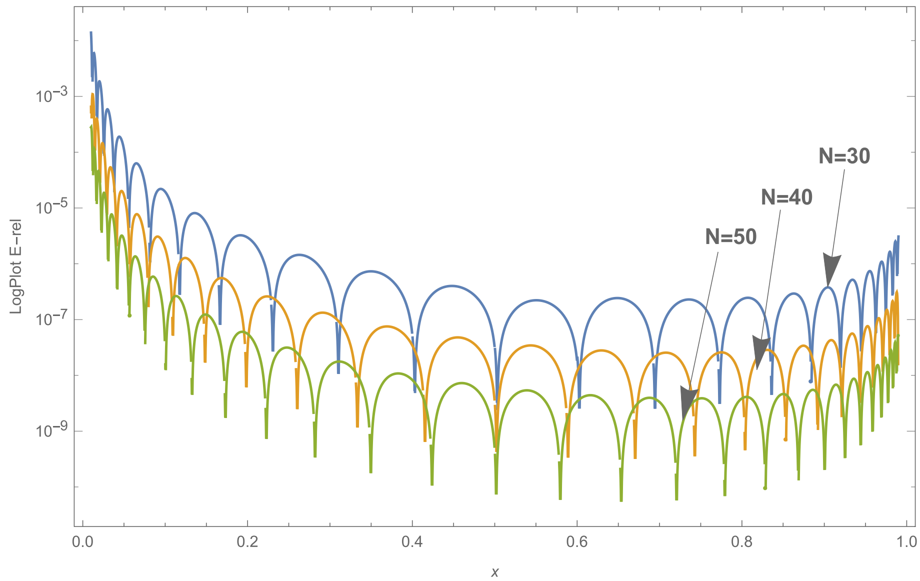

Fig. 2. LogPlot of the relative errors of Example 1 with different values of N.

Table 3
Relative errors in the solution of Example 2.

x	$N=10$	$N=20$	$N=30$	$N=40$	$N=50$
0.1	1.74×10^{-4}	1.96×10^{-6}	7.22×10^{-6}	2.35×10^{-6}	2.87×10^{-9}
0.3	4.23×10^{-5}	1.06×10^{-5}	1.05×10^{-6}	3.01×10^{-6}	1.66×10^{-6}
0.5	1.25×10^{-5}	1.02×10^{-6}	1.19×10^{-7}	1.88×10^{-8}	3.67×10^{-9}
0.7	4.65×10^{-5}	9.05×10^{-6}	1.01×10^{-6}	2.83×10^{-6}	1.59×10^{-6}
0.9	3.21×10^{-5}	4.79×10^{-6}	4.77×10^{-6}	2.05×10^{-6}	2.94×10^{-8}

Table 4
Errors in the solution of Example 2.

N	$h=\frac{\pi}{\sqrt{2N}}$	E_{abs}	$\|\|.\|\|_2$	RMS	$Cond(A)$
10	0.702481	6.40×10^{-4}	9.18×10^{-4}	1.96×10^{-4}	38.90
20	0.496729	1.73×10^{-4}	2.14×10^{-4}	3.31×10^{-5}	70.65
30	0.405578	1.38×10^{-4}	1.856×10^{-4}	2.35×10^{-5}	101.97
40	0.351241	9.90×10^{-5}	1.41×10^{-4}	1.55×10^{-5}	133.08
50	0.314159	7.34×10^{-5}	1.09×10^{-4}	1.08×10^{-5}	164.06

In our presented method, we take $d=\frac{\pi}{2}, \alpha=1$ and we applied our procedure for $N=10,20,30,40,$ and 50 and by using $h=(\pi d/(\alpha N))^{1/2}$ we can achieve h. The consistency of approximate and exact solution is shown in the Figures. By increasing N, the errors have been decreased in the Tables.

Example 1. We consider the following SBVP:

$$\begin{cases} y''(x)-\frac{3}{x}y'(x)=3x, & x\in[0,1] \\ y(0)=y(1)=0. \end{cases}$$

This problem has the exact solution $y(x)=x^4-x^3$, and by using (8) the Green's function is:

$$G(x,t)=\begin{cases} \frac{t^4(1-x^4)}{4t^3}, & 0\leqslant t\leqslant x; \\ \frac{x^4(1-t^4)}{4t^3}, & x\leqslant t\leqslant 1. \end{cases}$$

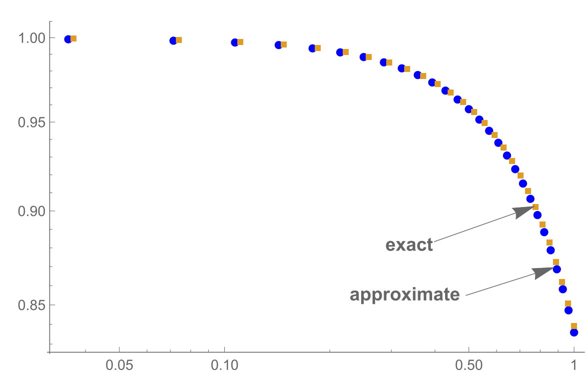

Fig. 3. Log–LogPlot of the exact and approximate solutions of Example 2 with $N=10$.

Table 5
Relative errors in the solution of Example 3.

x	$N=10$	$N=20$	$N=30$	$N=40$	$N=50$
0.1	8.67×10^{-5}	3.68×10^{-4}	1.58×10^{-4}	6.25×10^{-5}	1.01×10^{-6}
0.3	1.67×10^{-4}	2.53×10^{-4}	3.57×10^{-5}	7.04×10^{-5}	3.69×10^{-5}
0.5	1.26×10^{-3}	7.68×10^{-5}	8.63×10^{-6}	1.35×10^{-6}	2.62×10^{-7}
0.7	3.12×10^{-3}	2.87×10^{-5}	4.68×10^{-5}	5.33×10^{-5}	2.72×10^{-5}
0.9	6.01×10^{-3}	5.05×10^{-4}	2.84×10^{-5}	3.54×10^{-5}	1.73×10^{-6}

Table 6
Errors in the solution of Example 3.

N	$h=\frac{\pi}{\sqrt{2N}}$	E_{abs}	$\|\cdot\|_2$	RMS	$Cond(A)$
10	0.702481	4.59×10^{-2}	1.18×10^{-1}	2.52×10^{-2}	11.51
20	0.496729	2.56×10^{-2}	3.44×10^{-2}	5.31×10^{-3}	21.18
30	0.405578	1.47×10^{-2}	2.00×10^{-2}	2.54×10^{-3}	30.91
40	0.351241	9.36×10^{-3}	1.33×10^{-2}	1.47×10^{-3}	40.67
50	0.314159	6.60×10^{-3}	9.76×10^{-3}	9.67×10^{-4}	50.46

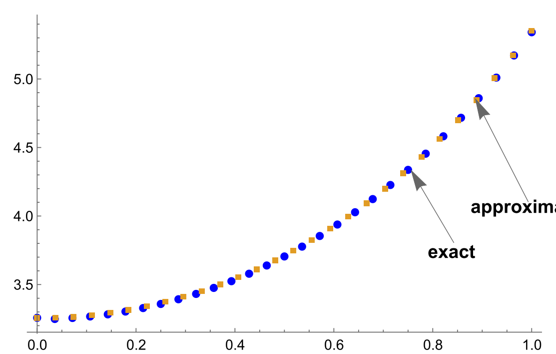

Fig. 4. Comparison between the exact and approximate solutions of Example 3 with $N=10$.

By increasing N, we tabulated the relative errors (25) in Table 1. Table 2 lists the errors (26)–(28). The graph of the approximate and exact solutions is in Fig. 1 which are coincide together. Fig. 2, illustrates LogPlot of the relative errors. In this example, we have $Cond(A)=\|A\|\|A^{-1}\|=1$ in the system (22) for a list of N.

Example 2. We consider the following SBVP:

$$\begin{cases} y''(x)+\frac{2}{x}y'(x)+1=0, x\in[0,1] \\ 6y(1)+3y'(1)=4, y(0)=1. \end{cases}$$

The exact solution is: $y(x)=1-\frac{x^2}{6}$. This may be transformed into the following form with homogeneous boundary conditions. If we use the relations (3) and (4), then we have:

$$\begin{cases} u''(x)+\frac{2}{x}u'(x)=\frac{4}{9x}-1,\ x\in[0,1] \\ u(0)=0,\quad 6u(1)+3u'(1)=0. \end{cases}$$

By using (8), the Green's function is

$$G(x,t)=\begin{cases} \left(\frac{-1}{2}+\frac{1}{x}\right)\left(-t^2\right), & 0\leqslant t\leqslant x; \\ \left(\frac{-1}{2}+\frac{1}{t}\right)\left(-t^2\right), & x\leqslant t\leqslant 1. \end{cases}$$

By increasing N, we tabulated the relative errors (25) in Table 3. Table 4 is consist of list of the errors (26)–(28), and Cond(A). The log–log plot of the approximate and exact solutions is in the Fig. 3.

Example 3. We consider the following SBVP:

$$\begin{cases} y''(x)+\frac{2}{x}y'(x)-4y=-2,\ x\in[0,1] \\ y'(0)=1,\quad y(1)=5.5, \end{cases}$$

The exact solution is: $y(x)=0.5+\frac{5\sinh(2x)}{x\sinh(2)}$. By using (8), the Green's function is

$$G(x,t)=\begin{cases} \frac{5e^4}{1-e^4}\left(-\frac{e^{2x}e^{-4}}{x}+\frac{e^{-2x}}{x}\right)\left(\frac{e^{2t}}{t}-\frac{e^{-2t}}{t}\right)t^2, & 0\leqslant t\leqslant x; \\ \frac{5e^4}{1-e^4}\left(\frac{e^{2x}}{x}-\frac{e^{-2x}}{x}\right)\left(-\frac{e^{2t}e^{-4}}{t}+\frac{e^{-2t}}{t}\right)t^2, & x\leqslant t\leqslant 1. \end{cases}$$

By increasing N, we tabulated the relative errors (25) in Table 5 and Table 6 is consist of list of the errors (26)–(28), and Cond(A). Fig. 4 shows the graph of the approximate and exact solutions which are coincide together.

6. Conclusion

In this paper, we have demonstrated that Sinc-Collocation method based on Green's function can be applied to solving a class of nonhomogeneous SBVPs. Numerical results indicate that by increasing N, the accuracy increases. In our approach, the convergence accuracy of the solution is $O\left(e^{(-\vartheta\sqrt{N})}\right)$, where $\vartheta>0$. This approach can be extended to the high dimensional (2D-3D) problems. According to our knowledge so far, we may need the use of Laplace transform to be combine with our approach.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A

The properties of the $G(x,t)$ are summarised as follows:

I. Boundary conditions:

$$\begin{cases} x=0: & a_0G(0,t)+a_1G_x(0,t)=0 \\ x=1: & b_0G(1,t)+b_1G_x(1,t)=0 \end{cases}$$

II. The condition of Continuity:

$$x=t,\ \lim_{x\to t^{+0}} G(x,t)=\lim_{x\to t^{-0}} G(x,t),$$

III. Jump discontinuity of the gradient:

$$\frac{\partial G}{\partial x}|_{t^{+0}}-\frac{\partial G}{\partial x}|_{t^{-0}}=-1.$$

Then a solution of the given BVPs (5)–(6) can be obtained as:

$$u(x)=-\int_0^1 G(x,t)f(t)dt.$$

Appendix B

$$\begin{aligned} &-\int_x^1 \tfrac{u_1(t)f(t)}{W(t)}dt=\int_1^x \tfrac{u_1(t)f(t)}{W(t)}dt \\ &=\int_0^{x-1} \tfrac{u_1(t+1)f(t+1)}{W(t+1)}dt \\ &=\int_0^x \tfrac{u_1(t+1)f(t+1)}{W(t+1)}dt-\int_{x-1}^x \tfrac{u_1(t+1)f(t+1)}{W(t+1)}dt \\ &=\int_0^x \tfrac{u_1(t+1)f(t+1)}{W(t+1)}dt-\int_0^1 \tfrac{u_1(t+x)f(t+x)}{W(t+x)}dt \end{aligned}$$

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