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Original article

New preconditioning and half-sweep accelerated overrelaxation solution for fractional differential equation

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ABSTRACT

The present paper investigates the approximate solution of a one-dimensional linear space-fractional diffusion equation using a new preconditioning matrix to develop an efficient half-sweep accelerated overrelaxation iterative method. The proposed method utilizes unconditionally stable implicit finite difference schemes to formulate the discrete approximation equation to the problem. The formulation employs the Caputo fractional derivative to treat the space-fractional derivative in the problem. The paper's focus is to assess the improvement in terms of the convergence rate of the solution obtained by the proposed iterative method. The numerical experiment illustrates the superiority of the proposed method in terms of solution efficiency against one of the existing preconditioned methods, preconditioned accelerated overrelaxation and implicit Euler method. The proposed method reveals the ability to compute the solution with lesser iterations and faster computation time than the preconditioned accelerated overrelaxation and implicit Euler method. The method introduced in the paper, half-sweep preconditioned accelerated overrelaxation, has the potential to solve a variety of space-fractional diffusion models efficiently. Future investigation will improve the absolute errors of the solutions.

1. Introduction

Fractional calculus has gained considerable popularity and importance for almost five decades now. It is mainly from various demonstrated applications in biological science, physical science and other branches of sciences. Fractional calculus has significantly contributed to the modelling of transmission of Covid-19 infection (Cui and Liu, 2022), pharmacokinetic compartments (Azizi, 2022), Meningitis with treatment and vaccination (Peter et al., 2022), tumour and immune cells interactions (Tang et al., 2022), mechanical behaviour of asphalt mastic (Lagos-Varas et al., 2022), fluid flow and heat transfer (Turkyilmazoglu, 2022), control behaviour of wearable exoskeletons (Sun et al., 2021) and control behaviour of a knee joint orthosis (Delavari and Jokar, 2021). Many different fractional differential equations (FDE) have arisen from the realistic applications of fractional calculus. FDE is a generalization of differential equations based on the established theory and application of fractional calculus. FDE can also be considered the extended partial differential equations by modifying the integer-order derivative into the fractional-order derivative.

The solutions of FDEs must be obtained to understand the fractional mathematical models. Various solution methods have been proposed to the literature, such as the finite difference method with collocation (Mesgarani et al., 2021;Safdari et al., 2020;Jaleb and Adibi, 2019), finite difference method with preconditioners

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(Barakitis et al., 2022; Shao and Kang, 2022; Sunarto et al., 2022; Sunarto et al., 2021), finite difference method with Lucas polynomials (Ali et al., 2022), Adomian decomposition method (Turkyilmazoglu, 2022; Ahmad et al., 2022; Turkyilmazoglu, 2021) and variational iteration method (Ibraheem et al., 2022). Following the high interest towards the finite difference method with preconditioners, this paper aims to investigate the approximate solution of a type of FDE, the space-fractional diffusion equation, using a new preconditioning matrix to develop an efficient half-sweep accelerated overrelaxation iterative method. This paper utilizes the Caputo fractional derivative to treat the space-fractional derivative because it allows the inclusion of traditional and conventional initial-boundary conditions in the formulation of the problem (Elsayed and Orlov, 2020). In addition, the Caputo space-fractional derivative's memories affect the dynamics of the considered variables (Sene, 2022). The importance of Caputo space-fractional can be seen in the modelling of biological models (Haghi and Ghanbari, 2022), sediment suspension in ice-covered channels (Wang et al., 2022), drug diffusion through the skin (Caputo and Cametti, 2021) and chaotic processes (Owolabi et al., 2020).

The paper's focus is to assess the improvement in terms of the convergence rate of the solution obtained by the proposed iterative method. Among various iterative methods that can be used to solve the generated system of equations from an FDE (She et al., 2023; AllaHamou et al., 2022; Wen et al., 2022; Sun et al., 2022; Tang and Huang, 2022), the paper proposes a modified accelerated overrelaxation iterative method using a new preconditioning matrix with a half-sweep iteration strategy. The paper's contribution is a new preconditioned iterative method that can solve a space-fractional diffusion equation at a good efficiency level. The following sections of the paper are organized: Section 2 formula tesa discrete approximation to a one-dimensional linear space-fractional diffusion equation using a half-sweep type finite difference method in the Caputo sense. Section 3 derives the proposed iterative method to solve the generated system of equations from the discretized problem. Section 4 illustrates the numerical results of solving several initial-boundary value problems using the proposed numerical method and the comparison analysis against the standard preconditioned accelerated overrelaxation method (Sunarto et al., 2016). The conclusion of the paper is stated in Section 5.

2. Half-sweep type finite difference method in caputosense

This section describes the formulation of a discrete approximation to a one-dimensional linear space-fractional diffusion equation using a half-sweep type finite difference method in the Caputo sense. The paper usesa general space fractional FDE in the formulation, which is given by (Reutskiy and Lin, 2018),

$$\frac{\partial U(x,t)}{\partial t} = c_1(x)\frac{\partial^\beta U(x,t)}{\partial x^\beta} + c_2(x)\frac{\partial U(x,t)}{\partial x} + c_3(x)U(x,t) + g(x,t), \tag{1}$$

and the solution is assumed to exist under the following initial and boundary conditions,

$$U(x,0) = I(x), U(0,t) = B_1(t), U(L,t) = B_2(t). \tag{2}$$

Based on Eq. (1), the variables $c_i, i = 1, 2,$ and 3 are either constants or functions in terms of x while $g(x,t)$ is a source function.

This paper utilizes half-sweep type implicit finite difference schemes to discretize Eq. (1) for the time derivative, integer-order space derivative and other functions (Ibrahim and Abdullah, 1995; Sunarto et al., 2021; Chew et al., 2021). Meanwhile, Caputo fractional derivative is applied to approximate the fractional-order space derivative. Below is the following established definition of Caputo fractional derivative used in the discretization (Oldham and Spanier, 2006):

Definition 1. Let x be the upper limit of the integral, and a real number β be the fractional order, such that $0 \le m-1 < \beta < m$ where m is a positive integer. Then, $f^{(m)}(\xi)$ represents the m-th order derivative of a smooth function $f(x)$. Hence, the Caputo fractional derivative of $f(x)$ is defined as

$$D_x^\beta f(x) = \frac{1}{\Gamma(m-\beta)}\int_0^x \frac{f^{(m)}(\xi)}{(x-\xi)^{\beta-m+1}}d\xi. \tag{3}$$

Combining half-sweep type finite difference schemes and Eq. (3) gives the following discrete approximation to the space-fractional derivative,

$$\frac{\partial^\beta U(x,t)}{\partial x^\beta} = \frac{1}{\Gamma(2-\beta)}\sum_{j=0,2,4,\cdots}^{i-2}\int_{jh}^{(j+1)h} \times \frac{U_{i-(j-2),n} - 2U_{i-j,n} + U_{i-(j+2),n}}{4h^2}(Ph-\xi)^\beta d\xi, \tag{4}$$

where for $i = 0, 2, 4, \ldots, s-2$ and $h = L/s$ where L and s represent the spatial interval and the numberof grid points, respectively. For the sake of simplicity, Eq. (4) can be simplified, and the space-fractional derivative can be equivalent to

$$\frac{\partial^\beta U(x,t)}{\partial x^\beta} = \rho\sum_{j=0,2,4,\cdots}^{i-2}\sigma_j^\beta\left(U_{i-(j-2),n} - 2U_{i-j,n} + U_{i-(j+2),n}\right), \tag{5}$$

where

$$\rho = \frac{(2h)^{-2}}{\Gamma(3-\beta)}, \tag{6}$$

and

$$\sigma_j^\beta = \left(\frac{j}{2}+1\right)^{2-\beta} - \left(\frac{j}{2}\right)^{2-\beta}. \tag{7}$$

Then, putting Eq. (5) together with the half-sweep finite differences for other derivatives such as first-order time derivative, first-order space derivative and source functions, Eq. (1) can be rewritten in the form of finite difference approximation equation in Caputo sense as follows,

$$k^{-1}\left(U_{i,n} - U_{i,n-1}\right) = c_1(x)\rho\sum_{j=0,2,4,\cdots}^{i-2}\sigma_j^\beta\left(U_{i-(j-2),n} - 2U_{i-j,n} + U_{i-(j+2),n}\right)$$

$$+c_2(x)(4h)^{-1}\left(U_{i+2,n} - U_{i-2,n}\right) + c_3(x)U_{i,n} + g_{i,n}. \tag{8}$$

Further arrangement and simplification can yield

$$c_2^*U_{i-2,n} + c_3^*U_{i,n} - c_2^*U_{i+2,n} - c_1^*\sum_{j=0,2,4,\cdots}^{i-2}\sigma_j^\beta\left(U_{i-(j-2),n} - 2U_{i-j,n} + U_{i-(j+2),n}\right)$$

$$= k^{-1}U_{i,n-1} + g_{i,n}, \tag{9}$$

where $c_1^* = c_1(x)\rho, c_2^* = c_2(x)(4h)^{-1}$ and $c_3^* = k^{-1} - c_3(x)$.

Based on Eq. (9), one may have the following equations subject to different values of j. For instance, when $j = 0$, Eq. (9) becomes

$$c_2^*U_{i-2,n} + c_3^*U_{i,n} - c_2^*U_{i+2,n} - c_1^*\sigma_0^\beta\left(U_{i+2,n} - 2U_{i,n} + U_{i-2,n}\right) = k^{-1}U_{i,n-1} + g_{i,n}, \tag{10}$$

and when $j = 2$, Eq. (9) becomes

$$c_2^*U_{i-2,n} + c_3^*U_{i,n} - c_2^*U_{i+2,n} - c_1^*\sigma_0^\beta\left(U_{i+2,n} - 2U_{i,n} + U_{i-2,n}\right)$$

$$-c_1^*\sigma_2^\beta(U_{i,n}-2U_{i-2,n}+U_{i-4,n})=k^{-1}U_{i,n-1}+g_{i,n}. \quad (11)$$

Hence, when the pattern continues for $j=4,6,\cdots$, one can easily obtain a general form of the equation that can generate a large-scale system of equations as follows,

$$-\tau_i+a_iU_{i-6,n}+b_iU_{i-4,n}+p_iU_{i-2,n}+q_iU_{i,n}++r_iU_{i+2,n}=f_{i,n}, \quad (12)$$

where

$$\tau_i=c_1^*\sum_{j=6,8,\dots}^{i-2}\sigma_j^\beta(U_{i-(j-2),n}-2U_{i-j,n}+U_{i-(j+2),n}), \quad (13)$$

$$a_i=-c_1^*\sigma_4^\beta, \quad (14)$$

$$b_i=-c_1^*\sigma_2^\beta+2c_1^*\sigma_4^\beta, \quad (15)$$

$$p_i=c_2^*-c_1^*\sigma_0^\beta+2c_1^*\sigma_2^\beta-c_1^*\sigma_4^\beta, \quad (16)$$

$$q_i=c_3^*+2c_1^*\sigma_0^\beta-c_1^*\sigma_2^\beta, \quad (17)$$

$$r_i=-c_2^*-c_1^*\sigma_0^\beta, \quad (18)$$

and

$$f_{i,n}=k^{-1}U_{i,n-1}+g_{i,n}. \quad (19)$$

When Eq. (12) takes all points bounded by a specified solution domain, the large-scale system of equations can be expressed in the form of a matrix equation,

$$M\widehat{U}=\widehat{f}, \quad (20)$$

where

$$M=\begin{bmatrix} q_2 & r_2 & & & \\ p_4 & q_4 & r_4 & & \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ & & a_{s-4} & b_{s-4} & p_{s-4} \\ & & & a_{s-2} & b_{s-2} \end{bmatrix}_{(s-2)\times(s-2)}, \quad (21)$$

$$\widehat{U}=\begin{bmatrix} U_2 \\ U_4 \\ \vdots \\ U_{s-4} \\ U_{s-2} \end{bmatrix}_{(s-2)\times 1}, \quad (22)$$

and

$$\widehat{f}=\begin{bmatrix} f_2-U_0p_2 \\ f_4 \\ \vdots \\ f_{s-4} \\ f_{s-2}-U_sp_{s-2} \end{bmatrix}_{(s-2)\times 1}. \quad (23)$$

Noted that the matrix dimensions of matrix M, $\widehat{U}$ and $\widehat{f}$ are $(s-2)\times(s-2)$, $(s-2)\times 1$, and $(s-2)\times 1$, respectively. This paper suggests that an efficient iterative method needs to be developed to solve a complex matrix equation like Eq. (20). Hence, this paper proposes a new preconditioning matrix that can enhance the convergence rate of the iterated solutions. Moreover, this paper develops a new iterative method called the half-sweep preconditioned accelerated overrelaxation.

3. Derivation of a preconditioned iterative method

This section is devoted to showing the derivation of the proposed preconditioned iterative method to solve the system of equations shown (Eq. (20)). From here, the paper shall use HSPAOR to stand for the proposed method to solve space-fractional diffusion problems. To begin the derivation, let's consider a transformed matrix equation that corresponds to Eq. (20) as follows,

$$A\widehat{U}=\widehat{f}. \quad (24)$$

Eq. (24) is obtained using the following matrix transformations with a new preconditioning matrix P,

$$A=PMP^T, \quad (25)$$

$$\widehat{f}=P\widehat{d}, \quad (26)$$

and

$$\widehat{U}=P^T\widehat{d}. \quad (27)$$

The preconditioning matrix P that is proposed in this paper has the form of

$$P=I+S, \quad (28)$$

where I is the identity matrix, and S has the form of

$$S=\begin{bmatrix} 0 & -\varphi_2 & 0 & 0 & 0 \\ 0 & 0 & -\varphi_4 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & -\varphi_{s-4} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{(s-2)\times(s-2)}. \quad (29)$$

Based on the coefficient matrix A that presents in the transformed matrix equation shown in Eq. (24), this paper considers a unique decomposition of matrix A that is given by

$$A=A_D-A_L-A_U, \quad (30)$$

where A_D, A_L, and A_U are the diagonal, the strictly lower triangular and the strictly upper triangular coefficients, respectively. Then, by strategically adding two accelerating parameters ω and θ, the HSPAOR method can be derived into

$$\widehat{U}^{(k+1)}=(A_D-\omega A_L)^{-1}[\omega A_U+(\theta-\omega)A_L+(1-\theta)A_D]\widehat{U}^{(k)} + \theta(A_D-\omega A_L)^{-1}\widehat{f}, \quad (31)$$

where $\widehat{U}^{(k+1)}$ and $\widehat{U}^{(k)}$ denotes the set of unknown points at $(k+1)$-th and (k)-th iterations, respectively. Based on the iterative method shown by Eq. (31), two accelerating parameters must be adjusted until some fixed values achieve the maximum convergence rate. Although the theory of estimating the optimum parameters exists, which can be referred to in (Hadjidimos, 1978), the theory is only valid for solving simple systems of linear equations.

Hence, to achieve the desired convergence rate and the objective of the numerical study, which is to investigate the numerical solutions, a manual selection of accelerating parameters is conducted by running the developed simulation program several times until the smallest number of iterations is obtained. The selection procedure can be described as follows. We initially let $\theta=1$ and use different values of ω within the range $(1,2)$. When the smallest number of iterations is obtained for some value of ω, by using the "optimum" value of ω, we increase the value of θ gradually until the final smallest number of iterations is obtained. The implementation of the HSPAOR method is programmed using the C++ programming language. The structure of the code and instructions are made thoroughly. Due to the copyright issue, the paper can only provide the following algorithm.

Algorithm 1: HSPAOR method to solve space-fractional diffusion equations

(i) Set the initial guess $\widehat{U}^{(k=0)} = 0$, and the tolerance error $\epsilon = 10^{-10}$.
(ii) For $i = 2, 4, \cdots, s-2$, iterate Eq. (31).
(iii) For $i = 1, 3, \cdots, s-1$, run linear interpolation module.
(iv) If $\left|\widehat{U}^{(k+1)} - \widehat{U}^{(k)}\right| \leq \epsilon$, then go to the next time-step or $n = n+1$.
(v) If the time-step reaches the final step or $n = N$, display outputs such as numerical solutions, the maximum number of iterations, program execution time, and maximum absolute errors.

4. Numerical experiment and results

Section 4 illustrates the proposed method's results by solving several initial-boundary value problems of space-fractional diffusion. Below are the following test problems considered in this paper.

Example 1. Consider the given one-dimensional linear time-dependent space-fractional diffusion equation (Khader, 2011),

$$\frac{\partial U(x,t)}{\partial t} = \Gamma(1.5)x^{0.5}\frac{\partial^{\beta} U(x,t)}{\partial x^{\beta}} + \left(x^2+1\right)\cos(t+1) - 2x \times \sin(t+1), \quad (32)$$

subjects to

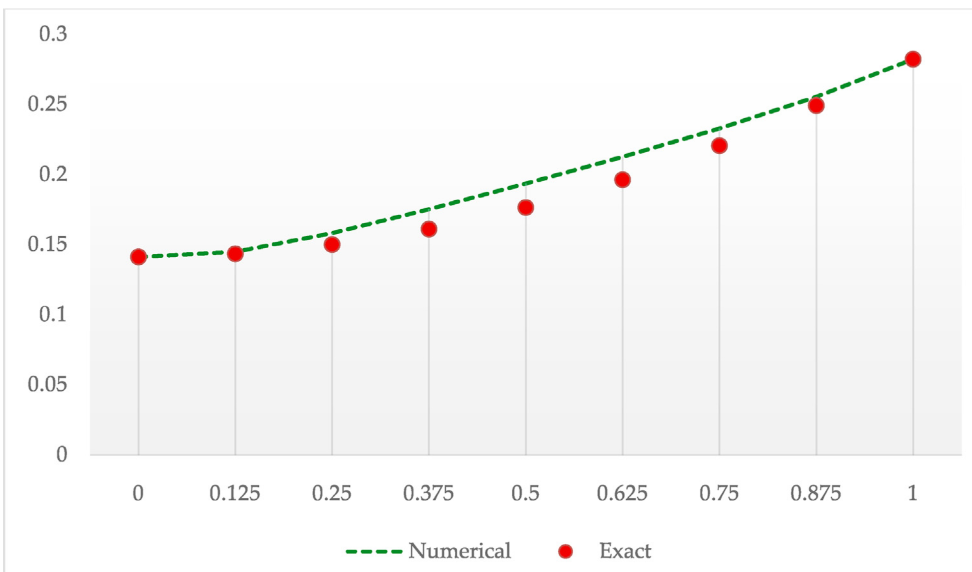

Fig. 1. Numerical solutions by HSPAOR against exact solutions of Example 1 at $\beta = 1.2$.

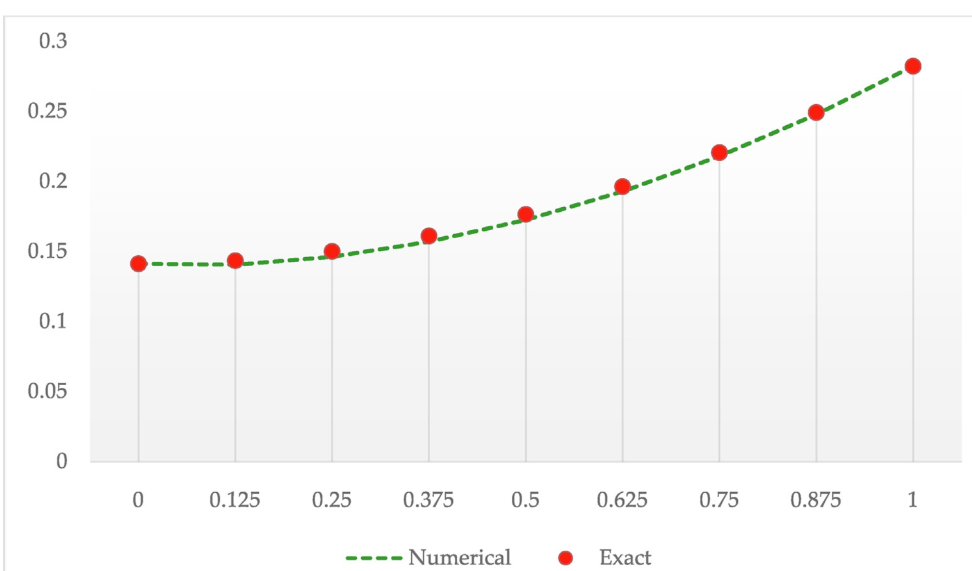

Fig. 2. Numerical solutions by HSPAOR against exact solutions of Example 1 at $\beta = 1.5$.

$$I(x) = (x^2 + 1)\sin(1), B_1(t) = \sin(t + 1), B_2(t) = 5\sin(t + 1). \tag{33}$$

Based on Eq. (32), the value of $\Gamma(1.5)x^{0.5}$ represents the diffusion coefficient, while the function $(x^2 + 1)\cos(t + 1) - 2x\sin(t + 1)$ is the source of diffusion. The accuracy of the numerical solution obtained by HSPAOR is compared to the exact solution,

$$U(x, t) = (x^2 + 1)\sin(t + 1). \tag{34}$$

Example 2. Consider another one-dimensional linear time-dependent space-fractional diffusion equation (Khader, 2011),

$$\frac{\partial U(x,t)}{\partial t} = \Gamma(1.2)x^{\beta}\frac{\partial^{\beta} U(x,t)}{\partial x^{\beta}} + 3x^2(2x - 1)e^{-t}, \tag{35}$$

subjects to

$$I(x) = x^2(1 - x), B_1(t) = B_2(t) = 0. \tag{36}$$

Based on Eq. (32), $\Gamma(1.2)x^{\beta}$ represents the diffusion coefficient, while $3x^2(2x - 1)e^{-t}$ is the source function. The accuracy of the numerical solution obtained by HSPAOR is compared to the exact solution,

$$U(x, t) = x^2(1 - x)e^{-t}. \tag{37}$$

The results considered take account of numerical solutions, the number of iterations to obtain the final solutions (k_f), the final time after completing the C++ program,which is measured in seconds (s) and the value of absolute errors. Fig. 1 until 6 show the numerical solutions obtained by HSPAOR after solving Examples 1 and 2 using $\beta = 1.2, 1.5,$ and 1.8. The solutions are compared to the exact solutions at various points and time level $T = 2.0$ seconds.

Based on Figs. 1 through 3, the effectiveness of HSPAOR in computing numerical solutions of Example 1 at various orders of space-fractional is illustrated. The numerical solutions are sufficiently

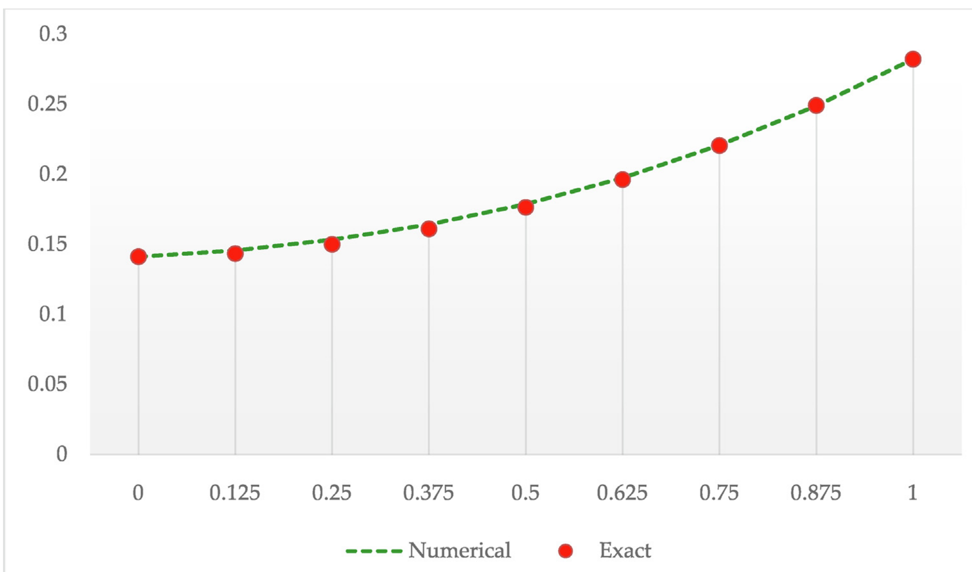

Fig. 3. Numerical solutions by HSPAOR against exact solutions of Example 1 at $\beta = 1.8$.

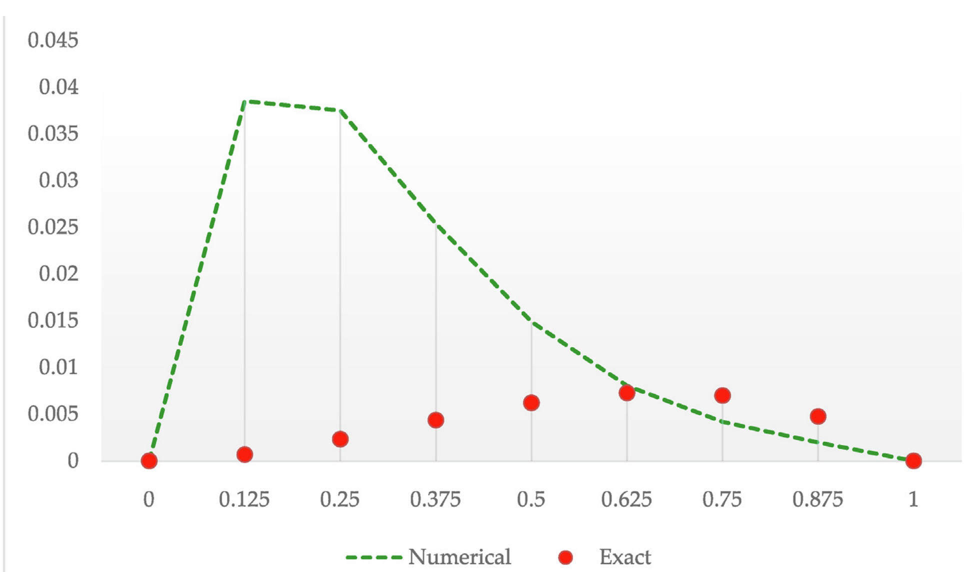

Fig. 4. Numerical solutions by HSPAOR against exact solutions of Example 2 at $\beta = 1.2$.

close to the provided exact solutions at $\beta = 1.2$ and well-fitted to the exact solutions at both $\beta = 1.5$ and 1.8. However, HSPAOR shows some disadvantages in computing the numerical solutions of Example 2 at $\beta = 1.2$ and 1.5 compared to the exact solutions, see Figs. 4 and 5. The accuracy of the solutions by HSPAOR is better when the value of space-fractional order is set to be greater than 1.5 or $\beta = 1.8$; for instance, see Fig. 6.

Next, comparison in terms of the number of iterations, program completion time and maximum absolute error between HSPAOR and two tested methods, such as the standard or full-sweep preconditioned accelerated overrelaxation (FSPAOR) (Sunarto et al., 2016) and implicit Euler (Meerschaert and Tadjeran, 2006) is shown in Tables 1 until 6. The comparison is conducted using three different values of space-fractional order, $\beta = 1.2, 1.5$, and 1.8, and five different numbers of domain points for the consistency of the solutions.

Based on Tables 1 until 6, the comparison results show that the HSPAOR method is more efficient than he FSPAOR and implicit Euler methods in solving Examples 1 and 2. The number of iterations and program completion time required by the HSPAOR method to obtain the final numerical solutions at all different points are significantly lesser than the other two tested methods. However, the absolute errors produced by the HSPAOR method are slightly larger than the FSPAOR and implicit Euler methods for Example 1 using $\beta = 1.5$ and 1.8 and Example 2 using $\beta = 1.8$. Furthermore, by observing the consistency of the numerical solutions with the increasing number of points in computation, this paper found that the absolute errors show some sign of gradual growth for Example 1 at $\beta = 1.2$ and Example 2 at all values of β.

To complete the numerical experiment, this paper compares the maximum absolute errors produced by the proposed HSPAOR method (with a time-step 0.2) with some numerical methods, including the methods that utilize the Chebyshev polynomial of degree n_p. The error comparison is made using a similar setting of Example 2 that has been done (Khader, 2011; Saadatmandi

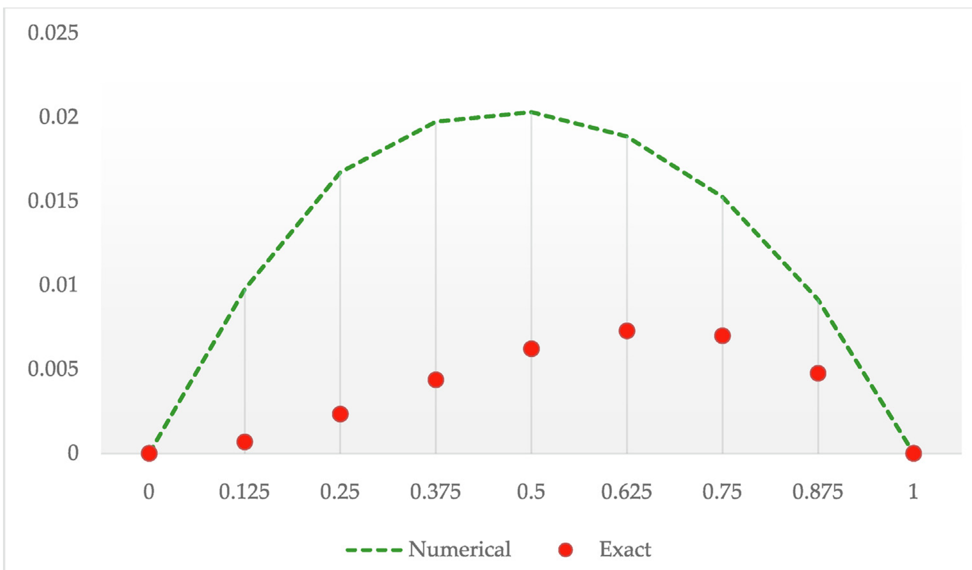

Fig. 5. Numerical solutions by HSPAOR against exact solutions of Example 2 at $\beta = 1.5$.

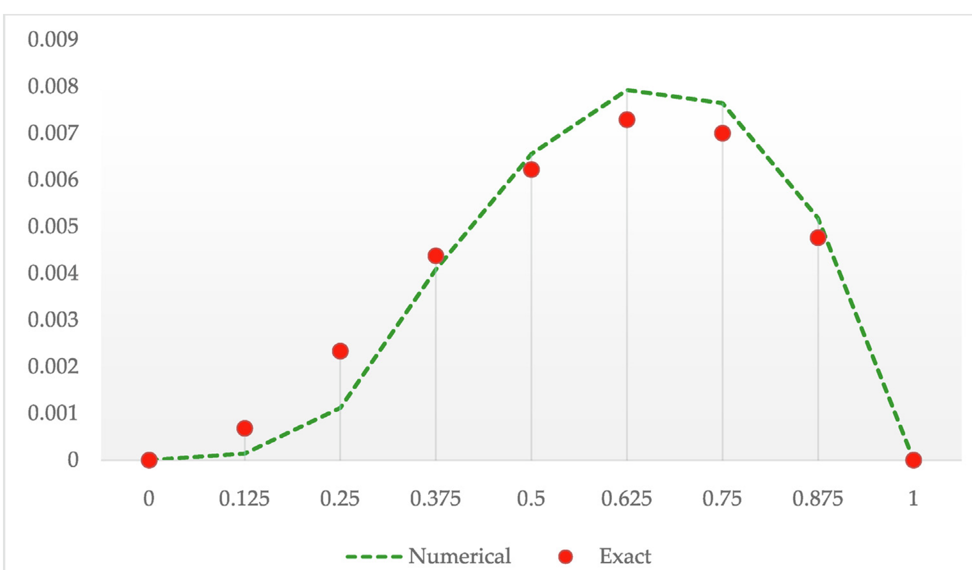

Fig. 6. Numerical solutions by HSPAOR against exact solutions of Example 2 at $\beta = 1.8$.

Table 1
Results comparison of solving Example 1using .$\beta = 1.2$.

s	Method	k	Seconds	Max Error
128	Implicit Euler	74	1.48	2.37e-02
	FSPAOR	33	0.73	2.37e-02
	HSPAOR	19	0.30	2.24e-02
256	Implicit Euler	152	11.64	2.44e-02
	FSPAOR	64	5.21	2.44e-02
	HSPAOR	35	2.73	2.37e-02
512	Implicit Euler	312	90.64	2.47e-02
	FSPAOR	127	35.22	2.47e-02
	HSPAOR	70	15.21	2.44e-02
1024	Implicit Euler	709	972.27	2.49e-02
	FSPAOR	272	342.76	2.49e-02
	HSPAOR	147	139.66	2.47e-02
2048	Implicit Euler	1647	3727.45	2.52e-02
	FSPAOR	597	1195.59	2.52e-02
	HSPAOR	318	452.46	2.49e-02

Table 2
Results comparison of solving Example 1 using $\beta = 1.5$.

s	Method	k	Seconds	Max Error
128	Implicit Euler	251	4.95	6.21e-04
	FSPAOR	77	1.84	6.21e-04
	HSPAOR	40	0.61	6.99e-04
256	Implicit Euler	666	51.01	5.69e-04
	FSPAOR	204	17.51	5.69e-04
	HSPAOR	100	7.040	6.21e-04
512	Implicit Euler	1780	550.52	5.35e-04
	FSPAOR	548	177.13	5.35e-04
	HSPAOR	261	49.26	5.69e-04
1024	Implicit Euler	4750	2970.31	5.13e-04
	FSPAOR	1469	873.87	5.13e-04
	HSPAOR	696	523.33	5.35e-04
2048	Implicit Euler	13,230	15348.70	5.09e-04
	FSPAOR	4012	4274.43	5.09e-04
	HSPAOR	1856	2132.82	5.24e-04

Table 3
Results comparison of solving Example 1 using $\beta = 1.8$.

s	Method	k	Seconds	Max Error
128	Implicit Euler	930	18.29	3.99e-04
	FSPAOR	234	5.56	3.99e-04
	HSPAOR	103	2.43	4.03e-04
256	Implicit Euler	3029	233.01	3.97e-04
	FSPAOR	769	66.34	3.97e-04
	HSPAOR	323	26.16	3.99e-04
512	Implicit Euler	9840	2755.31	3.96e-04
	FSPAOR	2528	828.27	3.96e-04
	HSPAOR	1067	305.81	3.97e-04
1024	Implicit Euler	46,847	7259.97	3.95e-04
	FSPAOR	11,783	2081.94	3.95e-04
	HSPAOR	5463	1005.63	3.96e-04
2048	Implicit Euler	187,322	28979.20	3.93e-04
	FSPAOR	47,253	8800.61	3.93e-04
	HSPAOR	22,125	4232.91	3.95e-04

and Dehghan, 2011; Azizi and Loghmani, 2013). Table 7 shows the comparison in terms of maximum absolute errors against the selected three methods.

Based on the findings through the numerical experiment, HSPAOR possesses the advantage in terms of computational efficiency, especially when a large system of equations is considered. The reason is that the iteration procedure by the preconditioned accelerated overrelaxation is highly efficient in computing the generated system of equations. Besides that, using a half-sweep strategy in formulating the finite difference approximation in the Caputo sense has successfully reduced the computational complexity in the developed program. However, to achieve a greater efficiency level, the accuracy of the solution becomes the trade-off. The disadvantage of the HSPAOR

Table 4
Results comparison of solving Example 2 using $\beta = 1.2$.

s	Method	k	Seconds	Max Error
128	Implicit Euler	57	1.42	5.44e-02
	FSPAOR	33	0.73	5.44e-02
	HSPAOR	19	0.30	5.16e-02
256	Implicit Euler	117	10.95	5.58e-02
	FSPAOR	64	5.21	5.58e-02
	HSPAOR	35	2.73	5.44e-02
512	Implicit Euler	249	81.84	5.58e-02
	FSPAOR	127	35.22	5.58e-02
	HSPAOR	70	15.21	5.28e-02
1024	Implicit Euler	560	853.89	5.65e-02
	FSPAOR	272	342.76	5.65e-02
	HSPAOR	147	139.66	5.32e-02
2048	Implicit Euler	1296	3157.00	5.80e-02
	FSPAOR	597	1195.59	5.80e-02
	HSPAOR	318	452.46	5.73e-02

Table 5
Results comparison of solving Example 2 using $\beta = 1.5$.

s	Method	k	Seconds	Max Error
128	Implicit Euler	182	4.41	1.80e-02
	FSPAOR	77	1.84	1.80e-02
	HSPAOR	40	0.61	1.73e-02
256	Implicit Euler	481	45.32	1.84e-02
	FSPAOR	204	17.51	1.84e-02
	HSPAOR	100	7.04	1.81e-02
512	Implicit Euler	1297	484.4	2.39e-02
	FSPAOR	548	177.13	2.39e-02
	HSPAOR	261	49.26	1.84e-02
1024	Implicit Euler	3493	2614.51	2.45e-02
	FSPAOR	1469	873.87	2.45e-02
	HSPAOR	696	523.33	1.86e-02
2048	Implicit Euler	9541	13859.30	2.92e-02
	FSPAOR	4012	4274.43	2.92e-02
	HSPAOR	1856	2132.82	1.86e-02

Table 6
Results comparison of solving Example 2 using $\beta = 1.8$.

s	Method	k	Seconds	Max Error
128	Implicit Euler	569	13.7	1.25e-04
	FSPAOR	234	5.56	1.25e-04
	HSPAOR	103	2.43	1.76e-04
256	Implicit Euler	1861	164.77	1.44e-04
	FSPAOR	769	66.34	1.44e-04
	HSPAOR	323	26.16	1.76e-04
512	Implicit Euler	6235	2027	1.53e-04
	FSPAOR	2528	828.27	1.53e-04
	HSPAOR	1067	305.81	1.82e-04
1024	Implicit Euler	29,937	5248.83	1.65e-04
	FSPAOR	11,783	2081.94	1.65e-04
	HSPAOR	5463	1005.63	1.84e-04
2048	Implicit Euler	121,482	22345.00	2.30e-04
	FSPAOR	47,253	8800.61	2.30e-04
	HSPAOR	22,125	4232.91	2.45e-04

method is revealed when it is used to solve Example 2 using $\beta = 1.2$ and 1.5. Since the development of HSPAOR is based on implicit finite difference schemes, the accuracy of HSPAOR is limited by the properties of implicit finite difference schemes, which are second-order accurate in space. This paper hypothesized that the magnitude of absolute errors could be reduced using higher-order finite difference schemes and different fractional definitions.

Table 7
Errors comparison of solving Example 2 using $\beta = 1.8$ at time-level $T = 2.0$ seconds for various points, x.

x	HSPAOR	(Khader, 2011), $n_p = 3$	(Saadatmandi and Dehghan, 2011)	(Azizi and Loghmani, 2013), $n_p = 5$
0	0	1.71e-04	0	0
0.1	5.87e-03	2.11e-05	2.89e-05	1.40e-07
0.2	6.98e-03	1.77e-04	1.09e-04	9.06e-07
0.3	6.31e-03	3.01e-04	2.20e-04	3.25e-08
0.4	5.10e-03	4.04e-04	3.40e-04	6.55e-08
0.5	3.83e-03	4.89e-04	4.45e-04	1.02e-08
0.6	2.67e-03	5.63e-04	5.15e-04	7.38e-09
0.7	1.71e-03	6.33e-04	5.27e-04	1.64e-07
0.8	9.54e-04	7.06e-04	4.60e-04	2.75e-08
0.9	3.91e-04	7.87e-04	2.91e-04	1.32e-07
1.0	0	8.83e-04	0	0

5. Conclusion

This paper successfully developed an efficient half-sweep accelerated overrelaxation iterative method using a new preconditioning matrix to solve several space-fractional diffusion problems. The Caputo fractional derivative is compatible with formulating a discrete approximation equation via implicit finite difference schemes. The numerical results showed the superiority of the proposed method in terms of solution efficiency against the standard preconditioned accelerated overrelaxation and implicit Euler methods. When the absolute errors by the proposed method are compared against several existing numerical methods, the errors are slightly larger than all considered methods. The magnitude of errors can be reduced by using higher-order finite difference schemes and different fractional definitions. Based on the performance of the proposed method in terms of efficiency, it has the potential to solve a variety of space-fractional diffusion models efficiently. Future investigation will improve the solutions' absolute errors so that the proposed method's reliability can be increased.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

Ahmad, M., Mishra, R., Jain, R., 2022. Analytical solution of one dimensional time fractional black-scholes equation through laplaceadomian decomposition method. Mathe. Eng., Sci. Aerospace. 13 (2), 373–386.
Ali, I., Haq, S., Aldosary, S.F., Nisar, K.S., Ahmad, F., 2022. Numerical solution of one- and two-dimensional time-fractional burgers equation via lucas polynomials coupled with finite difference method. Alex. Eng. J. 61 (8), 6077–6087. https://doi.org/10.1016/j.aej.2021.11.032.
AllaHamou, A., Hammouch, Z., Azroul, E., Agarwal, P., 2022. Monotone iterative technique for solving finite difference systems of time fractional parabolic equations with initial/periodic conditions. Appl. Numer. Math. 181, 561–593. https://doi.org/10.1016/j.apnum.2022.04.022.
Azizi, H., Loghmani, G.B., 2013. Numerical approximation for space-fractional diffusion equationsvia Chebyshev finite difference method. J. Fractional Appl. 4 (2), 303–311.
Azizi, T., 2022. Application of the fractional calculus in pharmacokinetic compartmental modeling. Commun. Biomathe. Sci. 5 (1), 63–77. https://doi.org/10.5614/cbms.2022.5.1.4.
Barakitis, N., Ekström, S., Vassalos, P., 2022. Preconditioners for fractional diffusion equations based on the spectral symbol. Num. Linear Algebra Appl. 29 (5). https://doi.org/10.1002/nla.2441. Article ID e2441.
Caputo, M., Cametti, C., 2021. Diffusion through skin in the light of a fractional derivative approach: Progress and challenges. J. Pharmacokinet Pharmacodyn. 48, 3–19. https://doi.org/10.1007/s10928-020-09715-y.
Chew, J.V.L., Sulaiman, J., Sunarto, A., 2021. Solving one-dimensional porous medium equation using unconditionally stable half-sweep finite difference and SOR method. Mathe. Stat. 9 (2), 166–171. https://doi.org/10.13189/ms.2021.090211.
Cui, T., Liu, P., 2022. Modeling the transmission phenomena of covid-19 infection with the effect of vaccination via noninteger derivative under real statistic. Fractals. 30 (5). https://doi.org/10.1142/S0218348X22401521. Article ID 2240152.
Delavari, H., Jokar, R., 2021. Intelligent fractional-order active fault-tolerant sliding mode controller for a knee joint orthosis. J. Intell. Robotic Syst.: Theory Appl. 102. https://doi.org/10.1007/s10846-021-01382-6. Article ID 39.
Elsayed, A.M., Orlov, V.N., 2020. Numerical scheme for solving time-space vibration string equation of fractional derivative. Mathematics 8. https://doi.org/10.3390/math8071069. Article ID 1069.
Hadjidimos, A., 1978. Accelerated overrelaxation method. Math. Comput. 32 (141), 149–157.
Haghi, T., Ghanbari, K., 2022. Existence and properties of positive solutions for Caputo fractional difference equation and applications. Computational Methods for. Diff. Eqs. 10 (3), 567–579. https://doi.org/10.22034/CMDE.2021.46257.1941.
Ibraheem, G.H., Turkyilmazoglu, M., Al-Jawary, M.A., 2022. Novel approximate solution for fractional differential equations by the optimal variational iteration method. J. Comput. Sci. 64. https://doi.org/10.1016/j.jocs.2022.101841. Article ID 101841.
Ibrahim, A., Abdullah, A.R., 1995. Solving the two dimensional diffusion equation by the fourpoint explicit decoupled group (EDG) iterative method. Int. J. Comput. Mathe. 58 (3–4), 253–263. https://doi.org/10.1080/00207169508804447.
Jaleb, H., Adibi, H., 2019. On a novel modification of the Legendre collocation method for solvingfractional diffusion equation. Comput. Methods Diff. Eqs. 7, 480–496.
Khader, M.M., 2011. On the numerical solutions for the fractional diffusion equation. Commun. Nonlinear Sci. Numer. Simul. 16 (6), 2535–2542. https://doi.org/10.1016/j.cnsns.2010.09.007.
Lagos-Varas, M., Movilla-Quesada, D., Raposeiras, A.C., Castro-Fresno, D., Muñoz-Cáceres, O., Andrés-Valeri, V.C., Rodríguez-Esteban, M.A., 2022. Viscoelasticity modelling of asphalt mastics under permanent deformation through the use of fractional calculus. Constr. Build. Mater. 329. https://doi.org/10.1016/j.conbuildmat.2022.127102. Article ID 127102.
Meerschaert, M.M., Tadjeran, C., 2006. Finite difference approximations for two-sided space-fractional partial differential equations. Appl. Numer. Math. 56 (1), 80–90. https://doi.org/10.1016/j.apnum.2005.02.008.
Mesgarani, H., Rashidinia, J., Aghdam, Y.E., Nikan, O., 2021. Numerical treatment of the spacefractional advection–dispersion model arising in groundwater hydrology. Comput. Appl. Mathe. 40. https://doi.org/10.1007/s40314-020-01410-5. Article ID 22.
Oldham, K.B., Spanier, J., 2006. The Fractional Calculus. Dover Publications, New York.
Owolabi, K.M., Gómez-Aguilar, J.F., Fernández-Anaya, G., Lavín-Delgado, J.E., Hernández-Castillo, E., 2020. Modelling of chaotic processes with Caputo fractional order derivative. Entropy 22 (9). https://doi.org/10.3390/e22091027. Article ID 1027.
Peter, O.J., Yusuf, A., Ojo, M.M., Kumar, S., Kumari, N., Oguntolu, F.A., 2022. A mathematical model analysis of Meningitis with treatment and vaccination in fractional derivatives. Int. J. Appl. Comput. Math. 8. https://doi.org/10.1007/s40819-022-01317-1. Article ID 117.
Reutskiy, S.Y., Lin, J., 2018. A semi-analytic collocation method for space fractional parabolic PDE. Int. J. Comput. Mathe. 95 (6–7), 1326–1339. https://doi.org/10.1080/00207160.2017.1420180.
Saadatmandi, A., Dehghan, M., 2011. A tau approach for solution of the space fractional diffusion equation. Comput. Math. Appl. 62 (3), 1135–1142. https://doi.org/10.1016/j.camwa.2011.04.014.
Safdari, H., Mesgarani, H., Javidi, M., Aghdam, Y.E., 2020. Convergence analysis of the spacefractional-order diffusion equation based on the compact finite difference

scheme. Comput. Appl. Mathe. 39. https://doi.org/10.1007/s40314-020-1078-z. Article ID 62.

Sene, N., 2022. Second-grade fluid with Newtonian heating under Caputo fractional derivative: analytical investigations via Laplace transforms. Mathe. Modell. Num. Simulat. Appl. 2 (1), 13–25. https://doi.org/10.53391/mmnsa.2022.01.002.

Shao, X., Kang, C., 2022. A preconditioner based on sine transform for space fractional diffusion equations. Appl. Num. Mathe. 178, 248–261. https://doi.org/10.1016/j.apnum.2022.04.001.

She, Z.H., Qiu, L.M., Qu, W., 2023. An unconditionally convergent RSCSCS iteration method for riesz space fractional diffusion equations with variable coefficients. Math. Comput. Simul 203, 633–646. https://doi.org/10.1016/j.matcom.2022.07.003.

Sun, A., Su, Y., Sun, J., 2022. Existence of solutions to a class of fractional differential equations. J. Nonlinear Model. Anal. 4, 409–442. https://doi.org/10.12150/jnma.2022.443.

Sun, J., Wang, J., Yang, P., Geng, Y., 2021. Model-free fractional-order adaptive back-stepping prescribed performance control for wearable exoskeletons. Int. J. Intell. Robot. Appl. 5, 590–605. https://doi.org/10.1007/s41315-021-00166-3.

Sunarto, A., Agarwal, P., Chew, J.V.L., Sulaiman, J., 2021. Approximation solution of the fractional parabolic partial differential equation by the half-sweep and preconditioned relaxation. Symmetry 13 (6). https://doi.org/10.3390/sym13061005. Article ID 1005.

Sunarto, A., Sulaiman, J., Saudi, A., 2016. Application of the full-sweep AOR iteration conceptfor space-fractional diffusion equation. J. Phys. Conf. Ser. 710. Article ID 012019.

Sunarto, A., Agarwal, P., Sulaiman, J., Lung, J.C.V., 2022. Numerical investigation on the solution of a space-fractional via preconditioned SOR iterative method. Progress Fractional Diff. Appl. 8 (2), 289–295. https://doi.org/10.18576/pfda/080208.

Tang, S.P., Huang, Y.M., 2022. A lopsided scaled DTS preconditioning method for the discrete space-fractional diffusion equations. Appl. Math. Lett. 131. https://doi.org/10.1016/j.aml.2022.108022.

Tang, T.Q., Shah, Z., Jan, R., Alzahrani, E., 2022. Modeling the dynamics of tumor–immune cells interactions via fractional calculus. Eur. Phys. J. Plus. 137. https://doi.org/10.1140/epjp/s13360-022-02591-0. Article ID 367.

Turkyilmazoglu, M., 2021. Nonlinear problems via a convergence accelerated decomposition method of adomian. CMES – Comput. Model. Eng. Sci. 127 (1), 1–22. https://doi.org/10.32604/cmes.2021.012595.

Turkyilmazoglu, M., 2022. Transient and passage to steady state in fluid flow and heat transfer within fractional models. Int. J. Numer. Meth. Heat Fluid Flow. https://doi.org/10.1108/HFF-04-2022-0262.

Wang, F., Li, Z., Zhang, Y., Guo, J., 2022. Fractional derivative modeling for sediment suspension in ice-covered channels. Environ. Sci. Pollut. Res. https://doi.org/10.1007/s11356-022-23011-x.

Wen, J., Liu, Z.X., Yue, C.W., Wang, S.J., 2022. Landweber iteration method for simultaneous inversion of the source term and initial data in a time-fractional diffusion equation. J. Appl. Math. Comput. 68, 3219–3250. https://doi.org/10.1007/s12190-021-01656-0.