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Certain recent fractional integral inequalities associated with the hypergeometric operators

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Abstract The principle aim of this paper is to establish some new (presumably) fractional integral inequalities whose special cases are shown to yield corresponding inequalities associated with Riemann–Liouville type fractional integral operators by using hypergeometric fractional integral operator. Some relevant connections of the results presented here with those earlier ones are also pointed out.

1. Introduction and preliminaries

In recent years the study of fractional integral inequalities involving functions of independent variables is an important research subject in mathematical analysis because the inequality technique is also one of the very useful tools in the study of special functions and theory of approximations. During the last two decades or so, several interesting and useful extensions of many of the fractional integral inequalities have been considered by several authors (see, for example, Cerone and Dragomir, 2007; Choi and Agarwal, 2014a,b,c,d ; see also the very recent work Anber and Dahmani, 2013). The above-mentioned works have largely motivated our present study.

For our purpose, we begin by recalling the well-known celebrated functional introduced by Chebyshev (1882) and defined by

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)g(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right), \tag{1.1}$$

where $f(x)$ and $g(x)$ are two integrable functions which are synchronous on $[a,b]$, i.e.,

$$(f(x)-f(y))(g(x)-g(y)) \geqslant 0, \tag{1.2}$$

for any $x, y \in [a,b]$.

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The functional (1.1) has attracted many researchers' attention due to diverse applications in numerical quadrature, transform theory, probability and statistical problems. Among those applications, the functional (1.1) has also been employed to yield a number of integral inequalities (see, e.g., Anastassiou, 2011; Dragomir, 2000; Sulaiman, 2011; for a very recent work, see also Wang et al., 2014).

In 1935, Grüss (1935) proved the inequality

$$|T(f,g)| \leqslant \frac{(M-m)(N-n)}{4}, \tag{1.3}$$

where $f(x)$ and $g(x)$ are two integrable functions which are synchronous on $[a,b]$, i.e.,

$$m \leqslant f(x) \leqslant M, \quad n \leqslant g(x) \leqslant N, \tag{1.4}$$

for any $m, M, n, N \in \mathbb{R}$ and $x, y \in [a,b]$.

In the sequel, Pólya and Szegö (1925) introduced the following inequality

$$\frac{\int_a^b f^2(x)dx \int_a^b g^2(x)dx}{\left(\int_a^b f(x)dx \int_a^b g(x)dx\right)^2} \leqslant \frac{1}{4}\left(\sqrt{\frac{MN}{mn}} + \sqrt{\frac{mn}{MN}}\right)^2, \tag{1.5}$$

Similarly, Dragomir and Diamond (2003) proved that

$$|T(f,g)| \leqslant \frac{(M-m)(N-n)}{4(b-a)^2\sqrt{mMnN}} \int_a^b f(x)dx \int_a^b g(x)dx, \tag{1.6}$$

where $f(x)$ and $g(x)$ are two positive integrable functions which are synchronous on $[a,b]$, i.e.,

$$0 < m \leqslant f(x) \leqslant M < \infty, \quad 0 < n \leqslant g(x) \leqslant N < \infty. \tag{1.7}$$

Here, motivated essentially by above works, the main objective of this paper is to establish certain new (presumably) Pólya–Szegö type inequalities associated with Gaussian hypergeometric fractional integral operators. Relevant connections of the results presented here with those involving Riemann–Liouville fractional integrals are also indicated. Nowadays, the fractional calculus (fractional integral and derivative operators) has become one of the most rapidly growing research subjects of all branches of science due to its numerous applications. Recently many authors have showed the far-reaching development of the fractional calculus by their remarkably large number of contributions (see, e.g., Bhrawy and Zaky, 2015a,b; Bhrawy and Abdelkawy, in press; Bhrawy et al., 2015; Cattani, 2010; Jumarie, 2009; Komatsu, 1966, 1967; Li et al., 2011, 2013; Liu et al., 2014; Saxena, 1967; Yang et al., 2013a,b; Yang and Baleanu, 2013, and the related references therein).

Here, we start by recalling the following definition.

Definition 1. Let $\alpha > 0$, $\mu > -1$, β, $\eta \in \mathbb{R}$, then a generalized fractional integral $I_t^{\alpha,\beta,\eta,\mu}$ (in terms of the Gauss hypergeometric function) of order α for a real-valued continuous function $f(t)$ is defined by Choi and Agarwal (2014b, p. 285, Eq. (1.8)):

$$I_t^{\alpha,\beta,\eta,\mu}\{f(t)\} = \frac{t^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)} \int_0^t \tau^{\mu}(t-\tau)^{\alpha-1}{}_2F_1 \left(\alpha+\beta+\mu, -\eta; \alpha; 1-\frac{\tau}{t}\right) f(\tau)d\tau, \tag{1.8}$$

where, the function ${}_2F_1(-)$ appearing as a kernel for the operator (1.8) is the Gaussian hypergeometric function defined by

$${}_2F_1(a,b;c;t) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}, \tag{1.9}$$

and $(a)_n$ is the Pochhammer symbol:

$(a)_n = a(a+1)\cdots(a+n-1), \ (a)_0 = 1.$

For $f(t) = t^{\lambda-1}$ in (1.8), we get (see Baleanu et al., 2014)

$$I_t^{\alpha,\beta,\eta,\mu}\{t^{\lambda-1}\} = \frac{\Gamma(\mu+\lambda)\Gamma(\lambda-\beta+\eta)}{\Gamma(\lambda-\beta)\Gamma(\lambda+\mu+\alpha+\eta)} t^{\lambda-\beta-\mu-1}. \tag{1.10}$$

where α, β, η, $\lambda \in \mathbb{R}$, $\mu > -1$, $\mu+\lambda > 0$ and $\lambda-\beta+\eta > 0$.

2. Certain fractional integral inequalities associate with hypergeometric operator

In this section, we establish certain Pólya–Szegö type integral inequalities for the synchronous functions involving the hypergeometric fractional integral operator (1.8), some of which are (new) presumably ones.

Theorem 1. *Let f and g be two positive integrable functions on $[0,\infty)$. Assume that there exist four positive integrable functions u_1, u_2, v_1 and v_2 on $[0,\infty)$ such that:*

(A_1) $0 < u_1(\tau) \leqslant f(\tau) \leqslant u_2(\tau), \quad 0 < v_1(\tau) \leqslant g(\tau) \leqslant v_2(\tau),$
$(\tau \in [0,t], \ t > 0).$

Then for $t > 0$ and $\alpha > 0$, the following inequality holds:

$$\frac{I_t^{\alpha,\beta,\eta,\mu}\{v_1v_2f^2\}(t) I_t^{\alpha,\beta,\eta,\mu}\{u_1u_2g^2\}(t)}{\left(I_t^{\alpha,\beta,\eta,\mu}\{(v_1u_1+v_2u_2)fg\}(t)\right)^2} \leqslant \frac{1}{4}. \tag{2.1}$$

Proof. To prove (2.1), we start from (A_1), for $\tau \in [0,t], t > 0$, we have

$$\frac{f(\tau)}{g(\tau)} \leqslant \frac{u_2(\tau)}{v_1(\tau)}, \tag{2.2}$$

which yields

$$\left(\frac{u_2(\tau)}{v_1(\tau)} - \frac{f(\tau)}{g(\tau)}\right) \geqslant 0. \tag{2.3}$$

Analogously, we have

$$\frac{u_1(\tau)}{v_2(\tau)} \leqslant \frac{f(\tau)}{g(\tau)}, \tag{2.4}$$

from which one has

$$\left(\frac{f(\tau)}{g(\tau)} - \frac{u_1(\tau)}{v_2(\tau)}\right) \geqslant 0. \tag{2.5}$$

Multiplying (2.3) and (2.5), we obtain

$$\left(\frac{u_2(\tau)}{v_1(\tau)} - \frac{f(\tau)}{g(\tau)}\right)\left(\frac{f(\tau)}{g(\tau)} - \frac{u_1(\tau)}{v_2(\tau)}\right) \geqslant 0,$$

or

$$\left(\frac{u_2(\tau)}{v_1(\tau)} + \frac{u_1(\tau)}{v_2(\tau)}\right)\frac{f(\tau)}{g(\tau)} \geqslant \frac{f^2(\tau)}{g^2(\tau)} + \frac{u_1(\tau)u_2(\tau)}{v_1(\tau)v_2(\tau)}. \tag{2.6}$$

After some manipulation (2.6) can be written as

$$(u_1(\tau)v_1(\tau)+u_2(\tau)v_2(\tau))f(\tau)g(\tau) \geqslant v_1(\tau)v_2(\tau)f^2(\tau)+u_1(\tau)u_2(\tau)g^2(\tau). \tag{2.7}$$

Now, multiplying both sides of (2.7) by

$$\frac{t^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)}\tau^{\mu}(t-\tau)^{\alpha-1}{}_2F_1\left(\alpha+\beta+\mu,-\eta;\alpha;1-\frac{\tau}{t}\right)$$

and integrating with respect to τ from 0 to t, we get

$$I_t^{\alpha,\beta,\eta,\mu}\{(u_1v_1+u_2v_2)fg\}(t) \geqslant I_t^{\alpha,\beta,\eta,\mu}\{v_1v_2f^2\}(t)+I_t^{\alpha,\beta,\eta,\mu}\{u_1u_2g^2\}(t).$$

Applying the AM–GM inequality, i.e., $a+b \geqslant 2\sqrt{ab},\ a,b\in\mathbb{R}^+$, we have

$$I_t^{\alpha,\beta,\eta,\mu}\{(u_1v_1+u_2v_2)fg\}(t) \geqslant 2\sqrt{I_t^{\alpha,\beta,\eta,\mu}\{v_1v_2f^2\}(t)I_t^{\alpha,\beta,\eta,\mu}\{u_1u_2g^2\}(t)},$$

which leads to

$$I_t^{\alpha,\beta,\eta,\mu}\{v_1v_2f^2\}(t)I_t^{\alpha,\beta,\eta,\mu}\{u_1u_2g^2\}(t) \leqslant \frac{1}{4}\left(I_t^{\alpha,\beta,\eta,\mu}\{(u_1v_1+u_2v_2)fg\}(t)\right)^2.$$

Therefore, we obtain the inequality (2.1) as requested. □

Theorem 2. *Let f and g be two positive integrable functions on $[0,\infty)$. Assume that there exist four positive integrable functions u_1,u_2, v_1 and v_2 satisfying (A_1) on $[0,\infty)$. Then for $t>0$ and $\alpha,\beta>0$, the following inequality holds:*

$$\frac{I_t^{\alpha,\beta,\eta,\mu}\{u_1u_2\}(t)I_t^{\gamma,\delta,\zeta,\nu}\{v_1v_2\}(t)I_t^{\alpha,\beta,\eta,\mu}\{f^2\}(t)I_t^{\gamma,\delta,\zeta,\nu}\{g^2\}(t)}{\left(I_t^{\alpha,\beta,\eta,\mu}\{u_1f\}(t)I_t^{\gamma,\delta,\zeta,\nu}\{v_1g\}(t)+I_t^{\alpha,\beta,\eta,\mu}\{u_2f\}(t)I_t^{\gamma,\delta,\zeta,\nu}\{v_2g\}(t)\right)^2} \leqslant \frac{1}{4}. \tag{2.8}$$

Proof. To prove (2.8), using the condition (A_1), we obtain

$$\left(\frac{u_2(\tau)}{v_1(\rho)}-\frac{f(\tau)}{g(\rho)}\right) \geqslant 0, \tag{2.9}$$

and

$$\left(\frac{f(\tau)}{g(\rho)}-\frac{u_1(\tau)}{v_2(\rho)}\right) \geqslant 0, \tag{2.10}$$

which imply that

$$\left(\frac{u_1(\tau)}{v_2(\rho)}+\frac{u_2(\tau)}{v_1(\rho)}\right)\frac{f(\tau)}{g(\rho)} \geqslant \frac{f^2(\tau)}{g^2(\rho)}+\frac{u_1(\tau)u_2(\tau)}{v_1(\rho)v_2(\rho)}. \tag{2.11}$$

Multiplying both sides of (2.11) by $v_1(\rho)v_2(\rho)g^2(\rho)$, we have

$$u_1(\tau)f(\tau)v_1(\rho)g(\rho)+u_2(\tau)f(\tau)v_2(\rho)g(\rho) \geqslant v_1(\rho)v_2(\rho)f^2(\tau)+u_1(\tau)u_2(\tau)g^2(\rho). \tag{2.12}$$

Multiplying both sides of (2.12) by

$$\frac{t^{-\alpha-\beta-\gamma-\delta-2(\mu+\nu)}}{\Gamma(\alpha)\Gamma(\gamma)}\tau^{\mu}(t-\tau)^{\alpha-1}\rho^{\nu}(t-\rho)^{\gamma-1} \times {}_2F_1\left(\alpha+\beta+\mu,-\eta;\alpha;1-\tfrac{\tau}{t}\right){}_2F_1\left(\gamma+\delta+\nu,-\zeta;\gamma;1-\tfrac{\rho}{t}\right)$$

and double integrating with respect to τ and ρ from 0 to t, we have

$$I_t^{\alpha,\beta,\eta,\mu}\{u_1f\}(t)I_t^{\gamma,\delta,\zeta,\nu}\{v_1g\}(t)+I_t^{\alpha,\beta,\eta,\mu}\{u_2f\}(t)I_t^{\gamma,\delta,\zeta,\nu}\{v_2g\}(t) \geqslant I_t^{\alpha,\beta,\eta,\mu}\{f^2\}(t)I_t^{\gamma,\delta,\zeta,\nu}\{v_1v_2\}(t)+I_t^{\alpha,\beta,\eta,\mu}\{u_1u_2\}(t)I_t^{\gamma,\delta,\zeta,\nu}\{g^2\}(t).$$

Applying the AM–GM inequality, we get

$$I_t^{\alpha,\beta,\eta,\mu}\{u_1f\}(t)I_t^{\gamma,\delta,\zeta,\nu}\{v_1g\}(t)+I_t^{\alpha,\beta,\eta,\mu}\{u_2f\}(t)I_t^{\gamma,\delta,\zeta,\nu}\{v_2g\}(t) \geqslant 2\sqrt{I_t^{\alpha,\beta,\eta,\mu}\{f^2\}(t)I_t^{\gamma,\delta,\zeta,\nu}\{v_1v_2\}(t)I_t^{\alpha,\beta,\eta,\mu}\{u_1u_2\}(t)I_t^{\gamma,\delta,\zeta,\nu}\{g^2\}(t)},$$

which leads to the desired inequality in (2.8). The proof is completed. □

Theorem 3. *Let f and g be two positive integrable functions on $[0,\infty)$. Assume that there exist four positive integrable functions u_1,u_2, v_1 and v_2 satisfying (A_1) on $[0,\infty)$. Then for $t>0$ and $\alpha,\beta>0$, the following inequality holds:*

$$I_t^{\alpha,\beta,\eta,\mu}\{f^2\}(t)I_t^{\gamma,\delta,\zeta,\nu}\{g^2\}(t) \leqslant I_t^{\alpha,\beta,\eta,\mu}\{(u_2fg)/v_1\}(t)I_t^{\gamma,\delta,\zeta,\nu}\{(v_2fg)/u_1\}(t) \tag{2.13}$$

Proof. From (2.2), we have

$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f^2(\tau)d\tau \leqslant \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\frac{u_2(\tau)}{v_1(\tau)}f(\tau)g(\tau)d\tau, \tag{2.14}$$

which implies

$$I_t^{\alpha,\beta,\eta,\mu}\{f^2\}(t) \leqslant I_t^{\alpha,\beta,\eta,\mu}\{(u_2fg)/v_1\}(t). \tag{2.15}$$

By (2.4), we get

$$\frac{1}{\Gamma(\beta)}\int_0^t (t-\rho)^{\beta-1}g^2(\rho)d\rho \leqslant \frac{1}{\Gamma(\beta)}\int_0^t (t-\rho)^{\beta-1}\frac{v_2(\rho)}{u_1(\rho)}f(\rho)g(\rho)d\rho,$$

from which one has

$$I_t^{\gamma,\delta,\zeta,\nu}\{g^2\}(t) \leqslant I_t^{\gamma,\delta,\zeta,\nu}\{(v_2fg)/u_1\}(t). \tag{2.16}$$

Multiplying (2.15) and (2.16), we get the desired inequality in (2.13). □

3. Special cases and concluding remarks

We now, briefly consider some consequences of the results derived in the previous sections. Following Curiel and Galué (1996), the operator (1.2) would reduce immediately to the extensively investigated Saigo, Erdélyi–Kober and Riemann–Liouville type fractional integral operators, respectively, given by the following relationships (see also Curiel and Galué, 1996 and Kiryakova, 1994):

$$I_{0,t}^{\alpha,\beta,\eta}\{f(t)\} = I_t^{\alpha,\beta,\eta,0}\{f(t)\} = \frac{t^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right)f(\tau)d\tau \quad (\alpha>0,\beta,\eta\in\mathbb{R}) \tag{3.1}$$

$$I^{\alpha,\eta}\{f(t)\} = I_t^{\alpha,0,\eta,0}\{f(t)\} = \{f(t)\} = \frac{t^{-\alpha-\eta}}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\tau^{\eta}f(\tau)d\tau \quad (\alpha>0,\eta\in\mathbb{R}), \tag{3.2}$$

and

$$R^{\alpha}\{f(t)\} = I_t^{\alpha,-\alpha,\eta,0}\{f(t)\} = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau)d\tau \quad (\alpha>0). \tag{3.3}$$

For example, if we set $\mu = 0$ in Theorem 1 and $\mu = \nu = 0$in Theorem 2 and 3, using (3.1), the inequality (2.1), (2.8) and (2.13) gives the following results involving Saigos fractional integral operators, which are believed to be new:

Corollary 1. *Let f and g be two positive integrable functions on* $[0,\infty)$*. Assume that there exist four positive integrable functions* u_1, u_2, v_1 *and* v_2 *satisfying* (A_1) *on* $[0,\infty)$*. Then for* $t > 0$ *and* $\alpha > 0$*, the following inequality holds:*

$$\frac{I_{0,t}^{\alpha,\beta,\eta}\{v_1 v_2 f^2\}(t) I_{0,t}^{\alpha,\beta,\eta}\{u_1 u_2 g^2\}(t)}{\left(I_{0,t}^{\alpha,\beta,\eta}\{(v_1 u_1 + v_2 u_2) fg\}(t)\right)^2} \leqslant \frac{1}{4}. \tag{3.4}$$

Corollary 2. *Let f and g be two positive integrable functions on* $[0,\infty)$*. Assume that there exist four positive integrable functions* u_1, u_2, v_1 *and* v_2 *satisfying* (A_1) *on* $[0,\infty)$*. Then for* $t > 0$ *and* $\alpha, \beta > 0$*, the following inequality holds:*

$$\frac{I_{0,t}^{\alpha,\beta,\eta}\{u_1 u_2\}(t) I_{0,t}^{\gamma,\delta,\zeta}\{v_1 v_2\}(t) I_{0,t}^{\alpha,\beta,\eta}\{f^2\}(t) I_{0,t}^{\gamma,\delta,\zeta}\{g^2\}(t)}{\left(I_{0,t}^{\alpha,\beta,\eta}\{u_1 f\}(t) I_{0,t}^{\gamma,\delta,\zeta}\{v_1 g\}(t) + I_{0,t}^{\alpha,\beta,\eta}\{u_2 f\}(t) I_{0,t}^{\gamma,\delta,\zeta}\{v_2 g\}(t)\right)^2} \leqslant \frac{1}{4}. \tag{3.5}$$

Corollary 3. *Let f and g be two positive integrable functions on* $[0,\infty)$*. Assume that there exist four positive integrable functions* u_1, u_2, v_1 *and* v_2 *satisfying* (A_1) *on* $[0,\infty)$*. Then for* $t > 0$ *and* $\alpha, \beta > 0$*, the following inequality holds:*

$$I_{0,t}^{\alpha,\beta,\eta}\{f^2\}(t) I_{0,t}^{\gamma,\delta,\zeta}\{g^2\}(t) \leqslant I_{0,t}^{\alpha,\beta,\eta}\{(u_2 fg)/v_1\}(t) I_{0,t}^{\gamma,\delta,\zeta}\{(v_2 fg)/u_1\}(t) \tag{3.6}$$

Similarly, if we set $\mu = \beta = 0$ in Theorem 1 and $\mu = \nu = \beta = \delta = 0$ in Theorem 2 and 3, using (3.2), the inequality (2.1), (2.8) and (2.13) gives the following results involving Erdélyi–Kober fractional integral operators, which are also believed to be new:

Corollary 4. *Let f and g be two positive integrable functions on* $[0,\infty)$*. Assume that there exist four positive integrable functions* u_1, u_2, v_1 *and* v_2 *satisfying* (A_1) *on* $[0,\infty)$*. Then for* $t > 0$ *and* $\alpha > 0$*, the following inequality holds:*

$$\frac{I_t^{\alpha,\eta}\{v_1 v_2 f^2\}(t) I_t^{\alpha,\eta}\{u_1 u_2 g^2\}(t)}{\left(I_t^{\alpha,\eta}\{(v_1 u_1 + v_2 u_2) fg\}(t)\right)^2} \leqslant \frac{1}{4}. \tag{3.7}$$

Corollary 5. *Let f and g be two positive integrable functions on* $[0,\infty)$*. Assume that there exist four positive integrable functions* u_1, u_2, v_1 *and* v_2 *satisfying* (A_1) *on* $[0,\infty)$*. Then for* $t > 0$ *and* $\alpha, \beta > 0$*, the following inequality holds:*

$$\frac{I_t^{\alpha,\eta}\{u_1 u_2\}(t) I_t^{\gamma,\zeta}\{v_1 v_2\}(t) I_t^{\alpha,\eta}\{f^2\}(t) I_t^{\gamma,\zeta}\{g^2\}(t)}{\left(I_t^{\alpha,\eta}\{u_1 f\}(t) I_t^{\gamma,\zeta}\{v_1 g\}(t) + I_t^{\alpha,\eta}\{u_2 f\}(t) I_t^{\gamma,\zeta}\{v_2 g\}(t)\right)^2} \leqslant \frac{1}{4}. \tag{3.8}$$

Corollary 6. *Let f and g be two positive integrable functions on* $[0,\infty)$*. Assume that there exist four positive integrable functions* u_1, u_2, ψ_1 *and* ψ_2 *satisfying* (A_1) *on* $[0,\infty)$*. Then for* $t > 0$ *and* $\alpha, \beta > 0$*, the following inequality holds:*

$$I_t^{\alpha,\eta}\{f^2\}(t) I_t^{\gamma,\zeta}\{g^2\}(t) \leqslant I_t^{\alpha,\eta}\{(u_2 fg)/v_1\}(t) I_t^{\gamma,\zeta}\{(v_2 fg)/u_1\}(t). \tag{3.9}$$

For another example, if we put $\mu = 0$ in Theorem 1 and $\mu, \nu = 0$ in Theorem 2 and 3, replace β by $-\alpha$ and β, δ by $-\alpha, -\gamma$ in Theorem 1 and 2, respectively, and use (3.3), the inequalities (2.1), (2.8) and (2.13) gives known results involving Riemann–Liouville fractional integral operators (see Ntouyas et al., submitted).

Furthermore, we also get some more special cases of Theorem 1–3, as follows:

Corollary 7. *Let f and g be two positive integrable functions on* $[0,\infty)$ *satisfying*

$$(A_2)\ 0 < m \leqslant f(\tau) \leqslant M < \infty, \quad 0 < n \leqslant g(\tau) \leqslant N < \infty, \quad (\tau \in [0,t],\ t > 0).$$

Then for $t > 0$ *and* $\alpha > 0$*, we have*

$$\frac{\left(I_t^{\alpha,\beta,\eta,\mu}\{f^2\}(t)\right)\left(I_t^{\alpha,\beta,\eta,\mu}\{g^2\}(t)\right)}{\left(I_t^{\alpha,\beta,\eta,\mu}\{fg\}(t)\right)^2} \leqslant \frac{1}{4}\left(\sqrt{\frac{mn}{MN}} + \sqrt{\frac{MN}{mn}}\right)^2. \tag{3.10}$$

Corollary 8. *Let f and g be two positive integrable functions on* $[0,\infty)$ *satisfying* (A_2)*. Then for* $t > 0$ *and* $\alpha, \beta > 0$*, we have*

$$\frac{t^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)} \frac{\left(I_t^{\alpha,\beta,\eta,\mu}\{f^2\}(t)\right)\left(I_t^{\gamma,\delta,\zeta,\nu}\{g^2\}(t)\right)}{\left(I_t^{\alpha,\beta,\eta,\mu}\{f\}(t) I_t^{\gamma,\delta,\zeta,\nu}\{g\}(t)\right)^2} \leqslant \frac{1}{4}\left(\sqrt{\frac{mn}{MN}} + \sqrt{\frac{MN}{mn}}\right)^2. \tag{3.11}$$

Corollary 9. *Let f and g be two positive integrable functions on* $[0,\infty)$ *satisfying* (A_2)*. Then for* $t > 0$ *and* $\alpha, \beta > 0$*, we have*

$$\frac{\left(I_t^{\alpha,\beta,\eta,\mu}\{f^2\}(t)\right)\left(I_t^{\gamma,\delta,\zeta,\nu}\{g^2\}(t)\right)}{I_t^{\alpha,\beta,\eta,\mu}\{fg\}(t) I_t^{\gamma,\delta,\zeta,\nu}\{fg\}(t)} \leqslant \frac{MN}{mn}. \tag{3.12}$$

4. Concluding remark

We conclude our present study with the remark that our main result here, being of a very general nature, can be specialized to yield numerous interesting fractional integral inequalities including some known results. Furthermore, they are expected to find some applications for establishing uniqueness of solutions in fractional boundary value problems in the fractional partial differential equations.

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