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ORIGINAL ARTICLE

Exp and modified Exp function methods for nonlinear Drinfeld–Sokolov system

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Abstract In this paper, Exp-function and its modification methods have been applied to obtain an exact solution of the nonlinear Drinfeld–Sokolov system (DS). Modification of the method was first introduced by the same authors. The prominent merit of this method is to facilitate the process of solving systems of partial differential equations. These methods are straightforward and concise by themselves; moreover, their applications are promising to obtain exact solutions of various partial differential equations. It is shown that the methods, with the help of symbolic computation, provide very effective and powerful mathematical tools for solving such systems.

1. Introduction

Mathematical modelings of many real phenomena lead to a non-linear ordinary or partial differential equations in various fields of physics and engineering. There are some methods to obtain approximate or exact solutions of these kinds of equations, such as the tanh method (Wazwaz, 2005; Malfliet and Hereman, 1996), sine–cosine method (Wazwaz, 2006), homotopy perturbation method (Biazar and Ghazvini, 2007; He, 2005), variational iteration method (He, 1999; He, 2000), Adomian decomposition method (Biazar et al., 2003), and many others (Wang, 1996; Abdou, 2007; Wang and Zhang, 2005; Wang et al., 2008). Most recently, a novel approach called the Exp-function method (He and Wu, 2006; Zhang, 2007; Biazar and Ayati, 2008) has been developed to obtain solutions of various nonlinear equations. The solution procedure of this method, by the help of any mathematical packages, say Matlab or Maple, is of utter simplicity. The modified version of this method was first presented in Biazar and Ayati (2009) by current authors. There, it was used to solve the system of partial differential equation directly and without change to ordinary differential equation.

In this paper, the nonlinear Drinfeld–Sokolov system is considered, in the following form, and is solved by the Exp function method

$$\begin{cases} u_t + (v^2)_x = 0, \\ v_t - av_{xxx} + 3bu_x v + 3cuv_x = 0. \end{cases} \tag{1}$$

where a, b, and c are constants. This system was introduced by Drinfeld and Sokolov as an example of a system of nonlinear equations possessing Lax pairs of a special form (Goktas and Hereman, 1997; Wazwaz, 2006).

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Let us introduce a complex variable ξ, as follows

$$\xi = kx + wt. \tag{2}$$

So, Eq. (1) turns to the following system of ordinary different equation,

$$\begin{cases} wu' + k(v^2)' = 0, \\ wv' - akv''' + 3bku'v + 3ckuv' = 0. \end{cases} \tag{3}$$

where k and w are constant to be determined. A simplified form of the first equation will be derived by taking integral from both sides of that. Let us consider the integral constant zero.

$$u = -k\frac{v^2}{w}. \tag{4}$$

Substituting Eq. (4) into the second equation of the system and integrating lead to

$$w^2v - awk^3v'' - (2b + c)k^2v^3 = 0. \tag{5}$$

where c, d, p, and q are positive integers which could be freely chosen, a_m for $m = -d, \ldots, c$ and b_n for $n = -q, \ldots, p$ are unknown constants to be determined. To find the values of c and p, we balance the linear terms of the highest order in Eq. (8) with the highest order nonlinear terms.

Similarly to find out the values of d and q, we balance the linear terms of the lowest order in Eq. (8) with the lowest order nonlinear terms.

3. Exp function method for the DS system

The Exp function method as well addressed in He and Wu (2006), Zhang (2007), Biazar and Ayati (2008), and in this part it will be applied to obtain the solution of the Drienfeld–Sokolov system.

We assume that the solution of Eq. (5) can be expressed in the form shown in the following form

$$u(\xi) = \frac{a_c \exp(c\xi) + \cdots + a_{-d}\exp(-d\xi)}{b_p \exp(p\xi) + \cdots + b_{-q}\exp(-q\xi)}, \tag{6}$$

In order to determine the constants c and p, we balance the linear term of the highest order in Eq. (5) with the highest order nonlinear term. By simple calculation, we have

$$v'' = \frac{c_1 \exp[(3p + c)\xi] + \cdots}{c_2 \exp[4p\xi] + \cdots}, \tag{7}$$

and

$$v^3 = \frac{c_3 \exp[3c\xi] + \cdots}{c_4 \exp[3p\xi] + \cdots} = \frac{c_3 \exp[(p + 3c)\xi] + \cdots}{c_4 \exp[4p\xi] + \cdots}. \tag{8}$$

By balancing the highest order terms of Exp-functions in Eqs. (7) and (8), we have

$$c + 3p = 3c + p, \tag{9}$$

which leads to the result:

$$p = c. \tag{10}$$

Similarly, we balance the lowest order terms in Eq. (5) to determine values of d and q, we obtain:

$$d = q. \tag{11}$$

It is possible to choose the values of c and d, too.

3.1. The choice of $p = c = 1$, and $q = d = 1$

We choose $p = c = 1$, and $q = d = 1$, the trial function, Eq. (6) converts to the following form

$$v(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1}\exp(-\xi)}{b_1 \exp(\xi) + b_0 + b_{-1}\exp(-\xi)}. \tag{12}$$

In the case $b_1 \neq 0$ Eq. (12) can be simplified as

$$v(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1}\exp(-\xi)}{\exp(\xi) + b_0 + b_{-1}\exp(-\xi)}. \tag{13}$$

Substituting Eq. (13) into Eq. (5), and taking the coefficients of $\exp(n\xi)$ in each term zero yield to a set of algebraic equations for a_1, a_0, a_{-1}, b_0, b_{-1}, k, and w. Solving this system of algebraic equations by the aid of Maple, or via any others, leads to

$$a_{-1} = 0,\ \ a_0 = a_0,\ \ a_1 = 0,\ \ b_{-1} = \frac{1}{8}\frac{a_0^2(2b + c)}{k^4a^2},\ \ b_0 = 0,\ \ k = k,\ \ w = ak^3. \tag{14}$$

where a_0 and k are free parameters. Substituting Eq. (14) into Eq. (13), we obtain the following exact solution

$$v(x,t) = \frac{a_0}{\exp(kx + ak^3t) + \frac{1}{8}\frac{a_0^2(2b+c)}{k^4a^2}\exp(-kx - ak^3t)}. \tag{15}$$

If we set $a_0 = \frac{2\sqrt{2}k^2a}{\sqrt{2b+c}}$, and $r = -ak^2$, Eq. (15) reduces to

$$v(x,t) = r\sqrt{\frac{2}{2b + c}}\sec h\left(\sqrt{\frac{-r}{a}}(x - rt)\right). \tag{16}$$

For $a_0 = \frac{2\sqrt{2}k^2a}{\sqrt{2b+c}}\mathrm{i}$, and $r = -ak^2$, we get

$$v(x,t) = -\mathrm{i}r\sqrt{\frac{2}{2b + c}}\csc h\left(\sqrt{\frac{-r}{a}}(x - rt)\right). \tag{17}$$

These solutions are the same as the Wazwaz's solution Wazwaz's (2006) solution.

3.2. The choice of $p = c = 2$, and $q = d = 1$

If we choose $p = c = 2$, and $q = d = 1$, Eq. (6) takes the following form:

$$v(\xi) = \frac{a_2 \exp(2\xi) + a_1 \exp(\xi) + a_0 + a_{-1}\exp(-\xi)}{\exp(2\xi) + b_1 \exp(\xi) + b_0 + b_{-1}\exp(-\xi)}. \tag{18}$$

Proceeding in a similar way as illustrated in Section 3.1, we can identify parameters, a_2, a_1, a_0, a_{-1}, b_1, b_0, b_{-1}, w, and k in Eq. (39) as the following

$$a_2 = 0,\ \ a_1 = a_1,\ \ a_0 = 0,\ \ a_{-1} = 0,\ \ b_1 = 0,\ \ b_0 = \frac{1}{8}\frac{a_1^2(2b + c)}{k^4a^2},\ \ b_{-1} = 0,\ \ w = ak^3,\ \ k = k. \tag{19}$$

$$a_2 = \pm\frac{1}{2}\frac{ak^2}{\sqrt{2b+c}},\ \ a_1 = \pm(1 + \sqrt{2})\sqrt{\pm\frac{a_0a}{\sqrt{2b+c}}}k,\ \ a_0 = a_0,\ \ a_{-1} = 0,\ \ b_1 = 0,\ \ b_0 = \pm\frac{2a_0}{k^2a}\sqrt{2b + c},\ \ b_{-1} = 0,\ \ w = -\frac{1}{2}ak^3,\ \ k = k. \tag{20}$$

Substituting Eqs. (19) and (20) into Eq. (18), we obtain the following exact solutions;

$$v(x,t)=\frac{a_1\exp(kx+ak^3t)}{\exp 2(kx+ak^3t)+\frac{1}{8}\frac{a_1^2(2b+c)}{k^4a^2}},\qquad(21)$$

Which is the same as (15).

$$v(x,t)=\frac{\pm\frac{1}{2}\frac{ak^2}{\sqrt{2b+c}}\exp(2kx-ak^3t)\pm(1+\sqrt{2})\sqrt{\pm\frac{a_0a}{\sqrt{2b+c}}}k\exp\left(kx-\frac{1}{2}ak^3t\right)+a_0}{\exp(2kx-ak^3t)\pm\frac{2a_0}{k^2a}\sqrt{2b+c}}.$$

3.3. The choice of $p = c = 2$, and $q = d = 2$

In this case, the trial function (6) can be expressed as follows

$$v(\xi)=\frac{a_2\exp(2\xi)+a_1\exp(\xi)+a_0+a_{-1}\exp(-\xi)+a_{-2}\exp(-2\xi)}{\exp(2\xi)+b_1\exp(\xi)+b_0+b_{-1}\exp(-\xi)+b_{-2}\exp(-2\xi)}.\qquad(23)$$

There are some free parameters in Eq. (23), for the sake of simplicity we take $b_1 = 0$ and $b_{-1} = 0$, the trial function, Eq. (24) can be simplified as follows:

$$v(\xi)=\frac{a_2\exp(2\xi)+a_1\exp(\xi)+a_0+a_{-1}\exp(-\xi)+a_{-2}\exp(-2\xi)}{\exp(2\xi)+b_0+b_{-2}\exp(-2\xi)}.\qquad(24)$$

Proceeding in a the same way as illustrated in 3.1, we can identify parameters, a_2, a_1, a_0, a_{-1}, a_{-2}, b_1, b_0, b_{-1}, b_{-2}, w, s , and $k$in Eq. (24) as the following

$$a_2=0,\ a_1=0,\ a_0=a_0,\ a_{-1}=0,\ a_{-2}=0,\ b_0=0,$$
$$b_{-2}=\frac{1}{128}\frac{a_0^2(2b+c)}{k^4a^2},\ w=4ak^3,\ k=k.\qquad(25)$$

$$a_2=0,\ a_1=a_1,\ a_0=0,\ a_{-1}=0,\ a_{-2}=0,$$
$$b_0=\frac{1}{8}\frac{a_1^2(2b+c)}{k^4a^2},\ b_{-2}=0,\ w=ak^3,\ k=k.\qquad(26)$$

$$a_2=\pm\frac{1}{2}\frac{ak^2}{\sqrt{2b+c}},\ a_1=a_1,\ a_0=\pm\frac{1}{4}\frac{a_1^2\sqrt{2b+c}}{k^2a},$$
$$a_{-1}=0,\ a_{-2}=0,\ b_0=-\frac{a_1^2(2b+c)}{k^4a^2},\ b_{-2}=0,$$
$$w=-\frac{1}{2}ak^3,\ k=k.\qquad(27)$$

Substituting Eqs. (25)–(27) into Eq. (24), the following exact solutions will be obtained;

$$v(x,t)=\frac{a_0}{\exp(2kx+8ak^3t)+\frac{1}{128}\frac{a_0^2(2b+c)}{k^4a^2}\exp(-2kx-8ak^3t)},\qquad(28)$$

Which is the same as Eq. (15).

$$v(x,t)=\frac{\pm\frac{1}{2}\frac{ak^2}{\sqrt{2b+c}}\exp(2kx-ak^3t)+a_1\exp(kx-\frac{1}{2}ak^3t)\pm\frac{1}{4}\frac{a_1^2\sqrt{2b+c}}{k^2a}}{\exp(2kx-ak^3t)-\frac{a_1^2(2b+c)}{k^4a^2}}.\qquad(29)$$

4. Application of the modified Exp function method to the DS system

In order to use the modified Exp function method, the solution of Eq. (3), is assumed to be expressed in the following forms

$$u(\xi)=\frac{a_c\exp(c\xi)+\cdots+a_{-d}\exp(-d\xi)}{b_p\exp(p\xi)+\cdots+b_{-q}\exp(-q\xi)},$$
$$v(\xi)=\frac{a'_m\exp(m\xi)+\cdots+a'_{-n}\exp(-n\xi)}{b'_l\exp(l\xi)+\cdots+b'_{-r}\exp(-r\xi)},\qquad(30)$$

where c, d, p, q, m, n, l, and r are positive integers and a_k, b_k, a'_k, and b'_k's are unknown constants to be determined. To find the values of parameters c, p, m, and l, the linear terms of the highest order will be balanced with the highest order nonlinear terms, in Eq. (3). Similarly, to determine the values of parameters d, q, n, and r, the linear terms of the lowest order will be balanced with the lowest order nonlinear terms in Eq. (3). By simple manipulations in the first equation in (3), it is acquired

$$u'=\frac{c_3\exp[(p+c)\xi]+\cdots}{c_4\exp[2p\xi]+\cdots}=\frac{c_3\exp[(p+c+3l)\xi]+\cdots}{c_4\exp[(2p+3l)\xi]+\cdots},\qquad(31)$$

$$vv'=\frac{c_3\exp[(2m+l)\xi]+\cdots}{c_4\exp[3l\xi]+\cdots}=\frac{c_3\exp[(2m+l+2p)\xi]+\cdots}{c_4\exp[(2p+3l)\xi]+\cdots}.\qquad(32)$$

By balancing the highest order of Exp-function in Eqs. (31) and (32), we derive:

$$2m+l+2p=c+p+3l.\qquad(33)$$

By the same way from the second equation in (3) we get

$$v'''=\frac{c_3\exp[(m+7l)\xi]+\cdots}{c_4\exp[8l\xi]+\cdots}=\frac{c_3\exp[(m+7l+2p)\xi]+\cdots}{c_4\exp[(8l+2p)\xi]+\cdots},\qquad(34)$$

$$uv'=\frac{d_3\exp[(c+m+l)\xi]+\cdots}{d_4\exp[(2l+p)\xi]+\cdots}=\frac{d_3\exp[(c+m+7l+p)\xi]+\cdots}{d_4\exp[(8l+2p)\xi]+\cdots},\qquad(35)$$

and

$$vu'=\frac{d_5\exp[(c+m+p)\xi]+\cdots}{d_6\exp[(2p+l)\xi]+\cdots}=\frac{d_5\exp[(c+m+p+7l)\xi]+\cdots}{d_6\exp[(2p+8l)\xi]+\cdots}.\qquad(36)$$

So

$$p+c+m+7l=m+7l+2p,\qquad(37)$$

Eqs. (33) and (37) lead to the following result:

$$m=l,\quad p=c$$

The values of d, q, n, and r can be determined in a similar way. By balancing the linear term of the lowest order in Eq. (3) , the result will be as following

$$d=q,\quad n=r$$

Values of c, d, m, and n can be chosen arbitrary. Considering the simplest case, let us take $p = c = 1$, $d = q = 1$, $m = l = 1$, and $n = r = 1$. Thus (30) becomes as

$$u(\xi)=\frac{a_1\exp(\xi)+a_0+a_{-1}\exp(-\xi)}{b_1\exp(\xi)+b_0+b_{-1}\exp(-\xi)},$$
$$v(\xi)=\frac{a'_1\exp(\xi)+a'_0+a'_{-1}\exp(-\xi)}{b'_1\exp(\xi)+b'_0+b'_{-1}\exp(-\xi)}.\qquad(38)$$

In the case $b_1\neq 0$ and $b'_1\neq 0$ the system (38) can be simplified to

$$u(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1}\exp(-\xi)}{\exp(\xi) + b_0 + b_{-1}\exp(-\xi)},$$
$$v(\xi) = \frac{a'_1 \exp(\xi) + a'_0 + a'_{-1}\exp(-\xi)}{\exp(\xi) + b'_0 + b'_{-1}\exp(-\xi)}. \tag{39}$$

Substituting Eq. (39) into Eq. (3), and taking zero to be the coefficients of $\exp(n\xi)$ in each term, a set of algebraic equations will be derived. These equations can be used to determine unknowns a_1, a_0, a_{-1}, b_0, b_{-1}, a'_1, a'_0, a'_{-1}, b'_0, b'_{-1}, w, and k Solving the system of algebraic equations with the aid of a mathematical package, say Maple 12, the following results will be obtained

$$w = -\frac{2ka_1'^2(c+2b)}{a},\ a_{-1} = -\frac{1}{12}\frac{(-a^2c + ba^2 - 2a_1'^2c^2 - 8a_1'^2b^2 - 8a_1'^2cb)b_0^2}{ac(c+2b)},$$
$$a_0 = -\frac{1}{3}\frac{(2a^2c + ba^2 - 2a_1'^2c^2 - 8a_1'^2b^2 - 8a_1'^2cb)b_0}{ac(c+2b)},$$
$$a_1 = -\frac{1}{3}\frac{(-a^2c + ba^2 - 2a_1'^2c^2 - 8a_1'^2b^2 - 8a_1'^2cb)}{ac(c+2b)},\ b_{-1} = \frac{1}{4}b_0^2, \tag{40}$$
$$a'_{-1} = -\frac{1}{4}(2b'_0 - b_0)b_0a'_1,\ a'_0 = (b'_0 - b_0)a'_1,\ b'_{-1} = \frac{1}{4}(2b'_0 - b_0)b_0.$$

Substituting these results into (39), the following exact solution will be derived

$$u(\xi) = -\frac{1}{3}\frac{(ba^2 - 2a_1'^2c^2 - 8a_1'^2b^2 - 8a_1'^2cb)}{ac(c+2b)} + \frac{1}{3}\frac{a^2c}{ac(c+2b)}\left(1 - \frac{3b_0}{\exp(\xi) + b_0 + \frac{1}{4}b_0^2\exp(-\xi)}\right),$$
$$v(\xi) = \frac{a'_1\exp(\xi) + (b'_0 - b_0)a'_1 - \frac{1}{4}(2b'_0 - b_0)b_0a'_1\exp(-\xi)}{\exp(\xi) + b'_0 + \frac{1}{4}(2b'_0 - b_0)b_0\exp(-\xi)}.$$

where

$$\xi = kx - \frac{2ka_1'^2(c+2b)}{a}t.$$

5. Conclusion

In this article, we have been looking for the exact solution of the nonlinear Drinfeld–Sokolov system. We achieved the solution by applying the exp function method and its modification. The free parameters can be determined using any related to initial or boundary conditions. The result shows that these methods are powerful tools for obtaining exact solution. The advantage of the modified Exp function over the Exp function method is that the solution of the system can be obtained directly and without changing system to ordinary differential equation. Applications of the Exp function method for other kinds of nonlinear equations are under study in our research group. The computations associated in this work were performed by using Maple 12.

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