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On the existence and uniqueness of solutions for a class of non-linear fractional boundary value problems

Arwa Ba Abdulla, Mohammed Al-Refai *, Ahmed Al-Rawashdeh

Department of Mathematical Sciences, UAE University, United Arab Emirates

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Abstract In this paper, we extend the maximum principle and the method of upper and lower solutions to study a class of nonlinear fractional boundary value problems with the Caputo fractional derivative $1 < \delta < 2$. We first transform the problem to an equivalent system of equations, including integer and fractional derivatives. We then implement the method of upper and lower solutions to establish existence and uniqueness results to the resulting system. At the end, some examples are presented to illustrate the validity of our results.

1. Introduction

Differential equations with fractional order are generalization of ordinary differential equations to non-integer order. In recent years, a great interest was devoted to study fractional differential equations, because of their appearance in various applications in Engineering and Physical Sciences, see Hilfer (2000), Luchko (2013), Mainardi (2010), Yang (2012), Yang and Baleanu (2013). Therefore, numerical and analytical techniques have been developed to deal with fractional differential equations (Agarwal et al., 2014; Al-Refai et al., 2014; Bhrawy and Zaky, 2015a,b; Nyamoradi et al., 2014; Li et al., 2011; Yang et al., 2013). The maximum principle and the method of lower and upper solutions are well established for differential equations of elliptic, parabolic and hyperbolic types (Pao, 1992; Protter and Weinberger, 1984). Recently, there are several studies devoted to extend, if possible, these results for fractional differential equations (Agarwal et al., 2010; Al-Refai and Hajji, 2011; Al-Refai, 2012; Furati and Kirane, 2008; Lakshmikantham and Vatsala, 2008; Luchko, 2009). It is noted that the extension is not a straightforward process, due to the difficulties in the definition and the rules of fractional derivatives. Therefore, the theory of fractional differential equations is not established yet and there are still many open problems in this area. Unlike, the integer derivative, there are several definitions of the fractional derivative, which are not equivalent in general. However, the most popular ones are the Caputo and Riemann–Liouville fractional derivatives. In this paper, we prove the existence and uniqueness of solutions to the fractional boundary value problem

* Corresponding author.
E-mail addresses: arwab@uaeu.ac.ae (A.B. Abdulla), m_alrefai@uaeu.ac.ae (M. Al-Refai), aalrawashdeh@uaeu.ac.ae (A. Al-Rawashdeh).
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$$D_{0^+}^{\delta} y + f(t, y, y') = 0, \quad 0 < t < 1,\ 1 < \delta < 2, \tag{1.1}$$

$$y(0) = a,\ y'(1) = b, \tag{1.2}$$

where f is continuous with respect to t on $[0, 1]$ and smooth with respect to y and y', and the fractional derivative is considered in the Caputo's sense. Several existence and uniqueness results for various classes of fractional differential equations have been established using the method of lower and upper solutions and fixed points theorems. The problem (1.1) with $f = f(t, y)$ and non-homogenous boundary conditions of Dirichlet type was studied by Al-Refai and Hajji (2011), where some existence and uniqueness results were established using the monotone iterative sequences of upper and lower solutions. In addition, the same problem (1.1) with $f(t, y) = f_0(t, y) + f_1(t, y) + f_2(t, y)$ was studied by Hu et al. (2013) using quasi-lower and quasi-upper solutions and monotone iterative technique. The problem (1.1) with $f = f(t, y)$ and homogeneous boundary conditions of Dirichlet type and $D_{0^+}^{\delta}$ is the standard Riemann–Liouville fractional derivative discussed by Bai and Lu (2005). They used certain fixed point theorems to establish the existence and multiplicity of positive solutions for the problem.

To the best of our knowledge, the method of monotone iterative sequences of lower and upper solutions has not been implemented for the problem (1.1)–(1.2), where the nonlinear term $f = f(t, y, y')$ depends on the variables y and y'. In order to apply the method of lower and upper solutions, we need some information about the fractional derivative of a function at its extreme points. While some estimates were obtained by Al-Refai (2012) for the fractional derivative $1 < \delta < 2$, these estimates require more information about the function, unlike the case when $0 < \delta < 1$. Therefore, we transform the problem (1.1)–(1.2) to an equivalent system of two equations and then we apply the method of lower and upper solutions to the new system.

This paper is organized as follows. In Section 2, we present some basic definitions and preliminary results. In Section 3, we establish the existence and uniqueness of solutions for an associated linear system of fractional equations using the Banch fixed point theorem. In Section 4, we establish the existence and uniqueness of maximal and minimal solution to the problem. Some illustrated examples are presented in Section 5. Finally, in Section 6, we present some concluding remarks.

2. Preliminary results

The left Caputo fractional derivative of order $\alpha > 0$, for $n - 1 < \alpha < n$, $n \in \mathbb{N}$ of a function f is defined by

$$(D_{0^+}^{\alpha} f)(t) = \left(I_{0^+}^{n-\alpha} \frac{d^n}{dt^n} f\right)(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)\,ds, & n-1 < \alpha < n \in \mathbb{N}, \\ f^{(n)}(t), & \alpha = n \in \mathbb{N}, \end{cases}$$

where Γ is the well-known Gamma function and $I_{0^+}^{\alpha}$ is the left Riemann–Liouville fractional integral defined by

$$(I_{0^+}^{\alpha} f)(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, & \alpha > 0, \\ f(t), & \alpha = 0. \end{cases} \tag{2.1}$$

For more details about the definition and properties of fractional derivatives, the reader is referred to Ortigueira (2011), Podlubny (1993). In the following, we transform the problem (1.1)–(1.2) to a system of differential equations, consisting of a fractional derivative and an integer derivative. Let $y_1 = y$, and $y_2 = y\prime = Dy$. Using the fact that $D_{0^+}^{\delta} y = D_{0^+}^{\delta-1}(Dy)$ for $1 < \delta < 2$, the system (1.1)–(1.2) is reduced to

$$Dy_1 - y_2 = 0, \quad 0 < t < 1, \tag{2.2}$$

$$D_{0^+}^{\alpha} y_2 + f(t, y_1, y_2) = 0, \quad 0 < t < 1,\ 0 < \alpha < 1, \tag{2.3}$$

$$y_1(0) = a,\ y_2(1) = b, \tag{2.4}$$

where $\alpha = \delta - 1$. For the above system we initially require that $y_1, y_2 \in C^1[0, 1]$ and f is continuous with respect to the variable t and smooth with respect to the variables y_1 and y_2.

We have the following definition of lower and upper solutions for the system (2.2)–(2.4).

Definition 2.1 (*Lower and Upper Solutions*). A pair of functions $(v_1, v_2) \in C^1[0, 1] \times C^1[0, 1]$ is called a pair of lower solutions of the problem (2.2)–(2.4), if they satisfy the following inequalities

$$Dv_1 - v_2 \le 0, \quad 0 < t < 1, \tag{2.5}$$

$$D_{0^+}^{\alpha} v_2 + f(t, v_1, v_2) \le 0, \quad 0 < t < 1,\ 0 < \alpha < 1, \tag{2.6}$$

$$v_1(0) \le a, \quad v_2(1) \le b. \tag{2.7}$$

Analogously, a pair of functions $(w_1, w_2) \in C^1[0, 1] \times C^1[0, 1]$ is called a pair of upper solutions of the problem (2.2)–(2.4), if they satisfy the reversed inequalities. In addition, if $v_1(t) \le w_1(t)$ and $v_2(t) \le w_2(t), \forall t \in [0, 1]$, we say that (v_1, v_2) and (w_1, w_2) are ordered pairs of lower and upper solutions.

The following important results will be used throughout the text.

Lemma 2.1. Al-Refai (2012). *Let $f \in C^1[0, 1]$ attain its absolute minimum at $t_0 \in (0, 1]$, then*

$$(D_{0^+}^{\alpha} f)(t_0) \le \frac{t_0^{-\alpha}}{\Gamma(1-\alpha)}[f(t_0) - f(0)] \le 0, \text{ for all } 0 < \alpha < 1.$$

Lemma 2.2. Changpin and Weihua (2007). *If $f \in C^n[0, 1]$ and $n - 1 < \alpha < n \in Z^+$, then $(D_{0^+}^{\alpha} f)(0) = 0$.*

We have the following new positivity result.

Lemma 2.3 (*Positivity Result*). *Let $\omega(t)$ be in $C^1[0, 1]$ that satisfies the fractional inequality*

$$D_{0^+}^{\alpha} \omega(t) + \mu(t)\omega(t) \ge 0,\ 0 < t < 1,\ 0 < \alpha < 1, \tag{2.8}$$

where $\mu(t) \ge 0$ and $\mu(0) \ne 0$. Then $\omega(t) \ge 0,\ \forall t \in [0, 1]$.

Proof. Assume by contradiction that $\omega(t) < 0$, for some $t \in [0, 1]$. As $\omega(t)$ is continuous on $[0, 1]$, $\omega(t)$ attains an absolute minimum value at $t_0 \in [0, 1]$ with $\omega(t_0) < 0$. If $t_0 \in (0, 1]$, then by Lemma 2.1, we have

$$\Gamma(1-\alpha)(D_{0^+}^{\alpha}\omega)(t_0) \le t_0^{-\alpha}[\omega(t_0) - \omega(0)] < 0.$$

Since $\Gamma(1 - \alpha) > 0$, for $0 < \alpha < 1$, we have $(D_{0^+}^{\alpha}\omega)(t_0) < 0$, and hence

$D^{\alpha}_{0^+}\omega(t_0) + \mu(t_0)\omega(t_0) < 0,$

which contradicts (2.8). If $t_0 = 0$, then by Lemma 2.2, $(D^{\alpha}_{0^+}\omega)(0) = 0$, and as $\mu(0) \neq 0$, we get

$(D^{\alpha}_{0^+}\omega)(0) + \mu(0)\omega(0) < 0,$

which contradicts (2.8). Hence the statement of the lemma is proved. □

Let $F[0,1]$ denote the set of all-real valued functions on $[0,1]$. We consider the order $\leq$ on $F[0,1] \times F[0,1]$, defined by $(f_1, f_2) \leq (g_1, g_2)$ if and only if, $f_1(x) \leq g_1(x)$ and $f_2(x) \leq g_2(x)$ for all $x \in [0,1]$. We have the following definition of comparable solutions of the problem (2.2)–(2.4).

Definition 2.2 (*Comparable Solutions*). Assume that $(u_1, u_2) \neq (v_1, v_2)$ are two solutions of the problem (2.2)–(2.4). We say that (u_1, u_2) and (v_1, v_2) are comparable solutions, if either $(u_1, u_2) \leq (v_1, v_2)$ or $(v_1, v_2) \leq (u_1, u_2)$.

The following result states the uniqueness of comparable solutions to the problem (2.2)–(2.4).

Theorem 2.1. *(Uniqueness of Comparable Solutions) Let* $(y_1(t), y_2(t)) \in C^1[0,1] \times C^1[0,1]$ *and* $(x_1(t), x_2(t)) \in C^1[0,1] \times C^1[0,1]$ *be comparable solutions of the problem (2.2)–(2.4). Assume that for any* $h_1, h_2 \in C^1[0,1]$ *there holds* $\frac{\partial f}{\partial h_1}(t, h_1, h_2) \leq 0$, *and* $\frac{\partial f}{\partial h_2}(t, h_1, h_2) \leq q$, *for some* $q < 0$. *Then* $(y_1, y_2) = (x_1, x_2)$, *for all* $t \in [0,1]$.

Proof. Since (y_1, y_2) and (x_1, x_2) are solutions of the problem (2.2)–(2.4), we have

$$D(x_1 - y_1) = x_2 - y_2,\ 0 < t < 1,$$
$$D^{\alpha}_{0^+}(x_2 - y_2) + f(t, x_1, x_2) - f(t, y_1, y_2) = 0,\ 0 < t < 1,\ 0 < \alpha < 1.$$

with $y_1(0) = x_1(0) = a,\ y_2(1) = x_2(1) = b$. As (y_1, y_2) and (x_1, x_2) are comparable solutions, we assume without loss of generality that $(y_1, y_2) \leq (x_1, x_2)$. Let $z_1 = x_1 - y_1 \geq 0$, and $z_2 = x_2 - y_2 \geq 0$. Applying the mean value theorem for the last equation we obtain

$$Dz_1 = z_2,\ 0 < t < 1, \tag{2.9}$$
$$D^{\alpha}_{0^+}z_2 + \frac{\partial f}{\partial y_1}(\rho_1)z_1 + \frac{\partial f}{\partial y_2}(\rho_2)z_2 = 0, \quad 0 < t < 1,\ 0 < \alpha < 1, \tag{2.10}$$

with $z_1(0) = 0$ and $z_2(1) = 0$, where $\rho_1 = \mu y_1 + (1-\mu)x_1$, 0.35em$\rho_2 = \nu y_2 + (1-\nu)x_2$ and $0 \leq \mu,\ \nu \leq 1$.

As $z_2 \in C^1[0,1]$ by Lemma 2.1, $D^{\alpha}_{0^+}z_2(0) = 0$, and by the continuity of $z_1(t)$ and $z_2(t)$ for $t \in [0,1]$, the last equation yields

$$0 = D^{\alpha}_{0^+}z_2(0) + \frac{\partial f}{\partial y_1}(\rho_1)z_1(0) + \frac{\partial f}{\partial y_2}(\rho_2)z_2(0) = \frac{\partial f}{\partial y_2}(\rho_2)z_2(0).$$

Since $\frac{\partial f}{\partial y_2}(t, y_1, y_2) \leq q < 0$, we have $z_2(0) = 0$.

Because $\frac{\partial f}{\partial y_1}(t, y_1, y_2) \leq 0$, and $z_1 \geq 0$, the Eq. (2.10) leads to

$$D^{\alpha}_{0^+}z_2 + \frac{\partial f}{\partial y_2}(\rho_2)z_2 = -\frac{\partial f}{\partial y_1}(\rho_1)z_1 \geq 0. \tag{2.11}$$

We have $\frac{\partial f}{\partial y_2}(t, y_1, y_2) \leq q < 0$, and $z_2 \geq 0$, thus $\frac{\partial f}{\partial y_2}(\rho_2)z_2 \leq qz_2 \leq 0$, and the Eq. (2.11) leads to

$$0 \leq D^{\alpha}_{0^+}z_2 + \frac{\partial f}{\partial y_2}(\rho_2)z_2 \leq D^{\alpha}_{0^+}z_2 + qz_2. \tag{2.12}$$

Applying the fractional integral operator $I^{\alpha}_{0^+}$ to the last inequality, we have

$$0 \leq I^{\alpha}_{0^+}D^{\alpha}_{0^+}z_2 + qI^{\alpha}_{0^+}z_2 = z_2(t) - z_2(0) + qI^{\alpha}_{0^+}z_2.$$

Since $z_2(0) = 0$, we have

$$0 \leq z_2(t) + q(I^{\alpha}_{0^+}z_2)(t),\ \forall t \in [0,1]. \tag{2.13}$$

In the following, we prove that $z_2(t) = 0,\ \forall t \in [0,1]$. Assume by contradiction that $z_2(t) \not\equiv 0$ in $[0,1]$. Since $z_2(1) = 0$, we have at $t = 1$,

$$0 \leq z_2(1) + q(I^{\alpha}_{0^+}z_2)(1) = q(I^{\alpha}_{0^+}z_2)(1)$$
$$= \frac{q}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}z_2(s)\,ds. \tag{2.14}$$

Since z_2 is continuous on $[0,1]$, and $z_2(t) \not\equiv 0$ on $[0,1]$, then the definite integral in the last equation is positive. Thus, $(I^{\alpha}_{0^+}z_2)(1) > 0$, which together with $q < 0$, leads to $q(I^{\alpha}_{0^+}z_2)(1) < 0$, which contradicts Eq. (2.14). Hence the assumption made is not correct and therefore $z_2(t) = 0,\ \forall t \in [0,1]$.

Substituting the last result in Eq. (2.9) yields $Dz_1 = 0$, which together with $z_1(0) = 0$, leads to $z_1 = 0,\ \forall t \in [0,1]$. Thus, $x_1 = y_1$ and $x_2 = y_2$ and the result of the theorem is proved. □

3. The linear system of fractional differential equations

In this section, we study the existence and uniqueness of solutions to the following linear initial and boundary value problems

$$\begin{cases} Dy_1(t) = g(t),\ 0 < t < 1, \\ y_1(0) = a, \end{cases} \tag{3.1}$$

$$\begin{cases} D^{\alpha}_{0^+}y_2(t) + \mu y_2(t) = f(t),\ 0 < t < 1,\ 0 < \alpha < 1, \\ y_2(1) = b, \end{cases} \tag{3.2}$$

where μ is a positive constant and $D^{\alpha}_{0^+}$ is the Caputo fractional derivative. These results will be used later on to establish the existence and uniqueness of solutions of the monotone iterative sequences of the nonlinear system (2.2)–(2.4). The existence and uniqueness of solution for the problem (3.1) is guaranteed provided $g(t)$ is continuous on $[0,1]$. We apply the Banach fixed point theorem to prove the existence and uniqueness of solutions to the problem (3.2). We have

Lemma 3.1. *Let* $f(t)$ *be in* $C[0,1]$. *Then* $y_2(t) \in C^1[0,1]$ *is a solution to the problem (3.2) if and only if, it is a solution to the integral equation*

$$y_2(t) = b + \int_0^1 G(t,s)[\mu y_2(s) - f(s)]\,ds, \tag{3.3}$$

where

$$G(t,s)=\begin{cases}\frac{(1-s)^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0<s<t<1,\\ \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0<t<s<1.\end{cases} \tag{3.4}$$

Proof. Applying the fractional integral operator $I_{0^+}^{\alpha}$ to the first equation in the system (3.2), we get

$$y_2(t)-y_2(0)+\mu(I_{0^+}^{\alpha}y_2)(t)=(I_{0^+}^{\alpha}f)(t),$$

which can be written as

$$y_2(t)=y_2(0)-\frac{\mu}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y_2(s)\,ds+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)\,ds.$$

By substituting $t=1$, in the last equation, we have

$$b=y_2(1)=y_2(0)-\frac{\mu}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}y_2(s)\,ds+\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}f(s)\,ds.$$

Thus,

$$y_2(0)=b+\frac{\mu}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}y_2(s)\,ds-\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}f(s)\,ds,$$

and

$$\begin{aligned}y_2(t)&=b+\frac{\mu}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}y_2(s)\,ds-\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}f(s)\,ds\\&\quad-\frac{\mu}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y_2(s)\,ds+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)\,ds\\&=b+\frac{1}{\Gamma(\alpha)}\int_0^1(\mu y_2(s)-f(s))(1-s)^{\alpha-1}ds\\&\quad-\frac{1}{\Gamma(\alpha)}\int_0^t(\mu y_2(s)-f(s))(t-s)^{\alpha-1}ds.\end{aligned} \tag{3.5}$$

The last equation can be written as

$$y_2(t)=b+\int_0^1G(t,s)[\mu y_2(s)-f(s)]\,ds,$$

where $G(t,s)$ is defined in (3.4).

Conversely, let $y_2(t)\in C^1[0,1]$ satisfy Eq. (3.3). Then y_2 satisfies Eq. (3.5), which can be written as

$$y_2(t)=b+\frac{1}{\Gamma(\alpha)}\int_0^1(\mu y_2(s)-f(s))(1-s)^{\alpha-1}\,ds-(I_{0^+}^{\alpha}(\mu y_2-f))(t).$$

Applying the fractional derivative operator $D_{0^+}^{\alpha}$ yields

$$(D_{0^+}^{\alpha}y_2)(t)=-(D_{0^+}^{\alpha}I_{0^+}^{\alpha}(\mu y_2-f))(t)=-\mu y_2(t)+f(t).$$

Thus, $(D_{0^+}^{\alpha}y_2)(t)+\mu y_2(t)=f(t)$. At $t=1$, from Eq. (3.5), we have $y_2(1)=b$, which completes the proof of the Theorem. □

In the following theorem, we establish the existence and uniqueness result of the system (3.2) using the Banach fixed point theorem.

Theorem 3.1. *Suppose that $f(t)\in C[0,1]$ and the constant μ satisfies*

$$0<\frac{2\mu}{\Gamma(\alpha+1)}<1, \tag{3.6}$$

then problem (3.2) possesses a unique solution.

Proof. For every $x\in C[0,1]$, define

$$Tx=b+\int_0^1G(t,s)[\mu x(s)-f(s)]\,ds.$$

Since $G(t,s)$ is Riemann integrable, it is clear that T is self mapping on $C[0,1]$. To show that problem (3.2) has a unique solution, we show that T is a contraction. Let $x_1(t)$ and $x_2(t)$ be in $C[0,1]$, then we have

$$\begin{aligned}\|Tx_1-Tx_2\|&=\max_{t\in[0,1]}|(Tx_1)(t)-(Tx_2)(t)|=\left\|\mu\int_0^1G(t,s)(x_1-x_2)\,ds\right\|\\&\le\|x_1-x_2\|\mu\max_{0\le t\le1}\int_0^1|G(t,s)|\,ds=\|x_1-x_2\|\mu\\&\quad\max_{0\le t\le1}\left\{\int_0^t-\frac{(1-s)^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds+\int_t^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right\}\\&=\|x_1-x_2\|\frac{\mu}{\Gamma(\alpha+1)}\max_{0\le t\le1}\{-1+t^{\alpha}+2(1-t)^{\alpha}\}\\&\le\|x_1-x_2\|\frac{2\mu}{\Gamma(\alpha+1)}.\end{aligned}$$

Since $\frac{2\mu}{\Gamma(\alpha+1)}<1$, we have that T is a contraction, therefore by the Banach fixed point theorem, the equation $Tx=x$ has a unique solution, and hence by Lemma 3.1, the proof is completed. □

4. Existence and uniqueness results

4.1. Monotone sequences of lower and upper solutions

In this section, we construct monotone iterative sequences of lower and upper solutions to the system (2.2)–(2.4). Then we use these sequences to establish an existence result.

Given pairs $V=(v_1^{(0)},v_2^{(0)})$ and $W=(w_1^{(0)},w_2^{(0)})$ of lower and upper solutions, respectively, to the problem (2.2)–(2.4) with $V\le W$. We define the set

$$[V,W]=\{(h_1,h_2)\in C^1[0,1]\times C^1[0,1]:v_1^{(0)}\le h_1\le w_1^{(0)},v_2^{(0)}\le h_2\le w_2^{(0)}\}.$$

We assume that the nonlinear term $f(t,y_1,y_2)$ satisfies the following conditions on $[V,W]$:

(A1) The function $f(t,h_1,h_2)$ is decreasing with respect to h_1, that is $\frac{\partial f}{\partial h_1}(t,h_1,h_2)\le0$ for all $(h_1,h_2)\in[V,W]$, and $t\in[0,1]$.
(A2) There exists a positive constant c, such that $\frac{\partial f}{\partial h_2}(t,h_1,h_2)\le c$, for all $(h_1,h_2)\in[V,W]$, and $t\in[0,1]$.

The following theorem describes the monotone iterative sequences of lower and upper pairs of solutions.

Theorem 4.1. *Assume that the conditions (A1) and (A2) are satisfied and consider the iterative sequence $U^{(k)}=(y_1^{(k)},y_2^{(k)})$, $k\ge0$ which is defined by*

$$Dy_1^{(k)}(t) = y_2^{(k-1)}(t),\ 0 < t < 1 \tag{4.1}$$

$$D_{0^+}^{\alpha} y_2^{(k)}(t) + cy_2^{(k)}(t) = cy_2^{(k-1)}(t) - f(t, y_1^{(k-1)}, y_2^{(k-1)}),\quad 0 < t < 1,\ 0 < \alpha < 1, \tag{4.2}$$

$$\text{with}\quad y_1^{(k)}(0) = a_k,\ y_2^{(k)}(1) = b_k. \tag{4.3}$$

We have

1. *If $U^{(0)} = V = (v_1^{(0)}, v_2^{(0)})$ and a_k, b_k are increasing sequences with $a_k \le a$, $b_k \le b$, then $U^{(k)} = (y_1^{(k)}, y_2^{(k)}) = (v_1^{(k)}, v_2^{(k)}) = V^{(k)}$ is an increasing sequence of lower pairs of solutions to the problem (2.2)–(2.4).*
2. *If $U^{(0)} = W = (w_1^{(0)}, w_2^{(0)})$ and a_k, b_k are decreasing sequences with $a_k \ge a$, $b_k \ge b$, then $U^{(k)} = (y_1^{(k)}, y_2^{(k)}) = (w_1^{(k)}, w_2^{(k)}) = W^{(k)}$ is a decreasing sequence of upper pairs of solutions to the problem (2.2)–(2.4). Moreover,*
3. *$(v_1^{(k)}, v_2^{(k)}) \le (w_1^{(k)}, w_2^{(k)})\ \ \forall k \ge 0$.*

Proof.

1. First, we use mathematical induction to show that $U^{(k)} = (v_1^{(k)}, v_2^{(k)})$ is an increasing sequence. For $k = 1$, we have

$$Dv_1^{(1)}(t) = v_2^{(0)}(t),\quad 0 < t < 1 \tag{4.4}$$

$$D_{0^+}^{\alpha} v_2^{(1)}(t) + cv_2^{(1)} = cv_2^{(0)} - f(t, v_1^{(0)}, v_2^{(0)}),\quad 0 < t < 1,\ 0 < \alpha < 1, \tag{4.5}$$

$$\text{with}\quad v_1^{(1)}(0) = a_1,\ v_2^{(1)}(1) = b_1. \tag{4.6}$$

Since $V = (v_1^{(0)}, v_2^{(0)})$ is a pair of lower solution, we have

$$Dv_1^{(0)} - v_2^{(0)} \le 0,\ 0 < t < 1 \tag{4.7}$$

$$D_{0^+}^{\alpha} v_2^{(0)} + f(t, v_1^{(0)}, v_2^{(0)}) \le 0,\ 0 < t < 1,\ 0 < \alpha < 1, \tag{4.8}$$

$$\text{and}\quad v_1^{(0)}(0) = a_0 \le a, v_2^{(0)}(1) = b_0 \le b. \tag{4.9}$$

Let $z_1 = v_1^{(1)} - v_1^{(0)}$ and by substituting Eq. (4.4) in Eq. (4.7), we have

$$0 \ge Dv_1^{(0)} - Dv_1^{(1)} = -D(v_1^{(1)} - v_1^{(0)}) = -Dz_1.$$

Thus $Dz_1 \ge 0$, with $z_1(0) = a_1 - a_0 \ge 0$. Since $Dz_1 \ge 0$, this means z_1 is non-decreasing which together with $z_1(0) \ge 0$, implies that $z_1 \ge 0$, and hence $v_1^{(1)} \ge v_1^{(0)}$. To prove that $v_2^{(1)} \ge v_2^{(0)}$, let $z_2 = v_2^{(1)} - v_2^{(0)}$ and by substituting Eq. (4.5) in Eq. (4.8), we have

$$0 \ge D_{0^+}^{\alpha} v_2^{(0)} - D_{0^+}^{\alpha} v_2^{(1)} - cv_2^{(1)} + cv_2^{(0)}$$
$$= -D_{0^+}^{\alpha}(v_2^{(1)} - v_2^{(0)}) - c(v_2^{(1)} - v_2^{(0)}) = -D_{0^+}^{\alpha} z_2 - cz_2.$$

Therefore $D_{0^+}^{\alpha} z_2 + cz_2 \ge 0$. By applying the positivity lemma, we have $z_2 \ge 0$, and hence $v_2^{(1)} \ge v_2^{(0)}$. Now, assume that $U^{(k-1)} \le U^{(k)}$, for $k = 0, 1, 2, \dots, n$. From Eq's. (4.4) and (4.5), we have

$$D(v_1^{(n+1)} - v_1^{(n)}) = v_2^{(n)} - v_2^{(n-1)},$$
$$D_{0^+}^{\alpha}(v_2^{(n+1)} - v_2^{(n)}) + c(v_2^{(n+1)} - v_2^{(n)}) = c(v_2^{(n)} - v_2^{(n-1)})$$
$$+ f(t, v_1^{(n-1)}, v_2^{(n-1)}) - f(t, v_1^{(n)}, v_2^{(n)}).$$

Let $z_1 = v_1^{(n+1)} - v_1^{(n)}$ and using the induction hypothesis, we have

$$Dz_1 = v_2^{(n)} - v_2^{(n-1)} \ge 0,$$

which together with $z_1(0) \ge 0$, proves that $v_1^{(n+1)} \ge v_1^{(n)}$ by the positivity lemma.Let $z_2 = v_2^{(n+1)} - v_2^{(n)}$ and applying the induction hypothesis, the conditions (A1) and (A2) and the mean value theorem, we have

$$D_{0^+}^{\alpha} z_2 + cz_2 = c(v_2^{(n)} - v_2^{(n-1)}) + (v_1^{(n-1)} - v_1^{(n)})\frac{\partial f}{\partial y_1}(\rho_1)$$
$$+ (v_2^{(n-1)} - v_2^{(n)})\frac{\partial f}{\partial y_2}(\rho_2) = (v_2^{(n-1)} - v_2^{(n)})\left(\frac{\partial f}{\partial y_2}(\rho_2) - c\right)$$
$$+ (v_1^{(n-1)} - v_1^{(n)})\frac{\partial f}{\partial y_1}(\rho_1) \ge 0,$$

where $\rho_1 = \mu v_1^{(n-1)} + (1-\mu)v_1^{(n)}, \rho_2 = \nu v_2^{(n-1)} + (1-\nu)v_2^{(n)}$ with $0 \le \mu, \nu \le 1$.

Again, by the positivity lemma, we have $z_2 \ge 0$ and hence $v_2^{(n+1)} \ge v_2^{(n)}$. Hence, $U^{(k)} \le U^{(k+1)}$.

Second, we prove that $(v_1^{(k)}, v_2^{(k)})$, for all $k \ge 0$ is a pair of lower solutions. Since the sequence $\{v_2^{(k)}\}$ is increasing and $Dv_1^{(k)} = v_2^{(k-1)}$, we have $Dv_1^{(k)} - v_2^{(k)} = v_2^{(k-1)} - v_2^{(k)} \le 0$, which together with $v_1^{(k)}(0) = a_k \le a$, proves that $v_1^{(k)}$ is a lower solution. From Eq. (4.2), we have

$$D_{0^+}^{\alpha} v_2^{(k)} + cv_2^{(k)} = cv_2^{(k-1)} - f(t, v_1^{(k-1)}, v_2^{(k-1)})$$
$$= -c(v_2^{(k)} - v_2^{(k-1)}) - f(t, v_1^{(k-1)}, v_2^{(k-1)}).$$

By adding $f(t, v_1^{(k)}, v_2^{(k)})$, applying the mean value theorem and using the fact that the sequences $\{v_1^{(k)}\}$ and $\{v_2^{(k)}\}$ are increasing, we have

$$D_{0^+}^{\alpha} v_2^{(k)} + f(t, v_1^{(k)}, v_2^{(k)}) = -c(v_2^{(k)} - v_2^{(k-1)}) + f(t, v_1^{(k)}, v_2^{(k)})$$
$$- f(t, v_1^{(k-1)}, v_2^{(k-1)}) = -c(v_2^{(k)} - v_2^{(k-1)})$$
$$+ \frac{\partial f}{\partial y_1}(\rho_1)(v_1^{(k)} - v_1^{(k-1)}) + \frac{\partial f}{\partial y_2}(\rho_2)(v_2^{(k)} - v_2^{(k-1)})$$
$$= \left(-c + \frac{\partial f}{\partial y_2}(\rho_2)\right)(v_2^{(k)} - v_2^{(k-1)})$$
$$+ \frac{\partial f}{\partial y_1}(\rho_1)(v_1^{(k)} - v_1^{(k-1)}),$$

where $\rho_1 = \zeta_1 v_1^{(k)} + (1-\zeta_1)v_1^{(k-1)}$, $\rho_2 = \zeta_2 v_2^{(k)} + (1-\zeta_2)v_2^{(k-1)}$, and $0 \le \zeta_1, \zeta_2 \le 1$. Applying the conditions (A1) and (A2), we have $D_{0^+}^{\alpha} v_2^{(k)} + f(t, v_1^{(k)}, v_2^{(k)}) \le 0$, which together with $v_2^{(k)}(1) = b_k \le b$, proves that $v_2^{(k)}$ is a lower solution.

2. Similar to the proof of (1). First, we apply induction arguments to prove that the two sequences $\{w_1^{(k)}\}$ and $\{w_2^{(k)}\}$ are decreasing. Then, we use these results to show that $(w_1^{(k)}, w_2^{(k)})$ is a pair of upper solutions for each $k \ge 0$.
3. Since $V = (v_1^{(0)}, v_2^{(0)})$ and $W = (w_1^{(0)}, w_2^{(0)})$ are ordered pairs of lower and upper solutions, we have $v_1^{(0)} \le w_1^{(0)}$ and $v_2^{(0)} \le w_2^{(0)}$. Hence the result is true for $n = 0$. Assume that $v_1^{(k)} \le w_1^{(k)}$ and $v_2^{(k)} \le w_2^{(k)}$, for all $k = 0, 1, 2, \dots, n$. we have $Dv_1^{(n+1)} = v_2^{(n)}$ and $Dw_1^{(n+1)} = w_2^{(n)}$. Thus

$$Dw_1^{(n+1)} - Dv_1^{(n+1)} = w_2^{(n)} - v_2^{(n)} \ge 0.$$

Let $z_1 = w_1^{(n+1)} - v_1^{(n+1)}$, thus $Dz_1 \geq 0$, which together with $w_1^{(n+1)}(0) \geq v_1^{(n+1)}(0)$, implies $z_1 \geq 0$, and hence $w_1^{(n+1)} \geq v_1^{(n+1)}$. We have

$$D_{0^+}^{\alpha}(w_2^{(n+1)} - v_2^{(n+1)}) + c(w_2^{(n+1)} - v_2^{(n+1)})$$
$$= c(w_2^{(n)} - v_2^{(n)}) + f(t, v_1^{(n)}, v_2^{(n)}) - f(t, w_1^{(n)}, w_2^{(n)}).$$

Let $z_2 = w_2^{(n+1)} - v_2^{(n+1)}$. Then z_2 satisfies

$$D_{0^+}^{\alpha} z_2 + cz_2 = c(w_2^{(n)} - v_2^{(n)}) + f(t, v_1^{(n)}, v_2^{(n)}) - f(t, w_1^{(n)}, w_2^{(n)}).$$

Applying the mean value theorem to the last equation yields

$$\begin{aligned} D_{0^+}^{\alpha} z_2 + cz_2 &= c(w_2^{(n)} - v_2^{(n)}) + \frac{\partial f}{\partial y_1}(\rho_1)(v_1^{(n)} - w_1^{(n)}) + \frac{\partial f}{\partial y_2}(\rho_2)(v_2^{(n)} - w_2^{(n)}) \\ &= (v_2^{(n)} - w_2^{(n)})\left(\frac{\partial f}{\partial y_2}(\rho_2) - c\right) + \frac{\partial f}{\partial y_1}(\rho_1)(v_1^{(n)} - w_1^{(n)}), \end{aligned} \tag{4.10}$$

for some $\rho_1 = \zeta_1 v_1^{(n)} + (1-\zeta_1)w_1^{(n)}$, $\rho_2 = \zeta_2 v_2^{(n)} - (1-\zeta_2)w_2^{(n)}$ and $0 \leq \zeta_1, \zeta_2 \leq 1$. By the induction hypothesis $w_1^{(n)} \geq v_1^{(n)}$ and $w_2^{(n)} \geq v_2^{(n)}$ and the conditions (A1) and (A2), we have $D_{0^+}^{\alpha} z_2 + cz_2 \geq 0$, which proves that $z_2 \geq 0$. Therefore, $w_2^{(n+1)} \geq v_2^{(n+1)}$, and the proof is completed. □

Remark 4.1. The existence and uniqueness of solutions to the sequences defined in (4.1)–(4.3) is guaranteed by Theorem 3.1 provided that $c < \frac{1}{2}\Gamma(1+\alpha)$.

Now, we state the convergence result of the two sequences of ordered pairs of lower and upper solutions described in Theorem 4.1.

Theorem 4.2. *Assume that the conditions (A1) and (A2) are satisfied, and consider the two iterative sequences* $V^{(k)} = (v_1^{(k)}, v_2^{(k)})$ *and* $W^{(k)} = (w_1^{(k)}, w_2^{(k)})$, *obtained from* (4.1)–(4.3), *with* $U^{(0)} = V = (v_1^{(0)}, v_2^{(0)})$ *and* $U^{(0)} = W = (w_1^{(0)}, w_2^{(0)})$, *respectively. Then the two sequences converge pointwise to* $V^* = (v_1^*, v_2^*)$ *and* $W^* = (w_1^*, w_2^*)$, *respectively with* $V^* \leq W^*$.

Proof. The two sequences $v_1^{(k)}$ and $v_2^{(k)}$ are increasing and bounded above by $w_1^{(0)}$ and $w_2^{(0)}$, respectively. Hence, they converge pointwise to v_1^* and v_2^*, respectively. By applying similar arguments, the two sequences $w_1^{(k)}$ and $w_2^{(k)}$ are decreasing and bounded below by $v_1^{(0)}$ and $v_2^{(0)}$, respectively. Hence, they converge pointwise to w_1^* and w_2^*, respectively.

Since $v_1^{(k)} \leq w_1^{(k)}$ and $v_2^{(k)} \leq w_2^{(k)}$, $\forall k \geq 0$, then $v_1^* \leq w_1^*$ and $v_2^* \leq w_2^*$. □

4.2. Existence and uniqueness of solutions

Applying standard arguments, one can verify the following result.

Lemma 4.1. *A pair of functions* $(y_1(t), y_2(t)) \in C^1[0,1] \times C^1[0,1]$ *is a solution to the problem*(2.2)–(2.4), *if and only if, it is a solution to the system of integral equations*

$$y_1(t) = a + \int_0^t y_2(s)\,ds,\ 0 < t < 1, \tag{4.11}$$

$$y_2(t) = \eta - \frac{1}{\Gamma(\alpha)}\int_0^t (1-s)^{\alpha-1} f(s, y_1(s), y_2(s))\,ds,$$
$$0 < t < 1,\ 0 < \alpha < 1, \tag{4.12}$$

where $\eta = b + \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f(s, y_1(s), y_2(s))\,ds$.

The following theorem proves the existence of solutions of the problem 4.11,4.12.

Theorem 4.3 (*Existence Result*). *Let* $V^* = (v_1^*, v_2^*)$ *and* $W^* = (w_1^*, w_2^*)$ *be the limits of the two sequences* $V^{(k)} = (v_1^{(k)}, v_2^{(k)})$ *and* $W^{(k)} = (w_1^{(k)}, w_2^{(k)})$ *defined by* (4.1)–(4.3) *with* $V^{(0)} = (v_1^{(0)}, v_2^{(0)})$ *and* $W^{(0)} = (w_1^{(0)}, w_2^{(0)})$, *respectively. Assume that* $\lim_{k\to\infty} a_k = a$ *and* $\lim_{k\to\infty} b_k = b$. *Then* V^* *and* W^* *are solutions to* 4.11,4.12.

Proof. We have that

$$Dv_1^{(k)} = v_2^{(k-1)}, \tag{4.13}$$

and $D_{0^+}^{\alpha} v_2^{(k)} + cv_2^{(k)} = cv_2^{(k-1)} - f(t, v_1^{(k-1)}, v_2^{(k-1)})$ (4.14)

Applying the integral operator I_{0^+} for Eq. (4.13), we have

$v_1^{(k)} - v_1^{(k)}(0) = I_{0^+}(v_2^{(k-1)})$, where $v_1^{(k)}(0) = a_k$.

Taking the limit and using the fact that $v_1^{(k)}$ converges pointwise to v_1^*, we have

$$v_1^* = a + \lim_{k\to\infty} I_{0^+}(v_2^{(k-1)}).$$

Since $v_2^{(k)}$ converges pointwise to v_2^*, bounded and Riemann integrable, then by the dominated convergence theorem, we have

$$v_1^* = a + I_{0^+}(v_2^*) = a + \int_0^t v_2^*\,ds,\ 0 < t < 1, \tag{4.15}$$

which proves that v_2^* is a solution to Eq. (4.11).

Similarly, applying the fractional integral operator $I_{0^+}^{\alpha}$ for the Eq. (4.14), we have

$$v_2^{(k)} - v_2^{(k)}(0) + cI_{0^+}^{\alpha}(v_2^{(k)}) = cI_{0^+}^{\alpha}(v_2^{(k-1)}) - I_{0^+}^{\alpha}(f(t, v_1^{(k-1)}, v_2^{(k-1)})).$$

Taking the limit and using the facts that $v_1^{(k)}$ and $v_2^{(k)}$ converge pointwise to v_1^* and v_2^*, respectively, they are bounded and Riemann integrable, and f is continuous, we have

$$v_2^* - v_2^*(0) + cI_{0^+}^{\alpha}(v_2^*) = cI_{0^+}^{\alpha}(v_2^*) - I_{0^+}^{\alpha}(f(t, v_1^*, v_2^*)),$$

which yields

$$v_2^*(t) = v_2^*(0) - I_{0^+}^{\alpha}(f(t, v_1^*, v_2^*)). \tag{4.16}$$

Now, at $t = 1$ we have $v_2^*(1) = v_2^*(0) - (I_{0^+}^{\alpha} f(t, v_1^*, v_2^*))(t = 1)$, and thus

$$v_2^*(0) = b + \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f(s, v_1^*(s), v_2^*(s))\,ds. \tag{4.17}$$

Substitute Eq. (4.17) in Eq. (4.16) to obtain the result. By similar arguments, one can show that (w_1^*, w_2^*) is also a solution to the problem 4.11,4.12. □

Remark 4.2. Since in general, we do not guarantee that $V^*,\ W^* \in C^1[0,1]\times C^1[0,1]$, we refer to V^* and W^* by the minimal and maximal solutions, respectively.

Theorem 4.4 (*Existence and Uniqueness Result*). *Let* $V^* = (v_1^*, v_2^*) \in C^1[0,1]\times C^1[0,1]$ *and* $W^* = (w_1^*, w_2^*) \in C^1[0,1]\times C^1[0,1]$ *be as in Theorem 4.3 and assume that they satisfy the conditions in Theorem 2.1 with* $\frac{\partial f}{\partial y_2}(t,y_1,y_2)\le q$,*for some* $q<0$.*Then* $v_1^*=w_1^*$ *and* $v_2^*=w_2^*$ *and the problem (2.2)–(2.4) has a unique solution on* $[V,W]$.

Proof. Since $V^*,\ W^* \in C^1[0,1]\times C^1[0,1]$ and satisfy $v_1^*\le w_1^*$ and $v_2^*\le w_2^*$, then (v_1^*,v_2^*) and (w_1^*,w_2^*) are comparable solution for the problem (2.2)–(2.4). Since $\frac{\partial f}{\partial y_2}(t,y_1,y_2)\le q<0$, by Theorem 2.1, we have $v_1^*=w_1^*$ and $v_2^*=w_2^*$. □

5. Illustrated examples

In this section, we present two examples to illustrate the validity of our results.

Example 5.1. Consider the linear fractional boundary value problem

$$D_{0^+}^{\frac{5}{3}}\ y(t) = \frac{1}{4}\ \sqrt[3]{t}\ y(t) - \frac{1}{4}\ y'(t),\ \ 0<t<1, \tag{5.1}$$

$$\text{with}\quad y(0)=1,\ \ y'(1)=0. \tag{5.2}$$

We first transform the problem to the following system

$$Dy_1(t)-y_2(t)=0,\ \ 0<t<1, \tag{5.3}$$

$$D_{0^+}^{\frac{2}{3}}\ y_2(t) - \frac{1}{4}\ \sqrt[3]{t}\ y_1(t) + \frac{1}{4}\ y_2(t)=0,\ \ 0<t<1, \tag{5.4}$$

$$\text{with}\quad y_1(0)=1,\ y_2(1)=0, \tag{5.5}$$

where $y_1(t)=y(t)$ and $y_2(t)=y'(t)$. It is clear that $V^{(0)}=(v_1^{(0)},v_2^{(0)})=(1,0)$ satisfies the definition of pair of lower solutions given in Eq's. (2.5)–(2.7). We now show that $W^{(0)}=(w_1^{(0)},w_2^{(0)})=(t+1,t)$ is a pair of upper solutions. We have

$$D(t+1)-t=1-t\ge 0,\ \ 0<t<1,$$

$$\text{and}\quad D_{0^+}^{\frac{2}{3}}\ t-\frac{1}{4}\ t^{\frac{4}{3}}-\frac{1}{4}\ t^{\frac{1}{3}}+\frac{1}{4}\ t=\frac{1}{\Gamma(\frac{4}{3})}\ t^{\frac{1}{3}}-\frac{1}{4}\ t^{\frac{4}{3}}-\frac{1}{4}\ t^{\frac{1}{3}}+\frac{1}{4}\ t$$

$$=t^{\frac{1}{3}}\left(\frac{1}{\Gamma(\frac{4}{3})}-\frac{1}{4}\ t-\frac{1}{4}+\frac{1}{4}\ t^{\frac{2}{3}}\right)\ge 0,\ \text{for}\ 0<t<1,$$

which together with $w_1^{(0)}(0)=1$ and $w_2^{(0)}(1)=1$, prove that $W^{(0)}=(t+1,t)$ is a pair of upper solutions of the system (5.3)–(5.5). In the last equation, we use the fact that $\frac{1}{\Gamma(\frac{4}{3})}>1$ and $-\frac{1}{4}\ t-\frac{1}{4}\ge -\frac{1}{2}$, for $0\le t\le 1$. Since $v_1^{(0)}=1\le 1+t=w_1^{(0)}$ and $v_2^{(0)}=0\le t=w_2^{(0)}$, $\forall t\in[0,1]$, we have $V^{(0)}$ and $W^{(0)}$ are ordered pairs of lower and upper solutions.

Now, from Eq. (5.4), we have $f(t,y_1,y_2)=-\frac{1}{4}\sqrt[3]{t}\ y_1(t)+\frac{1}{4}\ y_2(t)$ satisfying $\frac{\partial f}{\partial y_1}(t,y_1,y_2)=-\frac{1}{4}\ \sqrt[3]{t}\le 0$ and $\frac{\partial f}{\partial y_2}(t,y_1,y_2)=\frac{1}{4}$, hence we can choose $c=\frac{1}{4}<\frac{1}{2}\Gamma(1+\alpha), 0<\alpha<1$ and the result in Theorem 4.3 guarantees the existence of the minimal and maximal solutions to the problem.

Example 5.2. Consider the non-linear fractional boundary value problem

$$D^{\frac{3}{2}}\ y(t)-y^5(t)-\frac{1}{8}y'(t)=0,\ \ 0<t<1, \tag{5.6}$$

$$\text{with}\quad y(0)=0,\ y'(1)=1. \tag{5.7}$$

We transform the problem to the following system

$$Dy_1(t)-y_2(t)=0,\ \ 0<t<1, \tag{5.8}$$

$$D^{\frac{1}{2}}y_2(t)-y_1^5(t)-\frac{1}{8}y_2(t)=0,\ \ 0<t<1, \tag{5.9}$$

$$\text{with}\quad y_1(0)=0,\ y_2(1)=1, \tag{5.10}$$

where $y_1(t)=y(t)$ and $y_2(t)=y'(t)$. It is clear that $V^{(0)}=(v_1^{(0)},v_2^{(0)})=(0,0)$ is a pair of lower solutions. We now show that $W^{(0)}=(w_1^{(0)},w_2^{(0)})=(t^2,t)$ is a pair of upper solutions. We have

$$Dt^2-t=2t-t=t\ge 0,\ \ 0<t<1,$$

$$\text{and}\quad D_{0^+}^{\frac{1}{2}}t-t^{10}-\frac{1}{8}t=\frac{2}{\sqrt{\pi}}t^{\frac{1}{2}}-t^{10}$$

$$-\frac{1}{8}t=t^{\frac{1}{2}}\left(\frac{2}{\sqrt{\pi}}-t^{\frac{19}{2}}-\frac{1}{8}t^{\frac{1}{2}}\right)\ge 0,\ \ 0<t<1,$$

which together with $w_1^{(0)}(0)=0$ and $w_2^{(0)}(1)=1$, prove that $W^{(0)}=(t^2,t)$ is a pair of upper solutions of the system (5.8)–(5.10). In the last equation we use the fact that $\frac{2}{\sqrt{\pi}}>\frac{9}{8}$. Since $v_1^{(0)}=0\le w_1^{(0)}=t^2$ and $v_2^{(0)}=0\le w_2^{(0)}=t$, $\forall t\in[0,1]$, we have $V^{(0)}$ and $W^{(0)}$ are ordered pairs of lower and upper solutions. Now, from Eq. (5.9), we have $f(t,y_1,y_2)=-y_1^5(t)-\frac{1}{8}y_2(t)$ satisfying $\frac{\partial f}{\partial y_1}(t,y_1,y_2)=-5y_1^4\le 0$ and $\frac{\partial f}{\partial y_2}(t,y_1,y_2)=-\frac{1}{8}$, hence we can choose $c=\frac{1}{4}$ and the result in Theorem 4.4 guarantees the existence of unique solution to the problem in $[W^{(0)},V^{(0)}]$.

6. Concluding remarks

In this paper, a class of boundary value problems of fractional order $1<\delta<2$ has been discussed, where the fractional derivative is of Caputo's type. We proved that, under certain condition on the non-linear term in the equation, the problem has no comparable solutions. To establish existence and uniqueness results using the method of lower and upper solutions, we transform the problem to an equivalent system of differential equations including the fractional and integer derivatives. We generated a decreasing sequence of upper solutions that converges to a maximal solution of the system, as well as, an increasing sequence of lower solutions that converges to a minimal solution of the system. A new positivity result has been implemented to prove the monotonicity and convergence of the two sequences. Under the condition $\frac{\partial f}{\partial y_2}(t,y_1,y_2)\le q<0$, we guarantee that the maximal and minimal solutions coincide, and hence a uniqueness result is established. We have applied the Banach fixed point theorem to show that these sequences are well-defined and have unique solutions. The presented examples illustrate the validity of our results. Because of the non-sufficient information about the fractional derivative $1 < \delta < 2$ of a function at its extreme points, the current results cannot be obtained without transforming the original problem to a system of fractional derivatives of less order.

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