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ORIGINAL ARTICLE

Existence and uniqueness of solution for a fractional Riemann–Liouville initial value problem on time scales

Nadia Benkhettou [a], **Ahmed Hammoudi** [b], **Delfim F.M. Torres** [c,*]

[a] *Laboratoire de Mathématiques, Université de Sidi Bel-Abbès, B.P. 89, 22000 Sidi Bel-Abbès, Algeria*
[b] *Laboratoire de Mathématiques, Université de Ain Témouchentl, B.P. 89, 46000 Ain Témouchent, Algeria*
[c] *Center for Research and Development in Mathematics and Applications (CIDMA), Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal*

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Abstract We introduce the concept of fractional derivative of Riemann–Liouville on time scales. Fundamental properties of the new operator are proved, as well as an existence and uniqueness result for a fractional initial value problem on an arbitrary time scale.

1. Introduction

Let $\mathbb{T}$ be a time scale, that is, a nonempty closed subset of $\mathbb{R}$. We consider the following initial value problem:

$$ {}_{t_0}^{\mathbb{T}}D_t^{\alpha} y(t) = f(t, y(t)), \quad t \in [t_0, t_0 + a] = \mathcal{J} \subseteq \mathbb{T}, \quad 0 < \alpha < 1, \tag{1} $$

$$ {}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} y(t_0) = 0, \tag{2} $$

where ${}_{t_0}{}^{\mathbb{T}}D_t^{\alpha}$ is the (left) Riemann–Liouville fractional derivative operator or order α defined on $\mathbb{T}$, ${}_{t_0}{}^{\mathbb{T}}I_t^{1-\alpha}$ the (left) Riemann–Liouville fractional integral operator or order $1-\alpha$ defined on $\mathbb{T}$, and function $f : \mathcal{J} \times \mathbb{T} \to \mathbb{R}$ is a right-dense continuous function. Our main results give necessary and sufficient conditions for the existence and uniqueness of solution to problem (1)–(2).

2. Preliminaries

In this section, we collect notations, definitions, and results, which are needed in the sequel. We use $\mathcal{C}(\mathcal{J}, \mathbb{R})$ for a Banach space of continuous functions y with the norm $\|y\|_\infty = \sup\{|y(t)| : t \in \mathcal{J}\}$, where $\mathcal{J}$ is an interval. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. The reader interested on the calculus on time scales is referred to the books (Bohner and Peterson, 2001, 2003). For a survey, see (Agarwal et al., 2002). Any time scale $\mathbb{T}$ is a complete metric space with the distance $d(t,s) = |t-s|, t, s \in \mathbb{T}$.

* Corresponding author. Tel.: +351 234370668; fax: +351 234370066.
E-mail addresses: benkhettou_na@yahoo.com (N. Benkhettou), hymmed@hotmail.com (A. Hammoudi), delfim@ua.pt (D.F.M. Torres).
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Consequently, according to the well-known theory of general metric spaces, we have for $\mathbb{T}$ the fundamental concepts such as open balls (intervals), neighborhoods of points, open sets, closed sets, compact sets, etc. In particular, for a given number $\delta > 0$, the δ-neighborhood $U_\delta(t)$ of a given point $t \in \mathbb{T}$ is the set of all points $s \in \mathbb{T}$ such that $d(t,s) < \delta$. We also have, for functions $f : \mathbb{T} \to \mathbb{R}$, the concepts of limit, continuity, and the properties of continuous functions on a general complete metric space. Roughly speaking, the calculus on time scales begins by introducing and investigating the concept of derivative for functions $f : \mathbb{T} \to \mathbb{R}$. In the definition of derivative, an important role is played by the so-called jump operators (Bohner and Peterson, 2003).

Definition 1. Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$ we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$.

Remark 2. In Definition 1, we put $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(M) = M$ if $\mathbb{T}$ has a maximum M) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(m) = m$ if $\mathbb{T}$ has a minimum m), where $\emptyset$ denotes the empty set.

If $\sigma(t) > t$, then we say that t is right-scattered; if $\rho(t) < t$, then t is said to be left-scattered. Points that are simultaneously right-scattered and left-scattered are called isolated. If $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense; if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$.

The derivative makes use of the set $\mathbb{T}^\kappa$, which is derived from the time scale $\mathbb{T}$ as follows: if $\mathbb{T}$ has a left-scattered maximum M, then $\mathbb{T}^\kappa := \mathbb{T} \setminus \{M\}$; otherwise, $\mathbb{T}^\kappa := \mathbb{T}$.

Definition 3 (*Delta derivative (Agarwal and Bohner, 1999)*). Assume $f : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}^\kappa$. We define

$$f^\Delta(t) := \lim_{s \to t} \frac{f(\sigma(s)) - f(t)}{\sigma(s) - t}, \quad t \neq \sigma(s),$$

provided the limit exists. We call $f^\Delta(t)$ the delta derivative (or Hilger derivative) of f at t. Moreover, we say that f is delta differentiable on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. The function $f^\Delta : \mathbb{T}^\kappa \to \mathbb{R}$ is then called the (delta) derivative of f on $\mathbb{T}^\kappa$.

Definition 4. A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{C}_{rd}$. Similarly, a function $f : \mathbb{T} \to \mathbb{R}$ is called ld-continuous provided it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at right-dense points in $\mathbb{T}$. The set of ld-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{C}_{ld}$.

Definition 5. Let $[a, b]$ denote a closed bounded interval in $\mathbb{T}$. A function $F : [a, b] \to \mathbb{R}$ is called a delta antiderivative of function $f : [a, b) \to \mathbb{R}$ provided F is continuous on $[a, b]$, delta differentiable on $[a, b)$, and $F^\Delta(t) = f(t)$ for all $t \in [a, b)$. Then, we define the Δ-integral of f from a to b by

$$\int_a^b f(t)\Delta t := F(b) - F(a).$$

Proposition 6. (See Ahmadkhanlu and Jahanshahi (2012)) *Suppose $\mathbb{T}$ is a time scale and f is an increasing continuous function on the time-scale interval $[a, b]$. If F is the extension of f to the real interval $[a, b]$ given by*

$$F(s) := \begin{cases} f(s) & \text{if } s \in \mathbb{T}, \\ f(t) & \text{if } s \in (t, \sigma(t)) \notin \mathbb{T}, \end{cases}$$

then

$$\int_a^b f(t)\Delta t \leqslant \int_a^b F(t)dt.$$

We also make use of the classical gamma and beta functions.

Definition 7 (*Gamma function*). For complex numbers with a positive real part, the gamma function $\Gamma(t)$ is defined by the following convergent improper integral:

$$\Gamma(t) := \int_0^\infty x^{t-1} e^{-x} dx.$$

Definition 8 (*Beta function*). The beta function, also called the Euler integral of the first kind, is the special function $B(x, y)$ defined by

$$B(x, y) := \int_0^1 t^{x-1}(1 - t)^{y-1} dt, \quad x > 0, \quad y > 0.$$

Remark 9. The gamma function satisfies the following useful property: $\Gamma(t + 1) = t\Gamma(t)$. The beta function can be expressed through the gamma function by $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.

3. Main results

We introduce a new notion of fractional derivative on time scales. Before that, we define the fractional integral on a time scale $\mathbb{T}$. This is in contrast with (Benkhettou et al., 2015, in press-a, 2016), where first a notion of fractional differentiation on time scales is introduced and only after that, with the help of such a concept, the fraction integral is defined.

Definition 10. (Fractional integral on time scales) Suppose $\mathbb{T}$ is a time scale, $[a, b]$ is an interval of $\mathbb{T}$, and h is an integrable function on $[a, b]$. Let $0 < \alpha < 1$. Then the (left) fractional integral of order α of h is defined by

$${}_a^{\mathbb{T}}I_t^\alpha h(t) := \int_a^t \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)} h(s)\Delta s,$$

where Γ is the gamma function.

Definition 11. (Riemann–Liouville fractional derivative on time scales) Let $\mathbb{T}$ be a time scale, $t \in \mathbb{T}, 0 < \alpha < 1$, and $h : \mathbb{T} \to \mathbb{R}$. The (left) Riemann–Liouville fractional derivative of order α of h is defined by

$${}_a^{\mathbb{T}}D_t^\alpha h(t) := \frac{1}{\Gamma(1 - \alpha)} \left(\int_a^t (t - s)^{-\alpha} h(s)\Delta s \right)^\Delta. \quad (3)$$

Remark 12. If $\mathbb{T} = \mathbb{R}$, then Definition 11 gives the classical (left) Riemann–Liouville fractional derivative (Podlubny, 1999). For different extensions of the fractional derivative to time scales, using the Caputo approach instead of the Riemann–Liouville, see (Ahmadkhanlu and Jahanshahi, 2012; Bastos et al., 2011). For local approaches to fractional calculus on time scales we refer the reader to (Benkhettou et al., 2015, in press-a, 2016). Here we are only considering left operators. The corresponding right operators are easily obtained by changing the limits of integration in Definitions 10 and 11 from a to t (left of t) into t to b (right of t), as done in the classical fractional calculus (Podlubny, 1999). Here we restrict ourselves to the delta approach to time scales. Analogous definitions are, however, trivially obtained for the nabla approach to time scales by using the duality theory of (Caputo and Torres, 2015).

Along the work, we consider the order α of the fractional derivatives in the real interval $(0,1)$. We can, however, easily generalize our definition of fractional derivative to any positive real α. Indeed, let $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$. Then there exists $\beta \in (0,1)$ such that $\alpha = \lfloor \alpha \rfloor + \beta$, where $\lfloor \alpha \rfloor$ is the integer part of α, and we can set

$$ {}_a^{\mathbb{T}}D_t^{\alpha} h := {}_a^{\mathbb{T}}D_t^{\beta} h^{\Delta^{\lfloor \alpha \rfloor}}. $$

Fractional operators of negative order are defined as follows.

Definition 13. If $-1 < \alpha < 0$, then the (Riemann–Liouville) fractional derivative of order α is the fractional integral of order $-\alpha$, that is,

$$ {}_a^{\mathbb{T}}D_t^{\alpha} := {}_a^{\mathbb{T}}I_t^{-\alpha}. $$

Definition 14. If $-1 < \alpha < 0$, then the fractional integral of order α is the fractional derivative of order $-\alpha$, that is,

$$ {}_a^{\mathbb{T}}I_t^{\alpha} := {}_a^{\mathbb{T}}D_t^{-\alpha}. $$

3.1. Properties of the time-scale fractional operators

In this section we prove some fundamental properties of the fractional operators on time scales.

Proposition 15. *Let $\mathbb{T}$ be a time scale with derivative Δ, and $0 < \alpha < 1$. Then,*

$$ {}_a^{\mathbb{T}}D_t^{\alpha} = \Delta \circ {}_a^{\mathbb{T}}I_t^{1-\alpha}. $$

Proof. Let $h : \mathbb{T} \to \mathbb{R}$. From (3) we have

$$ {}_a^{\mathbb{T}}D_t^{\alpha}h(t) = \frac{1}{\Gamma(1-\alpha)}\left(\int_a^t (t-s)^{-\alpha}h(s)\Delta s\right)^{\Delta} $$
$$ = \left({}_a^{\mathbb{T}}I_t^{1-\alpha}h(t)\right)^{\Delta} = \left(\Delta \circ {}_a^{\mathbb{T}}I_t^{1-\alpha}\right)h(t). $$

The proof is complete. □

Proposition 16. *For any function h integrable on $[a,b]$, the Riemann–Liouville Δ-fractional integral satisfies ${}_a^{\mathbb{T}}I_t^{\alpha} \circ {}_a^{\mathbb{T}}I_t^{\beta} = {}_a^{\mathbb{T}}I_t^{\alpha+\beta}$ for $\alpha > 0$ and $\beta > 0$.*

Proof. By definition,

$$ \left({}_a^{\mathbb{T}}I_t^{\alpha} \circ {}_a^{\mathbb{T}}I_t^{\beta}\right)(h(t)) = {}_a^{\mathbb{T}}I_t^{\alpha}\left({}_a^{\mathbb{T}}I_t^{\beta}(h(t))\right) $$
$$ = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\left({}_a^{\mathbb{T}}I_t^{\beta}(h(s))\right)\Delta s $$
$$ = \frac{1}{\Gamma(\alpha)}\int_a^t \left((t-s)^{\alpha-1}\frac{1}{\Gamma(\beta)}\int_a^s (s-u)^{\beta-1}h(u)\Delta u\right)\Delta s $$
$$ = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t \int_a^s (t-s)^{\alpha-1}(s-u)^{\beta-1}h(u)\Delta u\Delta s $$
$$ = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t \left[\int_a^s (t-s)^{\alpha-1}(s-u)^{\beta-1}h(u)\Delta u + \int_s^t (t-s)^{\alpha-1}(s-u)^{\beta-1}h(u)\Delta u\right]\Delta s $$
$$ = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t \left[\int_a^t (t-s)^{\alpha-1}(s-u)^{\beta-1}h(u)\Delta u\right]\Delta s. $$

From Fubini's theorem, we interchange the order of integration to obtain

$$ \left({}_a^{\mathbb{T}}I_t^{\alpha} \circ {}_a^{\mathbb{T}}I_t^{\beta}\right)(h(t)) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t \left[\int_a^t (t-s)^{\alpha-1}(s-u)^{\beta-1}h(u)\Delta s\right]\Delta u $$
$$ = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t \left[\int_a^t (t-s)^{\alpha-1}(s-u)^{\beta-1}\Delta s\right]h(u)\Delta u $$
$$ = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t \left[\int_u^t (t-s)^{\alpha-1}(s-u)^{\beta-1}\Delta s\right]h(u)\Delta u. $$

By setting $s = u + r(t-u), r \in \mathbb{R}$, we obtain that

$$ \left({}_a^{\mathbb{T}}I_t^{\alpha} \circ {}_a^{\mathbb{T}}I_t^{\beta}\right)(h(t)) $$
$$ = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t \left[\int_0^1 (1-r)^{\alpha-1}(t-u)^{\alpha-1}r^{\beta-1}(t-u)^{\beta-1}(t-u)dr\right]h(u)\Delta u $$
$$ = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 (1-r)^{\alpha-1}r^{\beta-1}dr\int_a^t (t-u)^{\alpha+\beta-1}h(u)\Delta u $$
$$ = \frac{B(\alpha,\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t (t-u)^{\alpha+\beta-1}h(u)\Delta u = \frac{1}{\Gamma(\alpha+\beta)}\int_a^t (t-u)^{\alpha+\beta-1}h(u)\Delta u $$
$$ = {}_a^{\mathbb{T}}I_t^{\alpha+\beta}h(t). $$

The proof is complete. □

Proposition 17. *For any function h integrable on $[a,b]$ one has ${}_a^{\mathbb{T}}D_t^{\alpha} \circ {}_a^{\mathbb{T}}I_t^{\alpha}h = h$.*

Proof. By Propositions 15 and 16, we have

$$ {}_a^{\mathbb{T}}D_t^{\alpha} \circ {}_a^{\mathbb{T}}I_t^{\alpha}h(t) = \left[{}_a^{\mathbb{T}}I_t^{1-\alpha}\left({}_a^{\mathbb{T}}I_t^{\alpha}(h(t))\right)\right]^{\Delta} = \left[{}_a^{\mathbb{T}}I_t h(t)\right]^{\Delta} = h(t). $$

The proof is complete. □

Corollary 18. *For $0 < \alpha < 1$, we have ${}_a^{\mathbb{T}}D_t^{\alpha} \circ {}_a^{\mathbb{T}}D_t^{-\alpha} = Id$ and ${}_a^{\mathbb{T}}I_t^{-\alpha} \circ {}_a^{\mathbb{T}}I_t^{\alpha} = Id$, where Id denotes the identity operator.*

Proof. From Definition 14 and Proposition 17, we have that ${}_a^{\mathbb{T}}D_t^{\alpha} \circ {}_a^{\mathbb{T}}D_t^{-\alpha} = {}_a^{\mathbb{T}}D_t^{\alpha} \circ {}_a^{\mathbb{T}}I_t^{\alpha} = Id$; from Definition 13 and Proposition 17, we have that ${}_a^{\mathbb{T}}I_t^{-\alpha} \circ {}_a^{\mathbb{T}}I_t^{\alpha} = {}_a^{\mathbb{T}}D_t^{\alpha} \circ {}_a^{\mathbb{T}}I_t^{\alpha} = Id$. □

Definition 19. For $\alpha > 0$, let ${}_a^{\mathbb{T}}I_t^{\alpha}([a,b])$ denote the space of functions that can be represented by the Riemann–Liouville Δ integral of order α of some $\mathcal{C}([a,b])$-function.

Theorem 20. *Let $f \in \mathcal{C}([a,b])$ and $\alpha > 0$. In order that $f \in {}_a^{\mathbb{T}}I_t^{\alpha}([a,b])$, it is necessary and sufficient that*

$$ {}_a^{\mathbb{T}}I_t^{1-\alpha}f \in C^1([a,b]) \tag{4} $$

and

$$ \left({}_a^{\mathbb{T}}I_t^{1-\alpha}f(t)\right)\big|_{t=a} = 0. \tag{5} $$

Proof. Assume $f \in {}_a^{\mathbb{T}}I_t^{\alpha}([a,b]), f(t) = {}_a^{\mathbb{T}}I_t^{\alpha}g(t)$ for some $g \in \mathcal{C}([a,b])$, and

$$ {}_a^{\mathbb{T}}I_t^{1-\alpha}(f(t)) = {}_a^{\mathbb{T}}I_t^{1-\alpha}\left({}_a^{\mathbb{T}}I_t^{\alpha}g(t)\right). $$

From Proposition 16, we have

$$ {}_a^{\mathbb{T}}I_t^{1-\alpha}(f(t)) = {}_a^{\mathbb{T}}I_t g(t) = \int_a^t g(s)\Delta s. $$

Therefore, ${}_a^{\mathbb{T}}I_t^{1-\alpha}f \in \mathcal{C}([a,b])$ and

$$ \left({}_a^{\mathbb{T}}I_t^{1-\alpha}f(t)\right)\big|_{t=a} = \int_a^t g(s)\Delta s = 0. $$

Conversely, assume that $f \in \mathcal{C}([a,b])$ satisfies (4) and (5). Then, by Taylor's formula applied to function ${}_a^{\mathbb{T}}I_t^{1-\alpha}f$, one has

$$ {}_a^{\mathbb{T}}I_t^{1-\alpha}f(t) = \int_a^t \frac{\Delta}{\Delta s}{}_a^{\mathbb{T}}I_s^{1-\alpha}f(s)\Delta s, \quad \forall t \in [a,b]. $$

Let $\varphi(t) := \frac{\Delta}{\Delta t}{}_a^{\mathbb{T}}I_t^{1-\alpha}f(t)$. Note that $\varphi \in \mathcal{C}([a,b])$ by (4). Now, by Proposition 16, we have

$$ {}_a^{\mathbb{T}}I_t^{1-\alpha}(f(t)) = {}_a^{\mathbb{T}}I_t^{1}\varphi(t) = {}_a^{\mathbb{T}}I_t^{1-\alpha}\left[{}_a^{\mathbb{T}}I_t^{\alpha}(\varphi(t))\right] $$

and thus

$$ {}_a^{\mathbb{T}}I_t^{1-\alpha}(f(t)) - {}_a^{\mathbb{T}}I_t^{1-\alpha}\left[{}_a^{\mathbb{T}}I_t^{\alpha}(\varphi(t))\right] \equiv 0. $$

Then,

$$ {}_a^{\mathbb{T}}I_t^{1-\alpha}\left[f - {}_a^{\mathbb{T}}I_t^{\alpha}(\varphi(t))\right] \equiv 0. $$

From the uniqueness of solution to Abel's integral equation (Jahanshahi et al., 2015), this implies that $f - {}_a^{\mathbb{T}}I_t^{\alpha}\varphi \equiv 0$. Thus, $f = {}_a^{\mathbb{T}}I_t^{\alpha}\varphi$ and $f \in {}_a^{\mathbb{T}}I_t^{\alpha}[a,b]$. □

Theorem 21. *Let $\alpha > 0$ and $f \in \mathcal{C}([a,b])$ satisfy the condition in Theorem 20. Then,*

$$ \left({}_a^{\mathbb{T}}I_t^{\alpha} \circ {}_a^{\mathbb{T}}D_t^{\alpha}\right)(f) = f. $$

Proof. By Theorem 20 and Proposition 16, we have:

$$ {}_a^{\mathbb{T}}I_t^{\alpha} \circ {}_a^{\mathbb{T}}D_t^{\alpha}f(t) = {}_a^{\mathbb{T}}I_t^{\alpha} \circ {}_a^{\mathbb{T}}D_t^{\alpha}\left({}_a^{\mathbb{T}}I_t^{\alpha}\varphi(t)\right) = {}_a^{\mathbb{T}}I_t^{\alpha}\varphi(t) = f(t). $$

The proof is complete. □

3.2. Existence of solutions to fractional IVPs on time scales

In this section we prove the existence of a solution to the fractional order initial value problem (1)–(2) defined on a time scale. For this, let $\mathbb{T}$ be a time scale and $\mathcal{J} = [t_0, t_0 + a] \subset \mathbb{T}$. Then the function $y \in \mathcal{C}(\mathcal{J}, \mathbb{R})$ is a solution of problem (1)–(2) if

$$ {}_{t_0}^{\mathbb{T}}D_t^{\alpha}y(t) = f(t,y) \text{ on } \mathcal{J}, $$

$$ {}_{t_0}^{\mathbb{T}}I_t^{\alpha}y(t_0) = 0. $$

To establish this solution, we need to prove the following lemma and theorem.

Lemma 22. *Let $0 < \alpha < 1, \mathcal{J} \subseteq \mathbb{T}$, and $f : \mathcal{J} \times \mathbb{R} \to \mathbb{R}$. Function y is a solution of problem (1)–(2) if and only if this function is a solution of the following integral equation:*

$$ y(t) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}f(s,y(s))\Delta s. $$

Proof. By Theorem 21, ${}_{t_0}^{\mathbb{T}}I_t^{\alpha} \circ \left({}_{t_0}^{\mathbb{T}}D_t^{\alpha}(y(t))\right) = y(t)$. From (3) we have

$$ y(t) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}f(s,y(s))\Delta s. $$

The proof is complete. □

Our first result is based on the Banach fixed point theorem (Cronin, 1994).

Theorem 23. *Assume $\mathcal{J} = [t_0, t_0 + a] \subseteq \mathbb{T}$. The initial value problem (1)–(2) has a unique solution on $\mathcal{J}$ if the function $f(t,y)$ is a right-dense continuous bounded function such that there exists $M > 0$ for which $|f(t,y(t))| < M$ on $\mathcal{J}$ and the Lipshitz condition*

$$ \exists L > 0 : \forall t \in \mathcal{J} \text{ and } x,y \in \mathbb{R}, \quad \|f(t,x) - f(t,y)\| \leqslant L\|x-y\| $$

holds.

Proof. Let $\mathcal{S}$ be the set of rd-continuous functions on $\mathcal{J} \subseteq \mathbb{T}$. For $y \in \mathcal{S}$, define

$$ \|y\| = \sup_{t \in \mathcal{J}}\|y(t)\|. $$

It is easy to see that $\mathcal{S}$ is a Banach space with this norm. The subset of $\mathcal{S}(\rho)$ and the operator T are defined by

$$ \mathcal{S}(\rho) = \{X \in \mathcal{S} : \|X_s\| \leqslant \rho\} $$

and

$$ \mathrm{T}(y) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}f(s,y(s))\Delta s. $$

Then,

$$ |\mathrm{T}(y(t))| \leqslant \frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}M\Delta s \leqslant \frac{M}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}\Delta s. $$

Since $(t-s)^{\alpha-1}$ is an increasing monotone function, by using Proposition 6 we can write that

$$ \int_{t_0}^t (t-s)^{\alpha-1}\Delta s \leqslant \int_{t_0}^t (t-s)^{\alpha-1}ds. $$

Consequently,

$$ |\mathrm{T}(y(t))| \leqslant \frac{M}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}ds \leqslant \frac{M}{\Gamma(\alpha)}\frac{a^{\alpha}}{\alpha} = \rho. $$

By considering $\rho = \frac{Ma^{\alpha}}{\Gamma(\alpha+1)}$, we conclude that T is an operator from $\mathcal{S}(\rho)$ to $\mathcal{S}(\rho)$. Moreover,

$$ \begin{aligned} \|\mathrm{T}(x) - \mathrm{T}(y)\| &\leqslant \tfrac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}|f(s,x(s)) - f(s,y(s))|\Delta s \\ &\leqslant \tfrac{L\|x-y\|_{\infty}}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}\Delta s \\ &\leqslant \tfrac{L\|x-y\|_{\infty}}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}ds \\ &\leqslant \tfrac{L\|x-y\|_{\infty}}{\Gamma(\alpha)}\tfrac{a^{\alpha}}{\alpha} = \tfrac{La^{\alpha}}{\Gamma(\alpha+1)}\|x-y\|_{\infty} \end{aligned} $$

for $x, y \in \mathcal{S}(\rho)$. If $\frac{La^{\alpha}}{\Gamma(\alpha+1)} \leqslant 1$, then it is a contraction map. This implies the existence and uniqueness of the solution to the problem (1)–(2). □

Theorem 24. *Suppose $f : \mathcal{J} \times \mathbb{R} \to \mathbb{R}$ is a rd-continuous bounded function such that there exists $M > 0$ with $|f(t,y)| \leqslant M$ for all $t \in \mathcal{J}, y \in \mathbb{R}$. Then problem (1)–(2) has a solution on $\mathcal{J}$.*

Proof. We use Schauder's fixed point theorem (Cronin, 1994) to prove that T defined by (3) has a fixed point. The proof is given in several steps. *Step 1:* T is continuous. Let y_n be a sequence such that $y_n \to y$ in $\mathcal{C}(\mathcal{J}, \mathbb{R})$. Then, for each $t \in \mathcal{J}$,

$$\begin{aligned}|T(y_n)(t) \quad & -T(y)(t)| \\ & \leqslant \tfrac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} |f(s, y_n(s)) - f(s, y(s))| \Delta s \\ & \leqslant \tfrac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \sup_{s \in \mathcal{J}} |f(s, y_n(s)) - f(s, y(s))| \Delta s \\ & \leqslant \tfrac{\|f(\cdot, y_n(\cdot)) - f(\cdot, y(\cdot))\|_{\infty}}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \Delta s \\ & \leqslant \tfrac{\|f(\cdot, y_n(\cdot)) - f(\cdot, y(\cdot))\|_{\infty}}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} ds \\ & \leqslant \tfrac{\|f(\cdot, y_n(\cdot)) - f(\cdot, y(\cdot))\|_{\infty}}{\Gamma(\alpha)} \tfrac{a^{\alpha}}{\alpha} \\ & \leqslant \tfrac{a^{\alpha} \|f(\cdot, y_n(\cdot)) - f(\cdot, y(\cdot))\|_{\infty}}{\Gamma(\alpha+1)}.\end{aligned}$$

Since f is a continuous function, we have

$$|T(y_n)(t) - T(y)(t)|_{\infty} \leqslant \frac{a^{\alpha}}{\Gamma(\alpha+1)} \|f(\cdot, y_n(\cdot)) - f(\cdot, y(\cdot))\|_{\infty} \to 0 \text{ as } n \to \infty.$$

Step 2: the map T sends bounded sets into bounded sets in $\mathcal{C}(\mathcal{J}, \mathbb{R})$. Indeed, it is enough to show that for any ρ there exists a positive constant l such that, for each

$$y \in B_{\rho} = \{y \in \mathcal{C}(\mathcal{J}, \mathbb{R}) : \|y\|_{\infty} \leqslant \rho\},$$

we have $\|T(y)\|_{\infty} \leqslant l$. By hypothesis, for each $t \in \mathcal{J}$ we have

$$\begin{aligned}|T(y)(t)| \quad & \leqslant \tfrac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} |f(s, y(s))| \Delta s \\ & \leqslant \tfrac{M}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \Delta s \\ & \leqslant \tfrac{M}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} ds \\ & \leqslant \tfrac{Ma^{\alpha}}{\alpha\Gamma(\alpha)} = \tfrac{Ma^{\alpha}}{\Gamma(\alpha+1)} = l.\end{aligned}$$

Step 3: the map T sends bounded sets into equicontinuous sets of $\mathcal{C}(\mathcal{J}, \mathbb{R})$. Let $t_1, t_2 \in \mathcal{J}, t_1 < t_2, B_{\rho}$ be a bounded set of $\mathcal{C}(\mathcal{J}, \mathbb{R})$ as in Step 2, and $y \in B_{\rho}$. Then,

$$\begin{aligned}|T(y)(t_2) \quad & -T(y)(t_1)| \\ & \leqslant \tfrac{1}{\Gamma(\alpha)} \Big| \int_{t_0}^{t_1} (t_1 - s)^{\alpha-1} f(s, y(s)) \Delta s \\ & - \int_{t_0}^{t_2} (t_2 - s)^{\alpha-1} f(s, y(s)) \Delta s \Big| \\ & \leqslant \tfrac{1}{\Gamma(\alpha)} \Big| \int_{t_0}^{t_1} ((t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1} \\ & + (t_2 - s)^{\alpha-1}) f(s, y(s)) \Delta s \\ & \qquad - \int_{t_0}^{t_2} (t_2 - s)^{\alpha-1} f(s, y(s)) \Delta s \Big| \\ & \leqslant \tfrac{M}{\Gamma(\alpha)} \left| \int_{t_0}^{t_1} ((t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}) \Delta s + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \Delta s \right| \\ & \leqslant \tfrac{M}{\Gamma(\alpha)} \left| \int_{t_0}^{t_1} ((t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}) ds + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds \right| \\ & \leqslant \tfrac{M}{\Gamma(\alpha+1)} [(t_2 - t_1)^{\alpha} + (t_1 - t_0)^{\alpha} - (t_2 - t_0)^{\alpha}] + \tfrac{M}{\Gamma(\alpha+1)} (t_2 - t_1)^{\alpha} \\ & = \tfrac{2M}{\Gamma(\alpha+1)} (t_2 - t_1)^{\alpha} + \tfrac{M}{\Gamma(\alpha+1)} [(t_1 - t_0)^{\alpha} - (t_2 - t_0)^{\alpha}].\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3, together with the Arzela–Ascoli theorem, we conclude that $T : \mathcal{C}(\mathcal{J}, \mathbb{R}) \to \mathcal{C}(\mathcal{J}, \mathbb{R})$ is completely continuous. *Step 4:* a priori bounds. Now it remains to show that the set

$$\Omega = \{y \in \mathcal{C}(\mathcal{J}, \mathbb{R}) : y = \lambda T(y), 0 < \lambda < 1\}$$

is bounded. Let $y \in \Omega$. Then $y = \lambda T(y)$ for some $0 < \lambda < 1$. Thus, for each $t \in \mathcal{J}$, we have

$$y(t) = \lambda \left[\frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s, y(s)) \Delta s \right].$$

We complete this step by considering the estimation in Step 2. As a consequence of Schauder's fixed point theorem, we conclude that T has a fixed point, which is solution of problem (1)–(2). □

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