Journal of King Saud University – Science 31 (2019) 345–351

Contents lists available at ScienceDirect

Journal of King Saud University – Science

journal homepage: www.sciencedirect.com

Original article

Construction of circular strongly partially-balanced repeated measurements designs

Rida Jabeen, Rashid Ahmed *, H.M. Kashif Rasheed, Farrukh Shehzad

Department of Statistics, The Islamia University of Bahawalpur, Pakistan

ARTICLE INFO

Article history:
Received 16 May 2017
Accepted 21 March 2018
Available online 23 March 2018

MSC:
05B05
62K10
62K05

Keywords:
Circular strongly balanced repeated measurements designs
Circular strongly partially-balanced repeated measurements designs
Method of cyclic shifts
Repeated measurements designs
Residual effects

ABSTRACT

Strongly balanced repeated measurements designs are useful to balance out the residual effects. These designs are widely used in medicine, pharmacology, animal sciences and psychology. The situations where minimal strongly balanced repeated measurements designs cannot be constructed, strongly partially-balanced repeated measurements designs are preferred. This paper deals with the construction of circular strongly partially-balanced repeated measurements designs through method of cyclic shifts.

1. Introduction

A repeated measurements design is strongly balanced with respect to the first-order residual effects if each treatment is immediately preceded λ times by each other treatment (including itself). Williams (1949, 1950) initiated repeated measurements designs. Magda (1980) introduced the idea of a circular balance repeated measurements design when proper balance for different effects is considered. Cheng and Wu (1980) explained two different types of repeated measurements designs (RMD), the balanced uniform RMD and the strongly balanced uniform RMD. Afsarinejed (1990) presented some construction methods for repeated measurements design. Afsarinejed (1994) gave an easy method of constructing balanced and strongly balanced minimal repeated measurements designs with unequal period sizes. Iqbal and Jones (1994) constructed (i) efficient repeated measurements designs with equal and unequal period sizes using method of cyclic shifts, (ii) Strongly balanced repeated measurements designs, and (iii) combinatorial balanced designs for two unequal period sizes. Hedayat and Yang (2003) discussed the universal optimality of balanced cross over designs for $v \geq 3$ and $\lambda \leq (v-1)/2$. Sharma et al. (2003) introduced a general strategy of construction of balanced repeated measurements designs for odd number of treatments and their analysis. Iqbal and Tahir (2009) constructed CSBRMD (circular strongly balanced repeated measurements designs) for some classes. Iqbal et al. (2010) constructed some first- and second-order CBRMD (circular balanced repeated measurements designs). They also constructed some CSBRMDs. Their construction is for period sizes less than the number of treatments. Strongly balanced repeated measurement designs are widely used in medicine, pharmacology, animal sciences and psychology. The situations where minimal strongly balanced repeated measurements designs cannot be constructed, strongly partially-balanced repeated measurements designs are preferable. This paper deals with the construction of circular strongly partially-balanced repeated measurements designs through method of cyclic shifts. Chalikias and Kounias

* Corresponding author.
E-mail address: rashid701@hotmail.com (R. Ahmed).
Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

https://doi.org/10.1016/j.jksus.2018.03.014

(2012) extended the conditions of Cheng and Wu, in order to present a method to construct Φ-optimal designs for the case of $p = 3$ and $n = 0 \bmod 4$. Chalikias (2017) gave the impact review of two treatment repeated measurement designs with uncorrelated observations. Kounias and Chalikias (2015) extracts the universally optimal designs (strongly balanced on the periods and on the treatments) for the model with interactions.

The rest of the paper is organized as follows: In Section 2, we briefly explained the method of cyclic shifts. In Section 3, infinite series to generate CSPBRMDs are developed. In Sections 4 and 5, CSPBRMDs are constructed by method of cyclic shifts, Rule I and Rule II respectively.

2. Method of cyclic shifts

Method of cyclic shifts is explained here briefly. For detail, see Iqbal & Tahir (2009) and Iqbal et al. (2010).

Rule I: Let $S = [q_1, q_2, \ldots, q_{p-1}]$ be a set of shifts, where $0 \le q_i \le v - 1$. If each element $0, 1, 2, \ldots, v - 1$ appears an equal number of times, say λ' in a new set of shifts S^*, where $S^* = [q_1, q_2, \ldots, q_{p-1}, v - (q_1 + q_2 + \ldots + q_{p-1}) \bmod v]$ then it will be circular strongly balanced repeated measurement design. If S^* contains all of $0, 1, 2, \ldots, v - 1$ exactly λ_i times then it will be CSPBRMD.

Example 2.1. CSPBRMD is constructed for $v = 10$ and $p = 4$ through the following two sets of shifts. $S_1 = [2, 3, 8]$, $S_2 = [5, 6, 9]$

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}	B_{13}	B_{14}	B_{15}	B_{16}	B_{17}	B_{18}	B_{19}	B_{20}
0_3	1_4	2_5	3_6	4_7	5_8	6_9	7_0	8_1	9_2	0_0	1_1	2_2	3_3	4_4	5_5	6_6	7_7	8_8	9_9
2_0	3_1	4_2	5_3	6_4	7_5	8_6	9_7	0_8	1_9	5_0	6_1	7_2	8_3	9_4	0_5	1_6	2_7	3_8	4_9
5_2	6_3	7_4	8_5	9_6	0_7	1_8	2_9	3_0	4_1	1_5	2_6	3_7	4_8	5_9	6_0	7_1	8_2	9_3	0_4
3_5	4_6	5_7	6_8	7_9	8_0	9_1	0_2	1_3	2_4	0_1	1_6	2_3	3_4	4_5	5_6	6_7	7_8	8_9	9_0

Here ordered pairs {(0, 1), (1, 2), …, (8, 9), (9, 0), (0, 4), (1, 5), …, (5, 9), (6, 0), (7, 1), …, (9, 3)} do not appear together.

Rule II: Let $S_a = [q_{a1}, q_{a2}, \ldots, q_{a(p-1)}]$ and $S_b = [q_{b1}, q_{b2}, \ldots, q_{b(p-2)}]$t be sets of shifts, where $0 \le q_i \le v - 2$. If at least one set of shifts is of type $[q_1, q_2, \ldots, q_{(p-2)}]$t then S^* will be $[q_{a1}, q_{a2}, \ldots, q_{a(p-1)}, (v-1) - (q_{a1} + q_{a2} + \ldots + q_{a(p-1)}) \bmod (v-1), q_{b1}, q_{b2}, \ldots, q_{b(p-2)}]$. If each element $0, 1, 2, \ldots, v - 2$ appears an equal number of times, say λ' in a new set of shifts S^* then it will be circular strongly balanced repeated measurements design. If S^* contains all of $0, 1, 2, \ldots, v - 2$ exactly λ_i times then it will be CSPBRMD.

Example 2.2: Sets of shifts $S_1 = [1, 3, 8, 6]$ & $S_2 = [2, 4, 7]$t provide following CPBRMD for $v = 10$ with $p = 5$.

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}	B_{13}	B_{14}	B_{15}	B_{16}	B_{17}	B_{18}
0_0	1_1	2_2	3_3	4_4	5_5	6_6	7_7	8_8	0_9	1_9	2_9	3_9	4_9	5_9	6_9	7_9	8_9
1_0	2_1	3_2	4_3	5_4	6_5	7_6	8_7	0_8	2_0	3_1	4_2	5_3	6_4	7_5	8_6	0_7	1_8
4_1	5_2	6_3	7_4	8_5	0_6	1_7	2_8	3_0	6_2	7_3	8_4	0_5	1_6	2_7	3_7	4_8	5_0
3_4	4_5	5_6	6_7	7_8	8_0	0_1	1_2	2_3	4_6	5_7	6_8	7_0	8_1	0_2	1_3	2_4	3_5
0_3	1_4	2_5	3_6	4_7	5_8	6_0	7_1	8_2	9_4	9_5	9_6	9_7	9_8	9_0	9_1	9_2	9_3

Here ordered pairs {(0,5), (1,6), (2,7), (3,8), (4,0), (5,1), (6,2), (7,3), (8,4), (9,9)} do not appear together.

3. Infinite series to generate CSPBRMDs

Series 3.1: CSPBRMDs can be constructed for $v = 2mi + 2$, i integer and $p = 2m$, $m > 1$ through the following i sets of shifts.

$S_{j+1} = [mj + 2, mj + 3, \ldots, mj + m + 1, v - 2 - mj, v - 3 - mj, \ldots, v - m - mj]$; $j = 0, 1, \ldots, i - 2$.
$S_i = [v/2, (v+2)/2, v-1]$ for $p = 4$.
$S_i = [(v - (p - 2))/2, \ldots, (v - 6)/2, (v - 4)/2, v/2, (v+2)/2, (v+4)/2, (v+6)/2, \ldots, (v + p - 2)/2, v - 1]$ for $p > 4$.

Among v^2 ordered pairs, here $2v$ ordered pairs $\{(0, 1), (1, 2), \ldots, (v - 2, v - 1), (v - 1, 0), (0, (v - 2)/2), (1, v/2), \ldots, (v/2, v - 1), ((v + 2)/2, 0), ((v + 4)/2, 1), \ldots, (v - 1, (v - 4)/2)\}$ do not appear together while all other $v(v - 2)$ appear once.

Example 3.1(a). CSPBRMD can be constructed for $v = 4i + 2$ and $p = 4$ through the following i sets of shifts.

$S_{j+1} = [2j + 2, 2j + 3, v - 2 - 2j]$; $j = 0, 1, \ldots, i - 2$.
$S_i = [v/2, (v + 2)/2, v - 1]$

Example 3.1(b). CSPBRMD can be constructed for $v = 6i + 2$ and $p = 6$ through the following i sets of shifts.

$S_{j+1} = [3j + 2, 3j + 3, 3j + 4, v - 2 - 3j, v - 3 - 3j]$; $j = 0, 1, \ldots, i - 2$.
$S_i = [(v - 4)/2, v/2, (v + 2)/2, (v + 4)/2, v - 1]$

Example 3.1(c). CSPBRMD can be constructed for $v = 8i + 2$ and $p = 8$ through the following i sets of shifts.

$S_{j+1} = [4j + 2, 4j + 3, 4j + 4, 4j + 5, v - 2 - 4j, v - 3 - 4j, v - 4 - 4j]$; $j = 0, 1, \ldots, i - 2$.
$S_i = [(v - 6)/2, (v - 4)/2, v/2, (v + 2)/2, (v + 4)/2, (v + 6)/2, v - 1]$

Series 3.2: CSPBRMDs can be constructed for $v = 2m$, $m > 1$ integer and $p = 2m + 1$ through the following set of shifts.

$S_{j+1} = [0, 1, 2, \ldots, v - 1]$;

If sum of any two, three, ..., $v-1$ consecutive elements is 0 mod v, rearrange the elements to get the binary designs. Among v^2 ordered pairs, here v ordered pairs {(0, $v/2$), (1, $(v+2)/2$), ..., ($(v-2)/2$, $v-1$), ($v/2$, 0), ($(v+2)/2$, 1), ..., ($v-1$, $(v-2)/2$)} appear twice together while all other $v(v-1)$ appear once.

Example 3.2(a). CSPBRMD is constructed for $v = 6$ and $p = 7$ through the following set of shifts.

[0, 1, 4, 3, 2, 5]

Example 3.2(b). CSPBRMD is constructed for $v = 8$ and $p = 9$ through the following set of shifts.

[0, 1, 2, 3, 4, 5, 6, 7]

4. Construction of CSPBRMDs by Rule I

Construction 4.1: CSPBRMDs can be constructed for $v = ri + 1$, i odd and $p = r > 1$ odd through the following i sets of shifts.

$S_j = [q_{j1}, q_{j2}, \ldots, q_{j(r-1)}]$; $j = 1, 2, \ldots, i$.

where

- $0 \le q_{j1}, q_{j2}, \ldots, q_{j(r-1)} \le v-1$ but $\neq v/2$,
- S^* contains each of 0, 1, 2, ..., $v-1$ exactly once except $v/2$ which does not appear.
- $S^* = [q_{j1}, q_{j2}, \ldots, q_{j(r-1)}, v-(q_{j1} + q_{j2} + \ldots + q_{j(r-1)})]$

Among v^2 ordered pairs, here v ordered pairs {(0, $v/2$), (1, $(v+2)/2$), ..., ($(v-2)/2$, $v-1$), ($v/2$, 0), ($(v+2)/2$, 1), ..., ($v-1$, $(v-2)/2$)} do not appear together while all other $v(v-1)$ appear once. Designs constructed from this method for $p = 3$, $p = 5$, $p = 7$ and $p = 9$ for $v \le 100$ are presented in Tables A1–A4 respectively given in Appendix A.

Construction 4.2: CSPBRMDs can be constructed for $v = ri - 1$, i odd and $p = r$ through the following i sets of shifts.

$S_j = [q_{j1}, q_{j2}, \ldots, q_{j(r-1)}]$; $j = 1, 2, \ldots, i$.

where

- $0 \le q_{j1}, q_{j2}, \ldots, q_{j(r-1)} \le v-1$,
- S^* contains each of 0, 1, 2, ..., $v-1$ exactly once except $v/2$ which appears twice.
- $S^* = [q_{j1}, q_{j2}, \ldots, q_{j22}, v-(q_{j1} + q_{j2} + \ldots + q_{j22})]$

Among v^2 ordered pairs, here v ordered pairs {(0, $v/2$), (1, $(v+2)/2$), ..., ($(v-2)/2$, $v-1$), ($v/2$, 0), ($(v+2)/2$, 1), ..., ($v-1$, $(v-2)/2$)} appear twice together while all other $v(v-1)$ appear once. Designs constructed from this method for $p = 3$, $p = 5$, $p = 7$ and $p = 9$ for $v \le 100$ are presented in Tables A5–A8 respectively given in Appendix A.

Construction 4.3: CSPBRMDs can be constructed for $v = ri + 2$, i odd, $r > 1$(odd) and $p = r$ through the following i sets of shifts.

$S_j = [q_{j1}, q_{j2}, \ldots, q_{j(r-1)}]$; $j = 1, 2, \ldots, i$.

where,

- $0 \le q_{j1}, q_{j2}, \ldots, q_{j(r-1)} \le v-1$ but $\neq (v-1)/2, (v+1)/2$,
- S^* contains each of 0, 1, ..., $v-1$ exactly once except $(v-1)/2$, $(v+1)/2$ which does not appear.
- $S^* = [q_{j1}, q_{j2}, \ldots, q_{j(r-1)}, v-(q_{j1} + q_{j2} + \ldots + q_{j(r-1)}]$

Among v^2 ordered pairs, here $2v$ ordered pairs {(0, $(v-1)/2$), (1, $(v+1)/2$), ..., ($(v-1)/2$, $v-1$), ($(v+1)/2$, 0), ($(v+3)/2$, 1), ..., ($v-1$, $(v-1)/2$), (0, $(v+1)/2$), (1, $(v+3)/2$), ..., ($(v-3)/2$, $v-1$), ($(v-1)/2$, 0), ($(v+1)/2$, 1), ..., ($v-1$, $(v-1)/2$),} do not appear together while all other $v(v-2)$ appear once. Designs constructed from this method for $p = 5$, $p = 7$ and $p = 9$ for $v \le 100$ are presented in Tables A9–A11 respectively given in Appendix A.

Construction 4.4: CSPBRMDs can be constructed for $v = ri + 2$, i even, $r > 1$(odd) and $p = r$ through the following i sets of shifts.

$S_j = [q_{j1}, q_{j2}, \ldots, q_{j(r-1)}]$; $j = 1, 2, \ldots, i$.

where,

- $0 \le q_{j1}, q_{j2}, \ldots, q_{j(r-1)} \le v-1$ but $\neq 2, (v-4)/2$,
- S^* contains each of 0, 1, ..., $v-1$ exactly once except 2, $(v-4)/2$ which do not appear.
- $S^* = [q_{j1}, q_{j2}, \ldots, q_{j(r-1)}, v-(q_{j1} + q_{j2} + \ldots + q_{j(r-1)}]$

Among v^2 ordered pairs, $2v$ ordered pairs {(0, 2), (1, 3), ..., ($v-3$, $v-1$), ($v-2$, 0), ($v-1$, 1), (0, $(v-4)/2$), (1, $(v-2)/2$), (2, $v/2$), ..., ($(v+2)/2$, $v-1$), ($(v+4)/2$, 0), ($(v+6)/2$, 1), ..., ($v-1$, $(v-6)/2$)} do not appear together while all other $v(v-2)$ appear once. Designs constructed from this method for $p = 5$, $p = 7$ and $p = 9$ for $v \le 100$ are presented in Tables A12–A14 respectively given in Appendix A.

5. Construction of PSBRMDs by Rule II

Construction 5.1: CSPBRMDs can be constructed for $v = ri$, i even and $p = r > 3$(odd) through the following i sets of shifts.

$S_{j+1} = [q_{j1}, q_{j2}, \ldots, q_{j(r-1)}]$; $j = 1, 2, \ldots, i-1$.
$S_i = [q_{i1}, q_{i2}, \ldots, q_{i(r-2)}]$t

where

- $0 \le q_{j1}, q_{j2}, \ldots, q_{j(r-1)}, q_{i1}, q_{i2}, \ldots, q_{i(r-2)} \le v-2$ but $\neq v/2$.
- S^* contains each of 0, 1, ..., $v-2$ exactly once except $v/2$ which does not appear.
- $S^* = [q_{j1}, q_{j2}, \ldots, q_{j(r-1)}, v-1-(q_{j1} + q_{j2} + \ldots + q_{j(r-1)}), q_{i1}, q_{i2}, \ldots, q_{i(r-2)}]$

Here ordered pairs {(0, $(v-2)/2$), (1, $v/2$), ..., ($(v-2)/2$, $v-1$), ($v/2$, 0), ($(v+2)/2$, 1), ..., ($v-1$, $(v-4)/2$), ($v-1$, $v-1$)} does not appear together while all other appear once. Designs constructed from this method for $p = 5$, $p = 7$ and $p = 9$ for $v \le 60$ are presented in Tables B1–B3 respectively given in Appendix B.

Construction 5.2: CSPBRMDs can be constructed for $v = ri$, $i > 1$ (odd) and $p = r > 3$(odd) through the following i sets of shifts.

$S_{j+1} = [q_{j1}, q_{j2}, \ldots, q_{j(r-1)}]$; $j = 1, 2, \ldots, i-1$.
$S_i = [q_{i1}, q_{i2}, \ldots, q_{i(r-2)}]$t

where

- $0 \le q_{j1}, q_{j2}, \ldots, q_{j(r-1)}, q_{i1}, q_{i2}, \ldots, q_{i(r-2)} \le v-2$ but $\neq (v-1)/2$,
- S^* contains each of 0, 1, ..., $v-2$ exactly once except $(v-1)/2$ which does not appear.
- $S^* = [q_{j1}, q_{j2}, \ldots, q_{j(r-1)}, v-1-(q_{j1} + q_{j2} + \ldots + q_{j(r-1)}), q_{i1}, q_{i2}, \ldots, q_{i(r-2)}]$

Here ordered pairs {(0, $(v-1)/2$), (1, $(v+1)/2$), ..., ($(v-1)/2$, $v-1$), ($(v+1)/2$, 0), ($(v+3)/2$, 1), ..., ($v-1$, $(v-3)/2$), ($v-1$, $v-1$)} does not appear together while all other appear once. Designs constructed from this method for $p = 5$, $p = 7$ and $p = 9$ for $v \le 60$ are presented in Tables B4–B6 respectively given in Appendix B.

Construction 5.3: CSPBRMDs can be constructed for $v = ri + 1$, i even and $p = r > 3$(odd) through the following i sets of shifts.

$S_{j+1} = [q_{j1}, q_{j2}, \ldots, q_{j(r-1)}]$; $j = 1, 2, \ldots, i-1$.
$S_i = [q_{i1}, q_{i2}, \ldots, q_{i(r-2)}]$t

where

- $0 \leq q_{j1}, q_{j2}, \ldots, q_{j(r-1)}, q_{i1}, q_{i2}, \ldots, q_{i(r-2)} \leq v-2$ but $\neq 2$ and $(v-5)/2$.
- S^* contains each of 0,1,..., $v-2$ exactly once except 2 & $(v-5)/2$ which do not appear.
- $S^* = [q_{j1}, q_{j2}, \ldots, q_{j(r-1)}, v-1-(q_{j1}+q_{j2}+\ldots+q_{j(r-1)}), q_{i1}, q_{i2}, \ldots, q_{i(r-2)}]$

Here ordered pairs {(0, 2), (1, 3), ..., $(v-4, v-2)$, $(v-3, 0)$, $(v-2, 1)$, $(0, (v-5)/2)$, $(1, (v-3)/2)$,..., $(v-5, v-2)$, $(v-4, 0)$, $(v-3, 1)$, $(v-2, 2)$, $(v-1, v-1)$} do not appear together while all other $v(v-2)$ appear once. Designs constructed from this method for $p = 5$, $p = 7$ and $p = 9$ for $v \leq 60$ are presented in Tables B7–B9 respectively given in Appendix B.

Construction 5.4: CSPBRMDs can be constructed for $v = ri + 1$, $i > 1$(odd) and $p = r > 3$(odd) through the following i sets of shifts.

$S_{j+1} = [q_{j1}, q_{j2}, \ldots, q_{j(r-1)}]$; $j = 1, 2, \ldots, i-1$.
$S_i = [q_{i1}, q_{i2}, \ldots, q_{i(r-2)}]$t

where

- $0 \leq q_{j1}, q_{j2}, \ldots, q_{j(r-1)}, q_{i1}, q_{i2}, \ldots, q_{i(r-2)} \leq v-3$ but $\neq 1$.
- S^* contains each of 0, 1, ..., $v-2$ exactly once except 1 & $(v-2)$ which do not appear.
- $S^* = [q_{j1}, q_{j2}, \ldots, q_{j(r-1)}, v-1-(q_{j1}+q_{j2}+\ldots+q_{j(r-1)}), q_{i1}, q_{i2}, \ldots, q_{i(r-2)}]$

Here ordered pairs {(0, 1), (1, 2), ..., $(v-2, v-1)$, $(v-1, 0)$, $(0, (v-2)/2)$, $(1, v/2)$, ..., $(v/2, v-1)$, $((v+2)/2, 0)$, $((v+4)/2, 1)$, ..., $(v-1, (v-4)/2)$, $(v-1, (v-1)$} do not appear together while all other $v(v-2)$ appear once. Designs constructed from this method for $p = 5$, $p = 7$ and $p = 9$ for $v \leq 60$ are presented in Tables B10–B12 respectively given in Appendix B.

Conclusions and Ideas for Future Constructions: Series 3.1 works as a generator which generates the CSPBRMDs for every period size p (even). It provides CSPBRMDs for every v which is two plus multiple of p. Series 3.2 gives a design for a single value of v for each value of p, where $v = 2m$, $m > 1$ integer and $p = 2m + 1$. Constructions in Sections 4 and 5 are for p (odd). Generators to generate these designs are not developed yet for p (odd), therefore, these should be developed to generate (i) CSBRMDs in equal period sizes, (ii) CSBRMDs in unequal period sizes, (iii) CSPBRMDs in equal period sizes, (iv) CSPBRMDs in unequal period sizes, (v) CBRMDs in equal period sizes, (vi) CBRMDs in unequal period sizes, (vii) CWBRMDs (Circular weakly balanced repeated measurements designs) in equal period sizes, (viii) CWBRMDs in unequal period sizes.

Acknowledgment

Authors are highly grateful to the Reviewers for valuable corrections and suggestions. Authors are highly grateful also to the Higher Education Commission Research and Development Division National Research Program for Universities for providing grant for this research. No: 6294/Punjab/NRPU/R&D/HEC/2016.

Appendix A

Table A1.

v	p	Sets of shifts
4	3	[1,3]
10	3	[7,1] + [4,6] + [3,9]
16	3	[13,2] + [7,4] + [15,3] + [9,11] + [6,10]
22	3	[15,3] + [12,2] + [7,5] + [18,6] + [21,1] + [9,19] + [14,13]
28	3	[25,1] + [3,17] + [13,11] + [26,18] + [10,27] + [21,20] + [6,22] + [23,24] + [7,16]
34	3	[20,12] + [14,4] + [19,7] + [25,9] + [33,22] + [32,5] + [6,18] + [21,23] + [15,26] + [28,29] + [30,3]
40	3	[37,1] + [31,4] + [25,7] + [19,10] + [12,32] + [14,9] + [18,16] + [22,34] + [13,27] + [28,29] + [21,26] + [30,35] + [3,38]
46	3	[40,45] + [37,5] + [32,2] + [25,11] + [22,8] + [15,14] + [44,28] + [19,27] + [42,26] + [33,29] + [41,38] + [21,35] + [9,3] + [39,6] + [31,43]
52	3	[12,3] + [38,10] + [42,1] + [31,2] + [25,13] + [40,16] + [17,15] + [21,23] + [43,34] + [28,30] + [22,32] + [33,35] + [7,45] + [18,39] + [5,6] + [24,29] + [11,49]
58	3	[55,4] + [45,7] + [49,8] + [38,3] + [31,11] + [26,19] + [18,15] + [43,20] + [24,36] + [28,30] + [51,32] + [34,35] + [40,37] + [52,22] + [27,48] + [2,10] + [12,50] + [5,44] + [14,21]

Table A2.

v	p	Sets of shifts
6	5	[1,2,5,4]
16	5	[1,2,3,4] + [5,7,9,11] + [10,12,13,14]
26	5	[1,2,3,20] + [6,7,8,9] + [10,12,11,14] + [16,17,18,23] + [21,15,19,24]
36	5	[1,2,3,30] + [5,6,7,8] + [12,11,9,13] + [35,16,19,17] + [20,32,33,34] + [22,28,14,29] + [26,31,23,24]
46	5	[36,22,2,28] + [16,15,7,8] + [42,11,12,13] + [33,5,17,18] + [45,41,1,25] + [24,27,3,29] + [31,32,6,34] + [30,37,38,39] + [20,10,43,44]
56	5	[46,1,2,3] + [26,6,7,8] + [52,21,12,13] + [15,16,44,18] + [22,11,32,23] + [25,53,31,29] + [27,20,33,34] + [51,38,39,40] + [41,48,43,47] + [37,5,42,49] + [36,10,17,50]

Table A3.

v	p	Sets of shifts
8	7	[1,2,3,7,6,5]
22	7	[1,2,3,4,5,7] + [6,9,16,18,12,13] + [15,17,8,10,19,20]
36	7	[15,1,2,3,4,5] + [7,8,9,10,11,14] + [12,16,17,19,23,21] + [30,20,24,25,26,27] + [29,22,31,32,33,34]
50	7	[29,1,2,3,4,5] + [9,8,7,10,11,12] + [14,46,16,17,18,19] + [48,24,23,22,26,27] + [28,31,32,36,38,35] + [34,37,13,39,44,41] + [45,40,33,15,47,49]

Table A4.

v	p	Sets of shifts
10	9	[1,2,3,4,6,7,8,9]
28	9	[1,2,20,3,4,5,6,7] + [12,13,27,11,15,16,17,19] + [18,21,22,23,24,25,26,9]
46	9	[10,1,2,3,4,5,6,7] + [40,11,12,13,14,15,16,17] + [45,20,21,22,28,24,25,26] + [37,30,39,41,32,35,38,34] + [27,33,29,9,31,42,43,44]

Table A5.

v	p	Sets of shifts
2	3	[1,1]
8	3	[5,1] + [4,4] + [3,6]
14	3	[11,1] + [3,4] + [13,7] + [6,12] + [9,5]
20	3	[17,1] + [11,4] + [19,13] + [18,10] + [3,10] + [9,15] + [14,6]
26	3	[23,1] + [17,4] + [11,7] + [13,10] + [15,25] + [14,22] + [13,18] + [24,19] + [20,6]
32	3	[29,1] + [23,4] + [18,6] + [9,12] + [31,17] + [3,7] + [13,19] + [28,26] + [15,24] + [21,27] + [20,30]
38	3	[35,1] + [29,4] + [23,7] + [9,17] + [11,13] + [3,16] + [37,18] + [20,34] + [27,24] + [33,15] + [6,32] + [31,26] + [30,36]
44	3	[40,1] + [35,2] + [30,5] + [8,25] + [12,14] + [15,16] + [19,43] + [22,22] + [23,41] + [33,27] + [17,32] + [29,21] + [10,42] + [34,4] + [20,31]
50	3	[47,1] + [41,4] + [35,7] + [9,30] + [12,22] + [15,18] + [49,19] + [21,6] + [25,25] + [45,27] + [29,40] + [20,46] + [26,36] + [37,3] + [42,14] + [43,24] + [13,48]
56	3	[53,1] + [3,48] + [6,42] + [9,36] + [29,13] + [22,16] + [17,19] + [55,23] + [54,32] + [28,28] + [49,30] + [25,52] + [31,43] + [37,24] + [10,7] + [21,45] + [47,15] + [40,12] + [27,41]

Table A6.

v	p	Sets of shifts
4	5	[2,1,2,3]
14	5	[13,2,1,4] + [5,7,6,7] + [9,10,11,12]
24	5	[14,1,2,3] + [18,7,6,8] + [11,12,12,13] + [15,19,16,17] + [10,20,21,22]
34	5	[25,33,2,3] + [4,6,7,8] + [19,10,12,13] + [15,16,17,22] + [18,20,21,17] + [24,23,27,28] + [29,30,31,11]
44	5	[34,1,2,3] + [5,6,7,8] + [38,11,12,13] + [15,16,17,21] + [10,22,9,22] + [24,23,26,27] + [29,40,35,39] + [37,36,31,28] + [30,43,41,42]
54	5	[44,1,2,3] + [5,6,7,8] + [53,16,12,13] + [37,15,17,18] + [20,19,22,23] + [30,27,26,27] + [29,43,25,32] + [51,42,46,39] + [11,40,45,35] + [50,34,36,47] + [48,41,9,10]

Table A7.

v	p	Sets of shifts
6	7	[1,2,5,3,4,3]
20	7	[19,2,1,3,4,5] + [8,9,10,10,11,12] + [14,13,15,16,17,18]
34	7	[13,1,2,3,4,5] + [7,8,9,10,22,12] + [33,16,17,15,17,18] + [19,21,32,23,24,25] + [27,28,29,30,31,11]

Table A7. (*continued*)

v	p	Sets of shifts
48	7	[27,1,2,3,4,5] + [7,8,9,36,11,12] + [15,16,14,17,18,19] + [22,23,24,46,24,25] + [26,29,30,31,32,44] + [34,35,10,37,38,39] + [43,42,41,33,20,40]

Table A8.

v	p	Sets of shifts
8	9	[1,2,3,4,4,5,6,7]
26	9	[16,1,2,5,4,3,6,7] + [9,10,18,12,13,13,14,15] + [17,11,19,20,21,22,23,24]
44	9	[1,2,3,4,5,6,7,16] + [9,10,11,12,13,14,15,31] + [43,20,21,19,22,23,22,24] + [42,28,29,27,30,8,32,33] + [34,37,36,38,39,40,41,25]

Table A9.

v	p	Sets of shifts
7	5	[2,1,5,6]
17	5	[7,1,2,3] + [5,6,12,11] + [10,13,14,15]
27	5	[17,1,2,3] + [5,6,7,9] + [10,12,23,15] + [8,16,19,20] + [22,26,24,25]
37	5	[27,1,2,3] + [5,6,7,8] + [10,25,13,12] + [16,17,20,21] + [29,23,24,9] + [36,28,22,30] + [31,33,34,35]
47	5	[37,1,2,3] + [17,6,7,8] + [10,11,12,14] + [15,16,25,18] + [19,21,22,27] + [46,45,29,30] + [44,33,34,35] + [36,32,39,40] + [13,43,31,28]
57	5	[56,2,1,3] + [23,6,11,8] + [7,10,12,13] + [44,16,17,18] + [20,21,22,24] + [32,26,49,30] + [25,33,31,35] + [14,38,42,36] + [43,37,45,46] + [39,48,40,50] + [4,54,53,55]

Table A10.

v	p	Sets of shifts
9	7	[1,2,3,6,7,8]
23	7	[18,1,2,3,4,10] + [7,6,9,5,13,14] + [16,17,19,20,21,22]
37	7	[16,1,2,3,4,5] + [7,8,9,10,11,12] + [14,15,13,20,21,28] + [23,24,25,26,27,31] + [30,22,32,33,34,35]
51	7	[30,1,2,3,4,5] + [39,8,9,10,11,12] + [14,15,16,18,19,20] + [21,24,23,50,27,28] + [17,29,32,33,22,35] + [37,38,7,47,41,42] + [44,45,46,40,48,49]

Table A11.

v	p	Sets of shifts
11	9	[1,2,3,4,10,9,8,7]
29	9	[23,2,3,4,5,6,7,8] + [9,13,1,12,11,16,17,18] + [20,21,22,10,24,25,26,27]
47	9	[11,1,2,3,4,5,6,7] + [9,35,10,12,13,14,15,16] + [19,20,21,22,25,26,27,28] + [29,30,31,32,33,34,18,36] + [38,37,40,41,42,43,44,45]

Table A12.

v	p	Sets of shifts
12	5	[1,3,9,11] + [5,6,7,8]
22	5	[10,3,4,5] + [6,7,8,12] + [1,20,14,15] + [17,18,19,13]
32	5	[1,3,4,5] + [6,7,9,10] + [12,13,15,16] + [17,18,11,29] + [23,24,25,26] + [27,28,20,22]
42	5	[1,3,4,5] + [6,7,8,9] + [10,13,14,16] + [15,17,11,20] + [23,24,25,26] + [27,37,30,32] + [18,41,34,39] + [38,35,40,33]
52	5	[1,3,4,5] + [7,8,9,10] + [12,13,14,15] + [16,17,32,19] + [21,22,23,11] + [26,28,29,42] + [33,34,35,48] + [37,38,40,41] + [43,44,45,46] + [36,49,25,51]

Table A13.

v	p	Sets of shifts
16	7	[1,3,4,5,7,12] + [9,10,11,8,13,14]
30	7	[1,3,4,5,6,12] + [8,9,25,7,11,14] + [15,17,18,19,10,21] + [23,24,22,26,27,28]
44	7	[1,3,4,5,6,7] + [9,10,11,12,13,14] + [15,16,17,32,8,21] + [40,25,26,27,28,30] + [29,31,33,22,34,35] + [37,38,39,41,24,42]
58	7	[1,3,4,5,6,7] + [9,23,11,12,13,14] + [15,16,17,18,19,10] + [22,20,24,25,26,28] + [57,31,33,35,37,39] + [38,36,40,41,42,43] + [44,45,46,8,51,49] + [48,52,53,54,55,56]

Table A14.

v	p	Sets of shifts
20	9	[1,3,4,5,6,7,9,10] + [11,12,13,14,16,17,18,19]
38	9	[1,3,4,5,6,7,8,9] + [11,12,13,14,15,16,18,19] + [21,22,23,24,25,26,27,28] + [29,30,31,20,10,35,36,37]
56	9	[1,3,4,5,6,7,8,9] + [10,11,12,53,14,15,16,19] + [20,21,22,23,24,25,27,28] + [29,30,31,32,33,17,35,36] + [38,39,40,41,43,44,45,46] + [47,48,49,50,51,52,55,54]

Appendix B

Table B1.

v	p	Sets of shifts
10	5	[8,2,1,3] + [6,7,0]t
20	5	[9,1,2,3] + [5,6,7,8] + [11,13,18,15] + [16,17,14]t
30	5	[19,1,2,3] + [26,6,7,11] + [9,10,12,13] + [16,17,5,20] + [18,22,23,25] + [24,27,21]t
40	5	[29,1,2,3] + [5,6,7,8] + [32,11,12,9] + [15,16,10,18] + [21,22,23,24] + [26,25,28,38] + [37,17,33,34] + [36,31,30]
50	5	[39,1,2,3] + [19,6,7,8] + [10,12,13,14] + [15,16,17,18] + [20,21,22,11] + [26,27,28,29] + [40,5,33,34] + [36,30,38,48] + [41,42,43,23] + [45,46,31]t
60	5	[49,1,2,3] + [28,6,7,8] + [9,11,12,13] + [48,16,17,18] + [20,29,22,23] + [25,26,27,35] + [31,32,34,21] + [15,37,38,40] + [58,46,43,44] + [56,39,36,55] + [41,52,53,57] + [42,54,51]t

Table B2.

v	p	Sets of shifts
14	7	[1,2,3,4,5,11] + [8,9,10,6,12]t
28	7	[1,2,3,4,5,12] + [7,8,10,26,11,6] + [24,17,16,18,19,20] + [22,23,15,25,9]t
42	7	[20,1,2,3,4,5] + [7,8,9,10,11,24] + [38,15,16,17,18,19] + [22,23,12,26,25,27] + [40,39,31,32,33,34] + [36,35,14,30,28]t
56	7	[34,1,2,3,4,5] + [38,8,19,9,11,12] + [14,15,16,17,18,10] + [54,41,23,24,25,26] + [29,30,31,33,32,22] + [37,48,21,40,35,39] + [36,44,45,46,7,47] + [49,51,52,53,42]t

Table B3.

v	p	Sets of shifts
18	9	[15,1,2,3,4,5,6,7] + [10,11,12,13,14,0,16]t,
36	9	[34,1,2,3,4,5,6,7]+[32,10,12,11,13,14,15,16] + [19,20,21,22,24,23,25,26] + [28,29,27,31,9,33,0]t
54	9	[17,1,2,3,4,5,6,7] + [9,10,11,12,13,14,15,22] + [18,19,40,21,16,23,24,25] + [52,29,30,31,32,33,34,35] + [37,38,41,48,39,36,43,44] + [46,47,20,49,50,51,28]t

Table B4.

v	p	Sets of shifts
5	5	[0,1,3]t
15	5	[1,2,3,8] + [13,6,4,9] + [11,12,5]t
25	5	[14,1,2,3] + [18,7,6,8] + [11,10,13,23] + [16,17,19,20] + [21,22,5]t
35	5	[23,31,2,3] + [4,10,7,8] + [18,11,12,13] + [15,16,32,19] + [21,26,24,6] + [22,27,28,29] + [1,0,33]t
45	5	[34,1,2,3] + [5,6,7,8] + [39,10,12,13] + [15,16,17,21] + [28,32,23,24] + [26,27,20,29] + [31,40,33,37] + [36,11,42,43] + [41,38,9]t
55	5	[44,1,46,3] + [24,6,7,8] + [53,11,12,13] + [15,16,17,18] + [37,21,22,23] + [26,25,28,29] + [31,32,30,34,35] + [36,52,38,39] + [41,50,43,33] + [2,47,48,45] + [40,4,10]t

Table B5.

v	p	Sets of shifts
7	7	[0,1,2,4,5]t
21	7	[19,2,1,3,4,5] + [7,9,8,11,18,13] + [15,16,17,12,0]t
35	7	[13,1,2,3,4,5] + [7,8,9,21,11,12] + [19,33,15,16,25,18] + [22,23,24,20,26,27] + [29,30,31,32,14]t
49	7	[27,1,2,3,4,5] + [7,8,35,10,11,12] + [15,14,16,17,18,19] + [22,20,23,26,25,28] + [21,31,30,32,33,36] + [47,37,38,39,40,41] + [43,44,29,42,34]t
63	7	[41,1,2,3,4,5] + [7,9,10,11,12,13] + [14,15,16,18,17,19] + [44,22,23,24,20,26] + [29,28,56,32,33,34] + [35,37,38,40,48,8] + [43,21,45,46,47,59] + [50,51,52,53,54,55] + [30,58,39,60,61]t

Table B6.

v	p	Sets of shifts
9	9	[0,1,3,2,7,5,6]t
27	9	[16,1,2,3,5,4,6,7] + [10,12,9,11,14,15,24,17] + [19,20,21,22,23,0,25]t
45	9	[1,2,3,4,5,6,7,16] + [9,10,11,12,13,14,15,31] + [18,19,37,20,23,24,25,26] + [29,27,30,8,32,33,34,35] + [21,38,39,40,41,42,43]t

Table B7.

v	p	Sets of shifts
11	5	[1,8,5,6] + [7,4,9]t
21	5	[7,1,3,4] + [6,9,14,11] + [17,13,19,15] + [12,18,10]t
31	5	[17,1,10,27] + [6,7,8,9] + [11,12,14,19] + [16,18,15,20] + [22,23,24,25] + [3,28,29]t
41	5	[27,1,3,4] + [6,7,8,9] + [11,12,13,14] + [39,17,19,20] + [22,34,28,21] + [24,29,36,31] + [32,33,23,35] + [26,38,16]t
51	5	[37,1,3,4] + [16,7,8,9] + [46,12,13,14] + [25,17,18,19] + [20,22,48,27] + [26,6,49,31] + [11,28,34,35] + [30,39,40,41] + [36,43,44,45] + [47,24,29]t

Table B8.

v	p	Sets of shifts
15	7	[1,4,13,3,6,7] + [9,10,11,12,0]t
29	7	[1,3,4,5,6,9] + [8,7,16,11,13,14] + [23,17,18,19,20,21] + [10,24,25,26,27]t
43	7	[16,1,3,4,5,6] + [8,9,17,11,12,13] + [15,10,18,40,21,22] + [41,24,25,26,27,28] + [31,20,32,33,30,35] + [37,38,36,34,23]t
57	7	[30,1,3,4,5,6] + [8,9,45,11,12,13] + [55,16,17,18,19,20] + [10,21,24,25,27,28] + [22,31,32,29,34,35] + [37,38,39,40,36,42] + [44,46,47,43,49,51] + [50,52,53,54,15]t

Table B9.

v	p	Sets of shifts
19	9	[1,3,4,5,6,8,17,10] + [12,11,13,14,15,16,9]t
37	9	[29,1,3,4,5,6,7,8] + [11,10,12,13,14,30,17,18] + [20,21,23,22,24,25,26,27] + [0,15,31,32,33,34,35]t
55	9	[11,1,3,4,5,6,7,8] + [10,12,47,13,14,15,16,17] + [19,20,21,22,23,24,32,27] + [53,39,31,26,34,35,33,36] + [30,38,49,41,42,43,44,45] + [0,48,40,50,51,52,29]t

Table B10.

v	p	Sets of shifts
6	5	[0,2,3]t
16	5	[2,3,4,6] + [7,11,8,9] + [12,3,5]t
26	5	[11,2,3,4] + [17,6,8,9] + [12,21,13,14] + [16,18,19,22] + [7,20,23]t
36	5	[16,2,3,4] + [6,7,8,9] + [11,12,13,14] + [21,32,18,19] + [33,23,24,25] + [26,27,28,29] + [31,17,22]t

Table B10. (*continued*)

v	p	Sets of shifts
46	5	[31,2,3,4] + [6,7,8,9] + [41,12,13,14] + [16,17,18,19] + [11,42,23,24] + [26,27,28,29] + [32,33,34,36] + [22,37,38,40] + [21,30,39]t
56	5	[31,2,3,4] + [22,6,8,9] + [11,12,13,14] + [36,17,18,19] + [16,21,23,24] + [25,27,28,30] + [29,32,33,34] + [53,35,38,48] + [41,42,44,43] + [47,40,39,49] + [51,7,52]t

Table B11.

v	p	Sets of shifts
8	7	[0,2,3,5,4]t
22	7	[15,2,3,4,5,6] + [8,9,17,11,12,13] + [0,16,10,18,19]t
36	7	[8,2,3,4,5,6] + [9,10,11,12,13,15] + [29,17,16,18,20,19] + [22,23,24,25,26,27] + [14,30,31,32,33]t
50	7	[22,2,3,4,5,6] + [29,9,10,15,8,13] + [36,16,17,18,19,20] + [43,23,24,27,26,25] + [47,30,31,32,33,34] + [11,37,35,39,40,41] + [0,44,45,46,12]t

Table B12.

v	p	Sets of shifts
10	9	[0,2,3,4,5,6,7]t
28	9	[10,2,3,4,5,6,8,7] + [19,12,13,11,14,17,16,15] + [0,20,21,22,23,24,25]t
46	9	[2,3,4,5,6,7,8,10] + [19,11,12,13,14,15,17,16] + [36,20,21,22,26,25,24,23] + [30,29,9,31,32,33,34,35] + [27,38,39,40,41,42,43]t

References

Afsarinejed, K., 1990. Repeated measurements designs-a review. Commun. Stat. Theory Methods 19, 3985–4028.

Afsarinejed, K., 1994. Repeated measurements designs with unequal periods sizes. J. Ital. Stat. Soc. 2, 161–168.

Chalikias, M., Kounias, S., 2012. Extension and necessity of Cheng and Wu conditions. J. Stat. Plann. Inference 142 (7), 1794–1800.

Chalikias, M.S., 2017. Two treatment repeated measurement designs with uncorrelated observations: a compact review. J. Adv. Stat. 2 (1), 27.

Cheng, C.S., Wu, C.F., 1980. Balanced repeated measurements designs. Ann. Stat. 8, 1272–1283.

Hedayat, A.S., Yang, M., 2003. Universal optimality of balanced uniform crossover designs. Ann. Stat. 31, 978–983.

Iqbal, I., Jones, B., 1994. Efficient repeated measurements designs with equal and unequal period sizes. J. Stat. Plann. Inference 42, 79–88.

Iqbal, I., Tahir, M.H., 2009. Circular strongly balanced repeated measurements designs. Commun. Stat. Theory Methods 38, 3686–3696.

Iqbal, I., Tahir, M.H., Ghazali, S.S.A., 2010. Circular first-and second-order balanced repeated measurements designs. Commun. Stat. Theory Methods 39, 228–240.

Kounias, S., Chalikias, M.S., 2015. Optimal two treatment repeated measurement designs with treatment-period interaction in the model. Utilitas Math. 96, 243–261.

Magda, C.G., 1980. Circular balanced repeated measurements designs. Commun. Stat. Theory Methods 9, 1901–1918.

Sharma, V.K., Jaggi, S., Varghese, C., 2003. Minimal balanced repeated measurements designs. J. Appl. Stat. 30, 867–872.

Williams, E.R., 1949. Experimental designs balanced for the estimation of residual effects of treatments. Aust. J. Sci. Res. A2, 149–168.

Williams, E.R., 1950. Experimental designs balanced for pairs of residual effects. Aust. J. Sci. Res. A3, 351–363.