Journal of King Saud University – Science 31 (2019) 692–700

Contents lists available at ScienceDirect

Journal of King Saud University – Science

journal homepage: www.sciencedirect.com

Original article

Approximations of fractional integrals and Caputo derivatives with application in solving Abel's integral equations

Kamlesh Kumar, Rajesh K. Pandey *, Shiva Sharma

Department of Mathematical Sciences, Indian Institute of Technology (BHU) Varanasi, UP, India

ARTICLE INFO

Article history:
Received 16 December 2017
Accepted 31 December 2017
Available online 6 January 2018

Keywords:
Riemann-Liouville fractional integral
Caputo derivative
Quadratic scheme
Cubic scheme
Abel's integral equations

ABSTRACT

This paper presents two approximate methods such as Quadratic and Cubic approximations for the Riemann-Liouville fractional integral and Caputo fractional derivatives. The approximations error estimates are also obtained. Numerical simulations for these approximation schemes are performed with the test examples from literature and obtained numerical results are also compared. To establish the application of the presented schemes, the problem of Abel's inversion is considered. Numerical inversion of Abel's equation is obtained using Quadratic and Cubic approximations of the Caputo derivative. Test examples from literature are considered to validate the effectiveness of the presented schemes. It is observed that the Quadratic and Cubic approximations schemes produce the convergence of orders h^3 and h^4 respectively.

1. Introduction

Fractional derivatives have gained much attention in recent years and this could be due to its non-local nature compare to the traditional integer order derivatives. Fractional derivatives have played a significant role in analysing the behaviour of the physical phenomena through different domains of the science and engineering. Some of the pioneer contributions in these areas may be considered as biology (Robinson, 1981), viscoelasticity (Bagley and Torvik, 1983a,b), bioengineering (Magin, 2004), and many more can be found in Podlubny (1999) and Kilbas et al. (2006). Some of recent applications of the fractional derivatives in the emerging areas could be also noted as mathematical biology (Tripathi, 2011a; Tripathi and Anwar Bég, 2015a; Tripathi et al., 2015b; Bég et al., 2015) and heat and fluid flow (Arqub, 2017a).

Numerical integration is a basic tool for obtaining the approximate value of the definite integrals where the analytical integrations are difficult to evaluate. Numerical integrations for the fractional integrals also become important in developing the algorithms for solving applied problems defined using fractional derivatives. In recent years, numerical integrations of the fractional integrals and the fractional derivatives have attracted many researchers. The Adams–Bashforth–Moulton method for the fractional differential equations is discussed in Diethelm et al. (2002, 2004). Kumar and Agrawal (2006) presented quadratic approximation scheme for fractional differential equations. In Odibat (2006), Odibat presented a modified algorithm for approximation of fractional integral and Caputo derivatives and also obtained its error estimate. In Agrawal (2008) and Agrawal et al. (2012), Agrawal discussed the finite element approximation and fractional power series solution for the fractional variational problems. Pandey and Agrawal (2015) discussed a comparative study of different numerical methods such as linear, quadratic and quadratic-linear schemes for solving fractional variational problems defined in terms of the generalized derivatives. Recently, in Kumar et al. (2017), authors present three schemes for solving fractional integro-differential equations. Reproducing kernel algorithm are discussed for some time fractional partial differential equations in Arqub (2017b) and Arqub et al. (2015). Some more approximation schemes for solving fractional PDEs are elaborated in detail in Li and Zeng (2015).

In Odibat (2006), Odibat presented a scheme for approximating the Riemann-Liouville fractional integral and then obtained approximations for the Caputo derivatives. In this paper, we focus on the higher order approximations such as quadratic and cubic schemes to approximate the Riemann-Liouville fractional integral and Caputo derivatives. The numerical approximations are based

* Corresponding author.
E-mail addresses: kkp.iitbhu@gmail.com (K. Kumar), rkpandey.mat@iitbhu.ac.in (R.K. Pandey), shivasharma.math@gmail.com (S. Sharma).

Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

https://doi.org/10.1016/j.jksus.2017.12.017
1018-3647/© 2018 Production and hosting by Elsevier B.V. on behalf of King Saud University.

on the idea of dividing the whole interval into a set of small subintervals and between these two successive subintervals the unknown functions are approximated in terms of the quadratic and cubic polynomials. Thus, the numerical scheme presented for the approximation of the Riemann-Liouville fractional integral and Caputo derivatives are named as quadratic and cubic schemes. The error estimates for these approximations are also presented where we observe that the quadratic and cubic approximations achieve high convergence order. To validate these schemes, test examples are considered from the literature (Odibat, 2006). We also show that the obtained results using the proposed schemes preserve the results obtained by Odibat (2006). Further, the presented schemes are applied to solve the Abel's integral equations. The numerical approach for solving Abel's integral equations are recently studied by Jahanshahi et al. (2015), using the approximation scheme presented in Odibat (2006). Avazzadeh et al. (2011) used fractional calculus approach together with Chebyshev polynomials to solve Abel's integral equations. Saadatmandi and Dehghan (2008) applied collocation method to solve Abel's integral equations of first and second kind using shifted Legendre's polynomials. Li and Zhao (2013) studied the Abel's type integral equation using the Mikusinski's operator of fractional order. Badr (2012) presented the solution of Abel's integral using Jacobi polynomials. Further, Saleh et al. (2014) studied solution of generalized Abel's integral equation using Chebyshev Polynomials. The numerical results presented in Jahanshahi et al. (201), are considered here to validate and compare the results obtained by the presented schemes. Numerical simulations validate the presented schemes and show the advantage over existing method (Jahanshahi et al., 2015).

2. Definitions

The Riemann–Liouville fractional integral of order $\alpha > 0$ is defined as,

$$(\mathrm{I}^{\alpha}f)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \tag{1}$$

$$(\mathrm{I}^{0}f)(t) = f(t) \tag{2}$$

And the fractional derivative known as Riemann-Liouville fractional derivative of order $\alpha > 0$ is defined as,

$$(I_{\alpha}f)(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{n}\int_0^t (t-\tau)^{n-\alpha-1} f(\tau)d\tau, \quad n-1<\alpha \leqslant n, \tag{3}$$

where n is an integer. Another definition of fractional derivative introduced by Caputo, is defined as,

$$(\mathrm{D}^{\alpha}f)(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau)d\tau, \quad \text{for}, \quad m-1<\alpha\leq m \tag{4}$$

where, m is an integer. For more details, we refer the readers to Podlubny (1999) and Kilbas et al. (2006).

3. Numerical schemes

Here, two numerical schemes such as Quadratic and Cubic schemes are discussed. First, we divide the domain into several sub domains and then approximate the unknown function into each sub domain. Further, the approximations are obtained using Quadratic and Cubic polynomial approximations of the unknown function into each sub domains.

Here, we follow the simpler notations to the fractional integral and fractional derivatives and denote Riemann-Lowville fractional integral (Eq. (1)) and Caputo fractional derivative (Eq. (4)) as I-operator and D-operator respectively in the upcoming derivations of the numerical schemes. From Eq. (1) and (4), the approximation of the I-operator and D-operator can be expressed as,

$$(\mathrm{I}^{\alpha}f)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau \approx I(f,h,\alpha), \tag{5}$$

$$(\mathrm{D}^{\alpha}f)(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1} \mathrm{f}^{(m)}(\tau)\mathrm{d}\tau \approx D(f,h,\alpha) \tag{6}$$

$$(\mathrm{I}^{\alpha}f)(t) = I(f,h,\alpha) + E_I(f,h,\alpha), \tag{7}$$

$$(\mathrm{D}^{\alpha}f)(t) = D(f,h,\alpha) + E_D(f,h,\alpha). \tag{8}$$

where, $I(f,h,\alpha)$ and $D(f,h,\alpha)$ denote the approximation of the I-operator and D-operator respectively, and $E_I(f,h,\alpha)$, $E_D(f,h,\alpha)$ represent the error terms of their approximations. Now we present the Quadratic and Cubic approximation schemes of the I-operator and D-operator respectively as follows:

3.1. Quadratic scheme (S1)

In this subsection, the domain interval $[0,t]$ is distributed into even number of subintervals, $N=2n$ for $n \geqslant 1$, equal parts with uniform step size (or time interval) h, where $h=\frac{t}{2n}$ such that the node points are $t_i = ih, i = 0,1,2\ldots 2n$.

$$I^{\alpha}f(t) = \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)d\tau \approx IQ(f,h,\alpha), \tag{9}$$

$$D^{\alpha}_{p}f(t) \approx DQ(f,h,\alpha), \tag{10}$$

where, $IQ(f,h,\alpha)$, $DQ(f,h,\alpha)$ represent the quadratic approximation of the I and D-operators respectively and $E_{IQ}(f,h,\alpha)$, $E_{DQ}(f,h,\alpha)$ represent the error terms of the quadratic approximation such that,

$$E_{IQ}(f,h,\alpha) = I^{\alpha}f(t) - IQ(f,h,\alpha). \tag{11}$$

$$E_{DQ}(f,h,\alpha) = (\mathrm{D}^{\alpha}f)(t) - DQ(f,h,\alpha). \tag{12}$$

The function $f(\tau)$ is approximated over the interval $[t_{2i}, t_{2i+2}]$using the following formula (Pandey and Agrawal, 2015):

$$f_{i2} = \frac{-(\tau - t_{2i+1})}{2h}\left[1 - \frac{(\tau - t_{2i+1})}{h}\right] f_{2i} + \left[1 - \left(\frac{\tau - t_{2i+1}}{h}\right)^2\right] f_{2i+1} + \frac{(\tau - t_{2i+1})}{2h}\left[1 + \frac{(\tau - t_{2i+1})}{h}\right] f_{2i+2}. \tag{13}$$

In this case, results are presented as following lemmas.

Lemma 1. *Suppose that $f \in C^3[0,\delta]$, and the interval $[0,\delta]$ is divided into even number of sub intervals $[t_{2i}, t_{2i+2}]$ such that $t_i = ih$ with $h = \frac{\delta}{2n}, i = 0, 12, \ldots 2n$. Let f_{i2} is the quadratic polynomial approximation for f to the subintervals $[t_{2i}, t_{2i+2}]$ then the quadratic approximation $IQ(f,h,\alpha)$ of the I-operator is given by,*

(i)

$$IQ(f,h,\alpha) = \sum_{i=0}^{n-1}(A_{in}f(t_{2i}) + B_{in}f(t_{2i+1}) + C_{in}f(t_{2i+2})), \tag{14}$$

where

$$A_{in} = \frac{2^{\alpha}h^{\alpha}}{\Gamma(\alpha+3)}\Big\{(n-i-1)^{(\alpha+1)}(2-\alpha+4i-4n) +(n-i)^{\alpha}\Big(2+\alpha^2+4i^2+i(6-8n) +3\alpha(1+i-n)-6n+4n^2\Big)\Big\}, \quad (15)$$

$$B_{in} = \frac{2^{(\alpha+2)}h^{\alpha}}{\Gamma(\alpha+3)}\Big\{(n-i-1)^{(\alpha+1)}(\alpha-2i+2n) +(n-i)^{(\alpha+1)}(2+\alpha+2i-2n)\Big\}, \quad (16)$$

$$C_{jk} = -\frac{2^{\alpha}h^{\alpha}}{\Gamma(\alpha+3)}\Big\{(n-i)^{(\alpha+1)}(2+\alpha+4i-4n) +(n-i-1)^{\alpha}\Big(\alpha^2+2i-3\alpha i+4i^2-2n+3\alpha n-8in+4n^2\Big)\Big\}, \quad (17)$$

(ii) *and the approximation error* $E_{IQ}(f,h,\alpha)$ *has the form,*

$$|E_{IQ}(f,h,\alpha)| \leqslant C_{\alpha}f'''_{\infty}(t_{2n})^{\alpha}h^3 \quad (18)$$

where C_{α} *is a constant depending on* α.

Proof. From the definition of I-operator, we have,

$$I^{\alpha}f(t_{2n}) = \frac{1}{\Gamma(\alpha)}\int_0^{t_{2n}}(t_{2n}-\tau)^{\alpha-1}f(\tau)d\tau, \quad (19)$$

We approximate $f(\tau)$ over the interval $[t_{2i}, t_{2i+2}]$ using the quadratic polynomials (Pandey and Agrawal, 2015) as,

$$f_{i2} = \frac{-(\tau-t_{2i+1})}{2h}\left[1-\frac{(\tau-t_{2i+1})}{h}\right]f_{2i} + \left[1-\left(\frac{\tau-t_{2i+1}}{h}\right)^2\right]f_{2i+1} + \frac{(\tau-t_{2i+1})}{2h}\left[1+\frac{(\tau-t_{2i+1})}{h}\right]f_{2i+2}. \quad (20)$$

Evaluating Eq. (19) using Eq. (20), the desired approximation of $IQ(f,h,\alpha)$ as given in part (i) of the Lemma 1 is obtained.

For proof of the part (ii) of the Lemma 1, we use the following well known result of the interpolation by polynomials. □

Theorem 1. *Let* $g_n(t)$ *be the polynomial interpolating a function* $g \in C^{n+1}[a,b]$ *at the nodes* $t_0, t_1, t_2, t_3 \dots, t_n$ *lying in the interval* $[a,b]$. *Then for,* $g \in C^{n+1}[a,b]$, *there exists a* $\xi_t \in (a,b)$ *such that,* $E_n(t) = g(t) - g_n(t) = \frac{g^{(n+1)}(\xi_t)}{(n+1)!}\prod_{i=0}^{n}(t-t_i)$.

From Eq. (19) and (20) we have,

$$\begin{aligned}|E_{IQ}(f,h,\alpha)| &= |I^{\alpha}f(t) - IQ(f,h,\alpha)|, = |I^{\alpha}f(t_{2n}) - \frac{1}{\Gamma(\alpha)}\int_0^{t_{2n}}(t_{2n}-\tau)^{\alpha-1}f(\tau)d\tau|,\\ &= \frac{1}{\Gamma(\alpha)}|\int_0^{t_{2n}}(t_{2n}-\tau)^{\alpha-1}f(\tau)d\tau - \sum_{i=0}^{n-1}\int_{t_{2i}}^{t_{2i+2}}(t_{2n}-\tau)^{\alpha-1}f_{i2}(\tau)du|,\\ &= \frac{1}{\Gamma(\alpha)}|\sum_{i=0}^{n-1}\int_{t_{2i}}^{t_{2i+2}}(t_{2n}-\tau)^{\alpha-1}(f(\tau)-f_{i2}(\tau))d\tau|,\end{aligned} \quad (21)$$

Using Theorem 1, *and Eq.* (21) *we have,*

$$\begin{aligned}&\leqslant \frac{1}{6\Gamma(\alpha)}f'''_{\infty}|\sum_{i=0}^{n-1}\int_{t_{2i}}^{t_{2i+2}}(t_{2n}-\tau)^{\alpha-1}(\tau-t_{2i})(\tau-t_{2i+1})(\tau-t_{2i+2})d\tau|,\\ &\leqslant \frac{h^3}{9\sqrt{3}\Gamma(\alpha)}f'''_{\infty}|\sum_{i=0}^{n-1}\int_{t_{2i}}^{t_{2i+2}}(t_{2n}-\tau)^{\alpha-1}d\tau| = C_{\alpha}f'''_{\infty}(t_{2n})^{\alpha}h^3,\end{aligned} \quad (22)$$

where C_{α} *is constant depending on* α. *The proof is completed.*

Lemma 2. *Suppose that* $f \in C^{m+3}[0,\delta]$, *and the interval* $[0,\delta]$ *is divided into even number of sub intervals* $[t_{2i}, t_{2i+2}]$ *such that* $t_i = ih$ *with* $h = \frac{\delta}{2n}, i = 0, 12, \dots 2n$. *Let* $[t_{2i}, t_{2i+2}]$ *is the quadratic polynomial approximation for* $f^{(m)}$ *to the subintervals* $[t_{2i}, t_{2i+2}]$ *then the quadratic approximation* $DQ(f,h,\alpha)$ *of the D-operator is given by,*

(i)

$$DQ(f,h,\alpha) = \sum_{i=0}^{n-1}(\mathcal{A}_{in}f^{(m)}(t_{2i}) + \mathcal{B}_{in}f^{(m)}(t_{2i+1}) + \mathcal{C}_{in}f^{(m)}(t_{2i+2})) \quad (23)$$

where,

$$\mathcal{A}_{in} = \frac{2^{(m-\alpha)}h^{(m-\alpha)}}{\Gamma(m-\alpha+3)}\Big[(n-i-1)^{(m-\alpha+1)}(2-m+\alpha+4i-4n) +(n-i)^{(m-\alpha)}\Big\{\Big(2+(m-\alpha)^2+4i^2+i(6-8n) +3(m-\alpha)(1+i-n)-6n+4n^2\Big)\Big\}\Big], \quad (24)$$

$$\mathcal{B}_{in} = \frac{2^{(m-\alpha+2)}h^{(m-\alpha)}}{\Gamma(m-\alpha+3)}\Big[(n-i-1)^{(m-\alpha+1)}(m-\alpha-2i+2n) +(n-i)^{(m-\alpha+1)}(2+m-\alpha+2i-2n)\Big], \quad (25)$$

and,

$$\mathcal{C}_{in} = -\frac{2^{(m-\alpha)}h^{(m-\alpha)}}{\Gamma(m-\alpha+3)}\Big[(n-i)^{(m-\alpha+1)}(2+m-\alpha+4i-4n) +(n-i-1)^{(m-\alpha)}\Big\{(m-\alpha)^2+2i-3(m-\alpha)i+4i^2 -2n+3(m-\alpha)n-8in+4n^2\Big\}\Big]. \quad (26)$$

(ii) *And the approximation error* $E_{DQ}(f,h,\alpha)$ *has the form,*

$$|E_{DQ}(f,h,\alpha)| \leqslant C'_{\alpha}\|f^{(m+3)}\|_{\infty}t_{2n}^{(m-\alpha)}h^3 \quad (27)$$

where C'_{α} *is a constant depending only* α.

Proof. The proof of the part (i) and part (ii) of the lemma can be carried out following the similar steps and replacing α to $m-\alpha$ and $f(\tau)$ by $f^{(m)}(\tau)$ as described in the proof of the Lemma 1. □

3.2. Cubic scheme (S2)

Lemma 3. *Suppose that* $f \in C^4[0,\delta]$, *and the interval* $[0,\delta]$ *is divided into sub intervals* $[t_{3i}, t_{3i+3}]$ *such that* $t_i = ih$ *with* $h = \frac{\delta}{3n}, i = 0, 12, \dots 3n$. *Let* f_{i3} *is the cubic polynomial approximation for* f *to the subintervals* $[t_{3i}, t_{3i+3}]$ *then the cubic approximation* $IC(f,h,\alpha)$ *of the I-operator is given by,*

(i)

$$IC(f,h,\alpha) = \sum_{i=0}^{n-1}(D_{in}f(t_{3i}) + E_{in}f(t_{3i+1}) + F_{in}f(t_{3i+2}) + G_{in}f(t_{3i+3})), \quad (28)$$

where,

$$D_{in} = \frac{3^{\alpha}h^{\alpha}}{2\Gamma(\alpha+4)}\Big\{2(n-i-1)^{(1+\alpha)}(\alpha^2+\alpha(-4-9i+9n) +3(1+3i-3n)(2+3i-3n)) +(n-i)^{\alpha}(2\alpha^3+\alpha^2(12+11i-11n) +\alpha\Big(22+36(i-n)^2+55(i-n)\Big) +6(1+3i-3n)(2+3i-3n)(1+i-n))\Big\}, \quad (29)$$

$$E_{in} = \frac{3^{\alpha+2}h^{\alpha}}{2\Gamma(\alpha+4)}\{2(n-i)^{(1+\alpha)}(\alpha^2+5\alpha(1+i-n) +3(2+3i-3n)(1+i-n)) -(n-i-1)^{(\alpha+1)}(\alpha^2+\alpha(-3-8i+8n) +6(2+3i-3n)(i-n))\}, \quad (30)$$

$$F_{in} = \frac{3^{\alpha+2}h^{\alpha}}{2\Gamma(\alpha+4)}\{2(n-i-1)^{(\alpha+1)}(\alpha^2+5\alpha(n-i) +3(1+3i-3n)(i-n)) - (n-i)^{(1+\alpha)}(\alpha^2+\alpha(5+8i-8n) +6(1+3i-3n)(1+i-n))\}, \quad (31)$$

$$G_{in} = \frac{3^{\alpha}h^{\alpha}}{2\Gamma(\alpha+4)}\{2(n-i)^{(1+\alpha)}(\alpha^2+\alpha(5+9i-9n) +3(1+3i-3n)(2+3i-3n)) -(n-i-1)^{\alpha}(2\alpha^3+\alpha^2(1-11i+11n) +\alpha(3+36(i-n)^2+17(i-n)) -6(1+3i-3n)(2+3i-3n)(i-n))\}, \quad (32)$$

(ii) *and the approximation error $E_{IC}(f,h,\alpha)$ has the form,*

$$|E_{IC}(f,h,\alpha)| \leqslant D_{\alpha} f^{''''}_{\infty}(t_{3n})^{\alpha}h^4 \quad (33)$$

where D_{α} is a constant depending on α.

Proof. From Eq. (1), we have,

$$I^{\alpha}f(t_{3n}) = \sum_{i=0}^{n-1}\int_{t_{3i}}^{t_{3i+3}}(t_{3n}-\tau)^{\alpha-1}f(\tau), \quad (34)$$

We approximate $f(\tau)$ over the interval $[t_{3i}, t_{3i+3}]$using the cubic polynomials as,

$$f_{i3} = \left[-\frac{(\tau-t_{3i+1})(\tau-t_{3i+2})(\tau-t_{3i+3})}{6h^3}\right]f_{3i} + \left[\frac{(\tau-t_{3i})(\tau-t_{3i+2})(\tau-t_{3i+3})}{2h^3}\right]f_{3i+1} + \left[-\frac{(\tau-t_{3i})(\tau-t_{3i+1})(\tau-t_{3i+3})}{2h^3}\right]f_{3i+2} + \left[\frac{(\tau-t_{3i})(\tau-t_{3i+1})(\tau-t_{3i+2})}{6h^3}\right]f_{3i+3}, \quad (35)$$

Evaluating Eq. (34) using Eq. (35), the desired approximation of $IC(f,h,\alpha)$ as given in part (i) of the Lemma 1 is obtained. Proof of the second part of Lemma 3 is established here using Lemma 1.

From Eq. (34) and Eq. (35), we have,

$$\begin{aligned}|E_{IC}(f,h,\alpha)| &= |I^{\alpha}f(t) - IC(f,h,\alpha)|,\\ &= |I^{\alpha}f(t_{3n}) - \frac{1}{\Gamma(\alpha)}\int_0^{t_{3n}}(t_{3n}-\tau)^{\alpha-1}f(\tau)d\tau|,\\ &= \frac{1}{\Gamma(\alpha)}|\int_0^{t_{3n}}(t_{3n}-\tau)^{\alpha-1}f(\tau)d\tau - \sum_{i=0}^{n-1}\int_{t_{3i}}^{t_{3i+3}}(t_{3n}-\tau)^{\alpha-1}f_{i3}(\tau)du|,\\ &= \frac{1}{\Gamma(\alpha)}|\sum_{i=0}^{n-1}\int_{t_{3i}}^{t_{3i+3}}(t_{3n}-\tau)^{\alpha-1}(f(\tau)-f_{i3}(\tau))d\tau|,\end{aligned} \quad (36)$$

Using Lemma 1 and Eq. (36), we have,

$$\leq \frac{1}{24\Gamma(\alpha)}f^{''''}_{\infty}|\sum_{i=0}^{n-1}\int_{t_{3i}}^{t_{3i+3}}(t_{3n}-\tau)^{\alpha-1}(\tau-t_{3i})(\tau-t_{3i+1})(\tau - t_{3i+2})(\tau-t_{3i+2})d\tau|,$$

$$\leq \frac{h^4}{\Gamma(\alpha)}f^{''''}_{\infty}\sum_{i=0}^{n-1}\int_{t_{3i}}^{t_{3i+3}}(t_{3n}-\tau)^{\alpha-1}d\tau = D_{\alpha}f^{''''}_{\infty}(t_{3n})^{\alpha}h^4, \quad (37)$$

where D_{α} is constant depending on α.

This completes the proof. □

Lemma 4. *Suppose that $f \in C^{m+4}[0,\delta]$, and the interval $[0,\delta]$ is divided into sub intervals $[t_{3i}, t_{3i+3}]$ such that $t_i = ih$ with $h = \frac{\delta}{3n}, i = 0, 12, \ldots 3n$. Let f_{i3} is the cubic polynomial approximation for $f^{(m)}(\tau)$ to the subintervals $[t_{3i}, t_{3i+3}]$ then the cubic approximation $DC(f,h,\alpha)$ of the D-operator is given by,*

(i)

$$DC(f,h,\alpha) = \sum_{i=0}^{n-1}\Big(\mathcal{D}_{in}f^{(m)}(t_{3i}) + \mathcal{E}_{in}f^{(m)}(t_{3i+1}) + \mathcal{F}_{in}f^{(m)}(t_{3i+2}) +\mathcal{G}_{in}f^{(m)}(t_{3i+3})\Big), \quad (38)$$

where,

$$\mathcal{D}_{in} = \frac{3^{(m-\alpha)}h^{(m-\alpha)}}{2\Gamma(m-\alpha+4)}\{2(n-i-1)^{(m-\alpha+1)}\Big((m-\alpha)^2 +(m-\alpha)(-4-9i+9n)+3(1+3i-3n)(2+3i-3n)\Big) +(n-i)^{(m-\alpha)}\Big(2(m-\alpha)^3+(m-\alpha)^2(12+11i-11n) +(m-\alpha)\Big(22+36(i-n)^2+55(i-n)\Big) +6(1+3i-3n)(2+3i-3n)(1+i-n)\Big)\}, \quad (39)$$

$$\mathcal{E}_{in} = \frac{3^{(m-\alpha+2)}h^{(m-\alpha)}}{2\Gamma(m-\alpha+4)}\{2(n-i)^{(m-\alpha+1)}\Big((m-\alpha)^2 +5(m-\alpha)(1+i-n)+3(2+3i-3n)(1+i-n)\Big) -(n-i-1)^{(m-\alpha+1)}\Big((m-\alpha)^2+(m-\alpha)(-3-8i+8n) +6(2+3i-3n)(i-n)\Big)\}, \quad (40)$$

$$\mathcal{F}_{in} = \frac{3^{(m-\alpha+2)}h^{(m-\alpha)}}{2\Gamma(m-\alpha+4)}\{2(n-i-1)^{(m-\alpha+1)}\Big((m-\alpha)^2 +5(m-\alpha)(n-i)+3(1+3i-3n)(i-n)\Big) -(n-i)^{(m-\alpha+1)}((m-\alpha)^2+(m-\alpha)(5+8i-8n) +6(1+3i-3n)(1+i-n))\}, \quad (41)$$

$$\mathcal{G}_{in} = \frac{3^{(m-\alpha)}h^{(m-\alpha)}}{2\Gamma(m-\alpha+4)}\{2(n-i)^{(m-\alpha+1)}((m-\alpha)^2 +(m-\alpha)(5+9i-9n)+3(1+3i-3n)(2+3i-3n)) -(n-i-1)^{(m-\alpha)}(2(m-\alpha)^3+(m-\alpha)^2(1-11i+11n) +(m-\alpha)(3+36(i-n)^2+17(i-n)) -6(1+3i-3n)(2+3i-3n)(i-n))\}, \quad (42)$$

(ii) *and the approximation error $E_{DC}(f,h,\alpha)$ takes the form,*

$$|E_{DC}(f,h,\alpha)| \leqslant D'_{\alpha}||f^{(m+4)}||_{\infty}t_{3n}^{(m-\alpha)}h^4 \quad (43)$$

where D'_{α} is a constant depending only α.

Proof. The proof of the part (i) and part (ii) of above lemma can be carried out using the similar steps and replacing α to $m-\alpha$ and $f(\tau)$ by $f^{(m)}(\tau)$ as described in the proof of the Lemma 3. □

4. Results and discussions

Here, we consider the test example as illustrated by Odibat (2006), with $f(\tau) = \sin\tau$ in the I-operator for the comparison pur-

Table 1
Numerical results obtained using scheme S1 for I-operator, $I^{\alpha}f(t)$ (1) for $f(t) = \sin t$ and$\alpha = 0.5$.

n	h	$IQ(f, h, 0.5)$	$E_{IQ}(f, h, 0.5)$	$E_{IL}(f, h, 0.5)$ (Odibat, 2006)
10	0.05	0.6696838267942012	4.32783×10^{-7}	1.30405×10^{-4}
20	0.025	0.6696842212539105	3.83238×10^{-8}	3.32769×10^{-5}
40	0.0125	0.6696842561832945	3.39437×10^{-9}	8.4373×10^{-6}
80	0.00625	0.6696842592769013	3.00762×10^{-10}	2.1301×10^{-6}

Table 2
Numerical results obtained using scheme S1 for I-operator, $I^{\alpha}f(t)$ (1) for $f(t) = \sin t$ and $\alpha = 1$.

n	h	$IQ(f, h, 1)$	$E_{IQ}(f, h, 1)$	$E_{IL}(f, h, 1)$ (Odibat, 2006)
10	0.05	0.4596977100983376	1.59665×10^{-8}	9.57743×10^{-5}
20	0.025	0.4596976951295424	9.97682×10^{-10}	2.39428×10^{-5}
40	0.0125	0.4596976941942119	6.23516×10^{-11}	5.9856×10^{-6}
80	0.00625	0.4596976941357571	3.89683×10^{-12}	1.4964×10^{-6}

Table 3
Numerical results obtained using scheme S1 for of the I-operator, $I^{\alpha}f(t)$ (1) for $f(t) = \sin t$ and $\alpha = 1.5$.

n	h	$IQ(f, h, 1.5)$	$E_{IQ}(f, h, 1.5)$	$E_{IL}(f, h, 1.5)$ (Odibat, 2006)
10	0.05	0.2823225014367666	1.21065×10^{-7}	5.89010×10^{-5}
20	0.025	0.2823223880461549	7.67480×10^{-9}	1.47111×10^{-5}
40	0.0125	0.2823223808560551	4.84695×10^{-10}	3.6767×10^{-6}
80	0.00625	0.2823223804019575	3.05977×10^{-11}	9.191×10^{-7}

Table 4
Convergence order using scheme S1 for $I^{\alpha}f(t)$ (1) for $f(t) = \sin t$ and $\alpha = 0.5$.

$h = \frac{1}{2n}$	MAE (S1)	Convergence order
$\frac{1}{10}$	4.89232×10^{-6}	
$\frac{1}{20}$	4.32783×10^{-7}	3.49733
$\frac{1}{40}$	3.83238×10^{-8}	3.49733
$\frac{1}{80}$	3.39437×10^{-9}	3.49702
$\frac{1}{160}$	3.00762×10^{-10}	3.49645

Table 5
Convergence order using scheme S1 for$I^{\alpha}f(t)$ (1) for $f(t) = \sin t$ and $\alpha = 1$.

$h = \frac{1}{2n}$	MAE (S1)	Convergence order
$\frac{1}{10}$	2.55692×10^{-7}	
$\frac{1}{20}$	1.59665×10^{-8}	4.00129
$\frac{1}{40}$	9.97682×10^{-10}	4.00032
$\frac{1}{80}$	6.23516×10^{-11}	4.00008
$\frac{1}{160}$	3.89683×10^{-12}	4.00005

Table 6
Convergence order using scheme S1 for $I^{\alpha}f(t)$ (1) for $f(t) = \sin t$ and $\alpha = 1.5$.

$h = \frac{1}{2n}$	MAE (S1)	Convergence order
$\frac{1}{10}$	1.90186×10^{-6}	
$\frac{1}{20}$	1.21065×10^{-7}	3.97356
$\frac{1}{40}$	7.67480×10^{-9}	3.97951
$\frac{1}{80}$	4.84695×10^{-10}	3.98498
$\frac{1}{160}$	3.05977×10^{-11}	3.98558

pose. Lemmas 1 and 2 and Lemma 3 and 4 are applied for the approximation of the I and D-operators for different values of the fractional order α, and numerical results are obtained. The numerical results using Quadratic and Cubic approximation schemes for the I-operator is calculated for different values of the step size and fractional order α, and are placed in the Tables 1–3. For the comparison purpose, the similar values of the parameters such as fractional order α and step size are chosen as presented in Odibat (2006). It is clear from the Tables 1–3, that the scheme S1 works well and achieves the better accuracy compare to the linear scheme presented in Odibat (2006). From, Tables 1–3, it can be seen that the errors are getting reduced as we increase the number of subintervals. The convergence order of the scheme S1 for the results discussed in Tables 1–3 are presented in Tables 4–6 respectively. From Tables 4–6, it can be seen that the scheme S1 achieves the convergence order more than 3. Further, we observe that the scheme S2 works well and achieves the better accuracy compared to the scheme (Odibat, 2006), and the scheme S1. The results of scheme S2 are presented in Tables 7–9. Table 10 represents the convergence order of the scheme S2 for a particular case considered in Table 8.

Schemes S1 and S2 are also applied to approximate the Caputo derivative (D-operator). We consider the test function $f(\tau) = \sin\tau$, the fractional order $\alpha = 0.5$ and vary the step size to generate the numerical results. Numerical results using schemes S1 and S2 for approximations of D-operators are showed in Tables 11and 12 respectively. In the tables, MAE denotes the maximum absolute error and the convergence order is calculated as: Convergence order = lg[MAE(h)/MAE(h/2)]/lg(2).

It is noticed that exact value of the fractional integral $I^{\alpha}\sin t$ is calculated using the formula stated in Odibat (2006), as, $I^{\alpha}\sin t = t^{\alpha}\sum_{i=0}^{\infty}\frac{(-1)^{i}t^{2i+1}}{\Gamma(\alpha+2i+2)}t > 0$, and value at $t = 1$, is used to compute the error.

5. Application: Solving Abel's integral equation

To establish the application of the Quadratic and Cubic schemes for the D-operator as discussed in Section 4, we go through Abel integral equation of the first kind,

Table 7
The approximation results of the I-operator, $I^\alpha f(t)$ (1) for $f(t) = \sin t$ and $\alpha = 0.5$.

n	h	$IC(f,h,0.5)$	$E_{IC}(f,h,0.5)$	$E_{IL}(f,h,0.5)$ (Odibat, 2006)
10	1/30	0.6696842705520611	1.09744×10^{-8}	1.30405×10^{-4}
20	1/60	0.6696842602530784	6.75415×10^{-10}	3.32769×10^{-5}
40	1/120	0.6696842596262024	4.85388×10^{-11}	8.4373×10^{-6}
80	1/240	0.6696842596896193	1.11956×10^{-10}	2.1301×10^{-6}

Table 8
The approximation results of the I-operator, $I^\alpha f(t)$ (1) for $f(t) = \sin t$ and $\alpha = 1$.

n	h	$IC(f,h,1)$	$E_{IC}(f,h,1)$	$E_{IL}(f,h,1)$ (Odibat, 2006)
10	1/30	0.4596977012278377	7.09598×10^{-9}	9.57743×10^{-5}
20	1/60	0.4596976945752709	4.43411×10^{-10}	2.39428×10^{-5}
40	1/120	0.4596976941318603	6.23516×10^{-11}	5.9856×10^{-6}
80	1/240	0.4	3.89683×10^{-12}	1.4964×10^{-6}

Table 9
The approximation results of the I-operator, $I^\alpha f(t)$ (1) for $f(t) = \sin t$ and $\alpha = 1.5$.

n	h	$IC(f,h,1.5)$	$E_{IC}(f,h,1.5)$	$E_{IL}(f,h,1.5)$ (Odibat, 2006)
10	1/30	0.2823223847105428	4.33918×10^{-9}	5.89010×10^{-5}
20	1/60	0.2823223806431222	2.71762×10^{-10}	1.47111×10^{-5}
40	1/120	0.282322380395719	2.43592×10^{-11}	3.6767×10^{-6}
80	1/240	0.2823223804266486	5.52888×10^{-11}	9.191×10^{-7}

Table 10
Convergence order using scheme S2 for $I^\alpha f(t)$ (1) for $f(t) = \sin t$ and $\alpha = 1$.

$h = \frac{1}{3n}$	MAE (S2)	Convergence order
$\frac{1}{15}$	1.13626×10^{-7}	
$\frac{1}{30}$	7.09598×10^{-9}	4.00115
$\frac{1}{60}$	4.43411×10^{-10}	4.00029
$\frac{1}{120}$	2.77117×10^{-11}	4.00008
$\frac{1}{240}$	1.73189×10^{-12}	4.00008

$$f(t) = \int_0^t \frac{g(\tau)}{(t-\tau)^\alpha} d\tau, \quad 0 < \alpha < 1,\ 0 \leqslant t \leqslant \delta, \tag{44}$$

where, $f \in C^1[a,b]$ is given function satisfying $f(0) = 0$ and $g(\tau)$ is the unknown function. The solution to Eq. (44) can be obtained as,

$$g(t) = \frac{\sin(\alpha t)}{\pi} \int_0^t \frac{f'(\tau)}{(t-\tau)^{1-\alpha}} dt. \tag{45}$$

The solution given by Eq. (45) can also be presented in terms of the I and D-operators, using definition (Eq. (1)) as follows,

$$f(t) = \Gamma(1-\alpha) I^{1-\alpha} g(t) \tag{46}$$

Table 11
Numerical results obtained using scheme S1 for the D-operator, $D^\alpha f(\tau)$ (1) for $f(t) = \sin t$ and $\alpha = 0.5$.

k	h	$DQ(f,h,0.5)$	$E_{DQ}(f,h,0.5)$	$E_{DL}(f,h,0.5)$ (Odibat, 2006)
10	0.05	0.846057377964953	5.91241×10^{-7}	1.706097×10^{-4}
20	0.025	0.84605684138235	5.46582×10^{-8}	4.30544×10^{-5}
40	0.0125	0.846056791702752	4.9786×10^{-9}	1.08365×10^{-5}
80	0.00625	0.846056787174921	4.50768×10^{-10}	2.7222×10^{-6}

Table 12
the approximation results of the D-operator, $D^\alpha f(\tau)$ (1) for $f(t) = \sin t$ and $\alpha = 0.5$.

k	h	$DC(f,h,0.5)$	$E_{DC}(f,h,0.5)$	$E_{DL}(f,h,0.5)$ (Odibat, 2006)
10	1/30	0.846056800339386	1.36152×10^{-8}	1.706097×10^{-4}
20	1/60	0.846056787554831	8.30678×10^{-10}	4.30544×10^{-5}
40	1/120	0.846056786769922	4.57691×10^{-11}	1.08365×10^{-5}
80	1/240	0.846056786169989	5.54164×10^{-10}	2.7222×10^{-6}

Using the property D-operator is left inverse of I-operator and simplifying Eq. (46), it follows that,

$$g(t) = \frac{1}{\Gamma(1-\alpha)} D^{1-\alpha} f(t). \tag{47}$$

Now, we apply Lemma 2 and Lemma 4 to Eq. (47) to get the approximate solution of the Abel's integral equation given by Eq. (44).

Lemma 5. *Let* $0 < t < \delta$ *and suppose that the interval* $[0, \delta]$ *is subdivided into n sub intervals* $[t_{2i}, t_{2i+2}]$, $i = 1, 2, 3 \ldots n-1$ *of length* $h = \frac{\delta}{2n}$ *by using the nodes* $t_i = ih, i = 0, 1 \ldots 2n$. *Then the approximate solution* $\tilde{g}(t)$ *to the solution* $g(t)$ *of the Abel integral equation given by* Eq. (47) *can be expressed using scheme S1 as,*

$$\tilde{g}(t) = \sum_{i=0}^{n-1} (\mathcal{A}_{in} f'(t_{2i}) + \mathcal{B}_{in} f'(t_{2i+1}) + \mathcal{C}_{in} f'(t_{2i+2})), \tag{48}$$

where,

$$\mathcal{A}_{in} = \frac{2^{\alpha} h^{\alpha}}{\Gamma(\alpha+3)\Gamma(1-\alpha)} \{(n-i-1)^{(\alpha+1)}(2-\alpha+4i-4n) + (n-i)^{\alpha}(2+\alpha^2+4i^2+i(6-8n)+3\alpha(1+i-n) - 6n+4n^2)\}, \tag{49}$$

$$\mathcal{B}_{in} = \frac{2^{(\alpha+2)} h^{\alpha}}{\Gamma(\alpha+3)\Gamma(1-\alpha)} \{(n-i-1)^{(\alpha+1)}(\alpha-2i+2n) + (n-i)^{(\alpha+1)}(2+\alpha+2i-2n)\}, \tag{50}$$

$$\mathcal{C}_{in} = -\frac{2^{\alpha} h^{\alpha}}{\Gamma(\alpha+3)\Gamma(1-\alpha)} \{(n-i)^{(\alpha+1)}(2+\alpha+4i-4n) + (n-i-1)^{\alpha}(\alpha^2+2i-3\alpha i+4i^2-2n+3\alpha n-8in+4n^2)\}. \tag{51}$$

Moreover, if $f \in C^4[0t]$, then $g(t) = \tilde{g}(t) - \frac{1}{\Gamma(1-\alpha)} E(t)$ with

$$|E(t)| \leqslant S_{\alpha} f''''_{\infty} t^{\alpha} h^3, \tag{52}$$

where S_{α} is the constant depending only on α and $f''''_{\infty} = max_{x \in [0t]} |f''''(x)|$.

Proof. The solution of the Abel's integral equation (Eq. (44)) represented by Eq. (47) in the form of D-operator can be expressed as,

$$g(t) = \frac{\sin(\alpha\pi)\Gamma(\alpha)}{\pi} D^{1-\alpha} f(t) \tag{53}$$

The results can be obtained using Lemma 2 to Eq. (53) with some simple calculation. To validate the proposed approximation, an illustrative example from Jahanshahi et al. (2015), is considered and the approximate solution is obtained. □

Lemma 6. *Let* $0 < t < \delta$ *and suppose that the interval* $[0, \delta]$ *is subdivided into n sub intervals* $[t_{3i}, t_{3i+3}]$, $i = 0, 1, 2, 3 \ldots 3n-1$ *of length* $h = \frac{\delta}{3n}$ *by using the nodes* $t_i = ih, i = 0, \ldots 2n$. *Then the approximate solution* $\tilde{g}(t)$ *to the solution* $g(t)$ *of the Abel integral equation given by* Eq. (47) *can be expressed using scheme S2 as,*

$$\tilde{g}(t) = \sum_{i=0}^{n-1} (\mathcal{D}_{in} f'(t_{3i}) + \mathcal{E}_{in} f'(t_{3i+1}) + \mathcal{F}_{in} f'(t_{3i+2}) + \mathcal{G}_{in} f'(t_{3i+3})), \tag{54}$$

where,

$$\mathcal{D}_{in} = \frac{3^{\alpha} h^{\alpha}}{2\Gamma(1-\alpha)\Gamma(\alpha+4)} \{2(n-i-1)^{(1+\alpha)}(\alpha^2+\alpha(-4-9i+9n)+3(1+3i-3n)(2+3i-3n)) + (n-i)^{\alpha}(2\alpha^3 + \alpha^2(12+11i-11n)+\alpha(22+36(i-n)^2+55(i-n)) + 6(1+3i-3n)(2+3i-3n)(1+i-n))\}, \tag{55}$$

$$\mathcal{E}_{in} = \frac{3^{\alpha+2} h^{\alpha}}{2\Gamma(1-\alpha)\Gamma(\alpha+4)} \{2(n-i)^{(1+\alpha)}(\alpha^2+5\alpha(1+i-n) + 3(2+3i-3n)(1+i-n)) - (n-i-1)^{(\alpha+1)}(\alpha^2 + \alpha(-3-8i+8n)+6(2+3i-3n)(i-n))\}, \tag{56}$$

$$\mathcal{F}_{in} = \frac{3^{\alpha+2} h^{\alpha}}{2\Gamma(1-\alpha)\Gamma(\alpha+4)} \{2(n-i-1)^{(\alpha+1)}(\alpha^2+5\alpha(n-i) + 3(1+3i-3n)(i-n)) - (n-i)^{(1+\alpha)}(\alpha^2+\alpha(5+8i - 8n)+6(1+3i-3n)(1+i-n))\}, \tag{57}$$

$$\mathcal{G}_{in} = \frac{3^{\alpha} h^{\alpha}}{2\Gamma(1-\alpha)\Gamma(\alpha+4)} \{2(n-i)^{(1+\alpha)}(\alpha^2+\alpha(5+9i-9n) + 3(1+3i-3n)(2+3i-3n)) - (n-i-1)^{\alpha}(2\alpha^3 + \alpha^2(1-11i+11n)+\alpha(3+36(i-n)^2+17(i-n)) - 6(1+3i-3n)(2+3i-3n)(i-n))\}. \tag{58}$$

Moreover, if $f \in C^5[0t]$, then $g(t) = \tilde{g}(t) - \frac{1}{\Gamma(1-\alpha)} E(t)$ with

$$|E(t)| \leqslant S_{\alpha} f^{(5)}_{\infty} t^{\alpha} h^4, \tag{59}$$

where T_{α} is the constant depending only on α and $f''''\,'_{\infty} = max_{x \in [0t]} |f''''\,'(x)|$.

Proof. The proof can be acquired using some simple calculations to Eq. (47) together with the scheme S2 as discussed in Lemma 4. The results can be obtained using Lemma 4 to Eq. (53) with some simple calculation. To validate the proposed approximation, an illustrative example from Jahanshahi et al. (2015), is considered and numerical results are presented. □

Example 5.1. Consider the Abel's integral equation (Jahanshahi et al., 2015), $e^t - 1 = \int_0^t \frac{g(\tau)}{(t-\tau)^{1/2}} d\tau$. The exact solution for this problem is given by, $(t) = \frac{e^t}{\sqrt{\pi}} erf(\sqrt{t})$, where $erf(x)$ is error function, that is, $erf(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-\tau^2} d\tau$.

Example 5.1 is solved using the Lemma 5 and Lemma 6 and the obtained approximate results are presented in Tables 13-14 respectively. For solving this problem, the number of subintervals is considered as 10 and 100 and in each case the errors are obtained. From the Tables 13-14, it is clear that the error obtained by the proposed scheme is comparatively better even with the less number of subintervals than the method presented in Jahanshahi et al. (2015).

Example 5.2. Consider the following Abel integral equation (Jahanshahi et al., 2015), such that, $t = \int_0^t \frac{g(\tau)}{(t-\tau)^{4/5}} d\tau$, having exact solution, $g(t) = \frac{5}{4} \frac{\sin(\frac{\pi}{5})}{\pi} t^{4/5}$.

Lemma 5 and Lemma 6 are applied to solve the considered integral equation and the obtained numerical results are presented in Tables 15–16 respectively. The numerical results are obtained using the values of $n = 5, 10$ and the results are presented.

Table 13
Comparison of the exact solution, approximate solution using Lemma 5 and the respective errors for $n = 10, 100$.

t_i	Exact solution	Approx. sol. $n = 10$	Error $n = 10$	Error $n = 100$	Error (Jahanshahi et al., 2015) for $n = 100$
0.1	0.2152905021493694	0.2152905022928531	1.434×10^{-10}	1.72×10^{-10}	3.75×10^{-8}
0.2	0.3258840763232928	0.3258840781067156	1.783×10^{-9}	1.99×10^{-12}	2.61×10^{-8}
0.3	0.427565657562311	0.4275656656608028	8.098×10^{-9}	5.18×10^{-13}	2.14×10^{-7}

Table 14
Comparison of the exact solution, approximate solution using Lemma 6 and respective errors for $n = 10, 100$.

t_i	Exact solution	Approx. sol. $n = 10$	Error $n = 10$	Error $n = 100$	Error (Jahanshahi et al., 2015) for $n = 100$
0.1	0.2152905021493694	0.2152905021496013	2.31787×10^{-13}	4.34356×10^{-10}	3.75×10^{-8}
0.2	0.3258840763232928	0.325884076331575	8.28221×10^{-12}	6.19821×10^{-10}	2.61×10^{-8}
0.3	0.427565657562311	0.4275656576182749	5.5964×10^{-11}	7.65846×10^{-10}	2.14×10^{-7}

Table 15
Comparison of the exact solution, approximate solution using Lemma 5 and respective errors for $n = 5, 10$.

t_i	Exact solution	Approx. sol. $n = 5$	Error $n = 5$	Error $n = 10$	Error (Jahanshahi et al., 2015) for $n = 10$
0.4	0.112363903648632	0.1123639036486326	2.77556×10^{-16}	2.58127×10^{-15}	1×10^{-10}
0.5	0.1343243751756705	0.1343243751756709	3.60822×10^{-16}	3.13638×10^{-15}	1×10^{-10}
0.6	0.1554174667790617	0.155417466779062	3.33067×10^{-16}	3.60822×10^{-15}	$\leqslant 10^{-11}$

Table 16
Comparison of the exact solution, approximate solution using Lemma 6 and respective errors for $n = 5, 10$.

t_i	Exact solution	Approx. sol. $n = 5$	Error $n = 5$	Error $n = 10$	Error (Jahanshahi et al., 2015) for $n = 10$
0.4	0.1123639036486324	0.1123639036486273	5.06539×10^{-15}	2.35562×10^{-13}	1×10^{-10}
0.5	0.1343243751756705	0.1343243751756645	5.9952×10^{-15}	2.03615×10^{-13}	1×10^{-10}
0.6	0.1554174667790617	0.1554174667790548	6.93889×10^{-15}	1.7035×10^{-13}	$\leqslant 10^{-11}$

Absolute errors for each values of the subinterval n are also presented. Numerical results show that the presented schemes works well and produce the approximate solution to high accuracy.

6. Conclusions

We studied two approximation schemes namely Quadratic and Cubic schemes for Riemann-Liouville and Caputo derivatives. The error convergences for the presented schemes are obtained. The presented schemes are successfully validated on test cases. It is clear that the presented schemes show the advantages over the scheme discussed in Odibat (2006). Further, the presented schemes are applied to solve Abel's integral equation. The numerical results obtained by the presented schemes are appreciable as compare to the schemes presented in Jahanshahi et al. (2015). The schemes presented in the paper could be considered as the higher order approximation methods for the approximations of the fractional integrals and fractional derivatives.

Acknowledgements

Authors are thankful to the reviewers for their comments incorporated in the manuscript. Second author acknowledges National Board of Higher Mathematics, Department of Atomic Energy, Govt. of India for providing funding support through research project Ref. No. NBHM/R.P. 70/2015/Fresh/163.

References

Agrawal, O.P., 2008. A general finite element formulation for fractional variational problems. J. Math. Anal. Appl. 337, 1–12.

Agrawal, O.P., Hasan, M.M., Tangpong, X.W., 2012. A numerical scheme for a class of parametric problem of fractional variational calculus. J. Comput. Nonlinear Dyn. 7. 021005-1.

Arqub, O.A., 2017a. Numerical solutions for the Robin time-fractional partial differential equations of heat and fluid flows based on the reproducing kernel algorithm. J. Numer. Methods Heat Fluid Flow Int. https://doi.org/10.1108/HFF-07-2016-0278.

Arqub, O.A., 2017b. Fitted reproducing kernel Hilbert space method for the solutions of some certain classes of time-fractional partial differential equations subject to initial and Neumann boundary conditions. Comput. Math. Appl. 73 (6), 1243–1261.

Arqub, O.A., El-Ajou, A., Momani, S., 2015. Constructing and predicting solitary pattern solutions for nonlinear time-fractional dispersive partial differential equations. J. Comput. Phys. 293, 385–399.

Avazzadeh, Z., Shafiee, B., Loghmani, G.B., 2011. Fractional calculus for solving Abel's integral equations using chebyshev polynomials. Appl. Math. Sci. 5, 2207–2216.

Badr, A.A., 2012. Two numerical methods for Abel's integral equation with comparison. ARIMA J. 15 (2012), 1–10.

Bagley, R.L., Torvik, P.J., 1983a. A theoretical basis for the application of fractional calculus to viscoelasticity. J. of Rheology 27, 201–210.

Bagley, R.L., Torvik, P.J., 1983b. Fractional calculus a different approach to the analysis of viscoelastically damped structures. AIAA J. 21 (5), 741–748.

Bég, O.A., Tripathi, D., Sochi, T., Gupta, P.K., 2015. Adomian Decomposition Method (ADM) simulation of magneto-bio-tribological squeeze film with magnetic induction effects. J. Mech. Med. Biol. 15 (05), 1550072.

Diethelm, K., Ford, N.J., Freed, A., 2002. A predictor-corrector approach for the numerical solution of fractional differential equations. Nonlinear Dyn. 29, 3–22.

Diethelm, K., Ford, N.J., Freed, A., 2004. Detailed error analysis for fractional Adams method. Numer. Algorithms 36, 31–52.

Jahanshahi, S., Babolian, E., Torres, D.F.M., Vahidi, A., 2015. Solving Abel integral equations of first kind via fractional calculus. J. King Saud Univ. Sci. 27, 161–167. https://doi.org/10.1016/j.jksus.2014.09.004.
Kilbas, A.A., Srivastava, H.M., Trujillo, J.J., 2006. Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies. Elsevier, Amsterdam.
Kumar, P., Agrawal, O.P., 2006. An approximate method for numerical solution of fractional differential equations. Signal Process. 86, 2602–2610.
Kumar, K., Pandey, R.K., Sharma, S., 2017. Comparative study of three numerical schemes for fractional integro-differential equations. J. Comput. Appl. Math. 315, 287–302.
Li, C.P., Zeng, F., 2015. Numerical Methods for Fractional Calculus. CRC Press, Taylor and Francis Group.
Li, M., Zhao, W., 2013. Solving Abel's type integral equation with Mikusinski's operator of fractional order. Adv. Math. Phys., 4 Article ID 806984.
Magin, R.L., 2004. Fractional calculus in bioengineering. Critical Rev. Biomed. Eng. 32, 1–104.
Odibat, Z., 2006. Approximations of fractional integrals and Caputo fractional derivatives. Appl. Math. Comput. 178, 527–533.
Pandey, R. K., Agrawal, O. P., 2015. Numerical scheme for generalized isoperimetric constraint variational problems with A-operator. in: Proceeding of the 2015 ASME/IEEE International Conference on Mechatronic and Embedded Systems and Applications, Boston, MA, USA, August 2–5, Paper No. DETC2015-46570. doi: http://dx.doi.org/10.1115/DETC2015-46570.
Podlubny, I., 1999. Fractional Differential Equations. Academic Press, San Diego, CA.
Robinson, D.A., 1981. The use of control systems analysis in neurophysiology of eye movements. Ann. Rev. Neurosci. 4, 462–503.
Saadatmandi, A., Dehghan, M., 2008. A collocation method for solving Abel's integral equations of first and second kinds. Z. Naturforsch 63a (10), 752–756.
Saleh, M.H., Amer, S.M., Mohamed, D.S., Mahdy, A.E., 2014. Fractional calculus for solving generalized Abel's integral equations using Chebyshev polynomials. Int. J. Comput. Appl. 100 (8), 0975–8887.
Tripathi, D., 2011a. Numerical study on creeping flow of Burgers' fluids through a peristaltic tube. J. Fluids Eng. 133 (12), 121104.
Tripathi, D., 2011b. Peristaltic transport of fractional Maxwell fluids in uniform tubes: applications in endoscopy. Comput. Math. Appl. 62 (3), 1116–1126.
Tripathi, D., Anwar Bég, O., 2015. Mathematica numerical simulation of peristaltic biophysical transport of a fractional viscoelastic fluid through an inclined cylindrical tube. Comput. Methods Biomech. Biomed. Eng. 18 (15), 1648–1657.
Tripathi, D., Bég, O.A., Gupta, P.K., Radhakrishnamacharya, G., Mazumdar, J., 2015. DTM simulation of peristaltic viscoelastic biofluid flow in asymmetric porous media: a digestive transport model. J. Bionic Eng. 12 (4), 643–655.