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Optimal fourth- and eighth-order of convergence derivative-free modifications of King's method

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ABSTRACT

Starting by King's method, we propose a modified families of fourth- and eighth-order of convergence iterative methods for nonlinear equations. The fourth-order method requires at each iteration three function evaluations, while the eighth-order methods both need four function evaluations. The proposed methods are derivative-free. Based on the conjecture of Kung and Traub, the new methods attain the optimality with efficiency index 1.587 for the fourth-order method and 1.68 for the eighth-order methods. The convergence analyses of the methods are given, and comparisons with some well-known schemes having identical order of convergence demonstrate the efficiency of the present techniques.

1. Introduction

Searching out a solution of the equation $f(x) = 0$, where $f(x)$ is nonlinear is highly significant in mathematics. The second-order well-known Newton's iterative technique for solving nonlinear equations defined as Traub (1964)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \quad (1)$$

Many researchers have improved Newton's method in order to get more accurate results and higher order of convergence, see for example Behl et al. (2017), Chun (2008), Cordero et al. (2016), Pandey and Jaiswal (2017), Said Solaiman and Hashim (2019), Sharma and Goyal (2007), Waseem et al. (2018) and the references therein.

Recently, Kogan et al. (2017) proved that methods of order $p = 3$ are the most efficient methods among all one-point iterative methods without memory of order p. Besides, the efficiency index is a common method to compare the performance of different iterative methods. This index is defined as $p^{1/m}$, where p represents the convergence order and m is the number of functional evaluations needed at each iteration. Based on the conjecture of Kung and Traub (1974), the iterative scheme with m functional evaluations is optimal if its order of convergence equals 2^{m-1}. Many authors have constructed the optimal iterative methods of different convergence orders. The standard way for constructing optimal method is the composition technique together with the usage of some interpolations and approximations to minimize the needed functional evaluations at each iteration. Different optimal fourth-order iterative methods were constructed, see for example Argyros and Magreñán (2015), Behl et al. (2015a), Chun et al. (2012), Cordero et al. (2010), Sharma and Bahl (2015), Soleymani et al. (2012). Optimal eighth-order of convergence was also reached by many authors also, for instance Behl et al. (2016), Behl et al. (2018), Cordero et al. (2015), Geum et al. (2018),

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Sharma and Arora (2016), Singh and Jaiswal (2016). Neta and Chun (2014) and Chun and Neta (2016) presented a comparison of several families of optimal iterative methods which are of fourth- and eighth-order of convergence based on their basins of attraction. Chun and Neta (2017) presented a quantitative comparison of many optimal iterative techniques of order eight beside the visual comparison of the methods by graphing their dynamics.

Since, to find the derivative value is not always an easy task, as well as requires more computation time, many authors have proposed and implemented various derivative-free optimal methods, see Cordero et al. (2013), Lee and Kim (2012), Sharma and Goyal (2007), Yasmin et al. (2016), Zafar et al. (2015). One of the most famous optimal fourth-order iterative techniques is the method proposed by King (1973). But the main weakness is that finding the first derivative is needed at each iteration. Many authors have modified King's method. For example, Chun (2007) implemented some King's like methods of order four, but computing the first derivative within the iteration is needed also. Behl et al. (2015b) proposed a fourth-order derivative-free modification of King's method. Sharifi et al. (2014) implemented an optimal derivative-free fourth- and eighth-order modifications of King's method.

In this work, by modifying King's method, we propose a family of optimal fourth-order derivative-free iterative method for nonlinear equations. With the use of some approximations and the composition technique, we extend the new method to two new optimal schemes of order eight. The convergence analysis of all three methods are derived. The proposed optimal methods were tested on six different examples to show the efficiency of the methods with numerical comparison to other established methods of the same order.

The work of this paper is divided as follows. The new schemes are described below in Section 2. To show the order of convergence of the new schemes, thee convergence analysis is implemented in Section 3. The numerical examples with the comparisons with other techniques of identical orders are summarized in Section 4. Eventually, in Section 5 the conclusion is given.

2. The new methods

2.1. Derivative-free optimal fourth-order iterative method

We propose a modified family of optimal fourth-order derivative-free schemes. We start by Newton's two steps method of order four:

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)}. \end{cases} \tag{2}$$

This two steps method is not optimal as it needs the evaluation of two functions and two first derivatives. In order to reduce the number of functional evaluations, King (1973) replaced $f'(y_n)$ by the following approximation:

$$f'(y_n) = f'(x_n)\frac{f(x_n) + (\beta - 2)f(y_n)}{f(x_n) + \beta f(y_n)}. \tag{3}$$

Substituting (3) into (2) produces the famous optimal King's method of order four:

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \frac{f(y_n)}{f'(x_n)} \frac{f(x_n)+\beta f(y_n)}{f(x_n)+(\beta-2)f(y_n)}. \end{cases} \tag{4}$$

King's method needs the computation of two functions and one first derivative. Based on the conjecture of Kung and Traub (1974), King's method reached the optimality with efficiency index equals $(4)^{\frac{1}{3}} \approx 1.587$.

However, the computation of the first derivative is not always easy, in addition to that it costs extra time. To implement a derivative-free technique from King's method (4), the next approximations for $f'(x_n)$ in both steps will be considered

$$f'(x_n) \approx f[w_n, x_n], \tag{5}$$

$$f'(x_n) \approx g(x_n) = f[w_n, x_n] + 2(w_n - x_n)f[w_n, x_n, y_n] - f[y_n, w_n] + f[x_n, y_n]. \tag{6}$$

Here $w_n = x_n + f(x_n), f[x_n, y_n] = \frac{f(x_n)-f(y_n)}{x_n-y_n}$, and $f[w_n, x_n, y_n] = \frac{f[w_n,x_n]-f[x_n,y_n]}{w_n-y_n}$. The first approximation (5) was firstly used by Steffensen (1933), and the second one (6) is obtained with the help of divided differences approximation. After substituting (5) and (6) into (4) one can obtain the first essential finding of this work:

Algorithm 1. Given x_0, the approximate solution x_{n+1} of $f(x) = 0$ can be found by the following iterative scheme

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[w_n,x_n]}, \\ x_{n+1} = y_n - \frac{f(y_n)}{g(x_n)} \frac{f(x_n)+\beta f(y_n)}{f(x_n)+(\beta-2)f(y_n)}. \end{cases} \tag{7}$$

We call this family, modified King's method MK_4. The order of convergence of this family is four. In each iteration, MK_4 requires only three evaluations of functions and no derivative evaluation is needed. Based on the conjecture of Kung and Traub (1974), MK_4 reached the optimality with efficiency index equals $(4)^{\frac{1}{3}} \approx 1.587$.

2.2. Derivative-free optimal eighth-order iterative method

In order to extend MK_4 method given by Algorithm 1 to the eighth-order of convergence, we will use the composition technique. The additional step of Algorithm 1 is produced using the idea of Zafar et al. (2015) based on rational interpolation. Consider the following Algorithm, which is the second main finding of this work:

Algorithm 2. Given x_0, the approximate solution x_{n+1} of $f(x) = 0$ can be found by the following iterative scheme

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[w_n,x_n]}, \\ z_n = y_n - \frac{f(y_n)}{g(x_n)} \frac{f(x_n)+\beta f(y_n)}{f(x_n)+(\beta-2)f(y_n)}, \\ x_{n+1} = x_n - \frac{f(x_n)(m_1+m_2+m_3)}{m_1 f[w_n,x_n]+m_2 f[y_n,x_n]+m_3 f[z_n,x_n]}, \end{cases} \tag{8}$$

where

$$\begin{aligned} m_1 &= f(y_n)f(z_n)(z_n - y_n), \\ m_2 &= f(w_n)f(z_n)(w_n - z_n), \\ m_3 &= f(w_n)f(y_n)(y_n - w_n). \end{aligned}$$

We call this family, modified King's method MK_{8a}. The convergence order of this family is eight. In each iteration MK_{8a} requires only four evaluations of functions and no derivative evaluation is needed. Based on the conjecture of Kung and Traub (1974), MK_{8a} reached the optimality with efficiency index equals $(8)^{\frac{1}{4}} \approx 1.68$.

2.3. Another derivative-free optimal eighth-order iterative method

Another extension of MK_4 method can be achieved by adding Newton's technique as a third step of MK_4 method, and then using the derivative of the second degree Padé approximation for $f'(z_n)$. This approximation was firstly proposed by Cordero et al. (2013). The following algorithm is the third essential finding of this work:

Algorithm 3. Given x_0, the approximate solution x_{n+1} of $f(x)=0$ can be found by the following iterative scheme

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[w_n,x_n]}, \\ z_n = y_n - \frac{f(y_n)}{g(x_n)}\frac{f(x_n)+\beta f(y_n)}{f(x_n)+(\beta-2)f(y_n)}, \\ x_{n+1} = z_n - \frac{f(z_n)}{c_2-c_1c_4}, \end{cases} \tag{9}$$

where

$$\begin{aligned} c_1 =& \; f(z_n),\\ c_2 =& \; f[y_n,z_n]-c_3(y_n-z_n)+c_4 f(y_n),\\ c_3 =& \; f[y_n,z_n,w_n]+c_4 f[y_n,w_n],\\ c_4 =& \; \frac{f[y_n,z_n,x_n]-f[y_n,z_n,w_n]}{f[y_n,w_n]-f[y_n,x_n]}. \end{aligned}$$

We call this family, modified King's method MK_{8b}. The order of convergence of this family is eight. Each iteration of MK_{8b} requires only four evaluations of functions and there is no derivative evaluation needed. Based on the conjecture of Kung and Traub (1974), MK_{8b} reached the optimality with efficiency index equals $(8)^{\frac{1}{4}} \approx 1.68$.

3. Convergence analysis

The convergence analysis of the proposed methods will be discussed in the following theorems.

Theorem 4. *Consider that α is a root of a sufficiently differentiable function $f: I \subseteq \mathbb{R} \to \mathbb{R}$ in an open interval I, and let x_0 be sufficiently close to α. Also let $e_n = x_n - \alpha$ be the error at the n^{th} iteration. The method defined in Algorithm 1 is of fourth-order of convergence.*

Proof. By the Taylor series expansion of $f(x)$ about $x=\alpha$ one obtains

$$f(x_n) = [c_1e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + \cdots], \tag{10}$$

where $c_k = \frac{f^{(k)}(\alpha)}{k!}$, $k = 1,2,3,\ldots$. Furthermore,

$$\begin{aligned} w_n =& \; x_n + f(x_n),\\ =& \; \alpha + e_n + c_1e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + \cdots. \end{aligned} \tag{11}$$

Expanding $f(w_n)$ about α, one obtains

$$f(w_n) = (c_1 + c_1^2)e_n + (c_2 + 3c_1c_2 + c_1^2c_2)e_n^2 + \cdots. \tag{12}$$

Hence, from Eqs. (10)–(12) we obtain

$$\begin{aligned} f[w_n,x_n] =& \; \frac{f(w_n)-f(x_n)}{w_n-x_n} = \frac{f(w_n)-f(x_n)}{f(x_n)},\\ =& \; c_1 + (2c_2 + c_1c_2)e_n + (c_2^2 + 3c_3 + 3c_1c_3 + c_1^2c_3)e_n^2 + \cdots. \end{aligned} \tag{13}$$

Now, substituting Eqs. (10) and (13) into the first step of scheme (7), we have

$$\begin{aligned} y_n =& \; x_n - \frac{f(x_n)}{f[w_n,x_n]},\\ =& \; \alpha + \frac{(c_1c_2+c_2)e_n^2}{c_2} - \frac{(2c_2^2+2c_1c_2^2+c_1^2c_2^2-2c_1c_3-3c_1^2c_3-c_1^3c_3)e_n^3}{c_1^2} + \cdots. \end{aligned} \tag{14}$$

Expanding $f(y_n)$ about α, with the use of Eq. (14) one obtains

$$f(y_n) = (c_2 + c_1c_2)e_n^2 + \frac{(-2c_2^2 - 2c_1c_2^2 - c_1^2c_2^2 + 2c_1c_3 + 3c_1^2c_3)e_n^3}{c_1^3} + \cdots. \tag{15}$$

Also, we have

$$\begin{aligned} g(x_n) =& \; f[w_n,x_n] + 2(w_n - x_n)f[w_n,x_n,y_n] - f[y_n,w_n] + f[x_n,y_n],\\ =& \; c_1 + 2(c_2 + c_1c_2)e_n + (2c_2^2 + 3c_3 + 5c_1c_3 + 2c_1^2c_3)e_n^2 + \cdots. \end{aligned} \tag{16}$$

After substituting Eqs. (10), (15) and (16) into the second step of scheme (7), we obtain

$$x_{n+1} = \alpha + \frac{(1+c_1)^2c_2((1+2\beta+2(-1+\beta)c_1)c_2^2 - c_1c_3)e_n^4}{c_1^3} + \cdots. \tag{17}$$

This implies that

$$e_{n+1} = \frac{(1+c_1)^2c_2((1+2\beta+2(-1+\beta)c_1)c_2^2 - c_1c_3)e_n^4}{c_1^3} + O(e_n^5).$$

Hence, MK_4 method proposed in Algorithm 1 has fourth-order of convergence. □

Theorem 5. *Consider the same assumptions in Theorem 4, then the method defined by Algorithm 2 is of eighth-order of convergence.*

Proof. From Theorem 4, we have

$$z_n = \alpha + \frac{(1+c_1)^2c_2((1+2\beta+2(-1+\beta)c_1)c_2^2 - c_1c_3)e_n^4}{c_1^3}. \tag{18}$$

Expanding $f(z_n)$ about α, with the use of Eq. (18), we have

$$\begin{aligned} f(z_n) =& \; [c_2^3 + 2\beta c_2^3 + 6\beta c_1c_2^3 + (6\beta-3)c_1^2c_2^3 + 2(\beta-1)c_1^3c_2^3\\ & - c_1c_2c_3 - 2c_1^2c_2c_3 - c_1^3c_2c_3]e_n^4c_1^{-2} + \cdots. \end{aligned} \tag{19}$$

From Eqs. (11), (12), (14), (15), (18), and (19) we have

$$m_1 = -\frac{(1+c_1)^4c_2^3(c_2^2 + 2\beta c_2^2 - 2c_1c_2^2 + 2\beta c_1c_2^2 - c_1c_3)e_n^8}{c_1^3} + \cdots, \tag{20}$$

$$m_2 = \frac{(1+c_1)^4(c_2^3 + 2\beta c_2^3 - 2c_1c_2^3 + 2\beta c_1c_2^3 - c_1c_2c_3)e_n^6}{c_1} + \cdots, \tag{21}$$

$$m_3 = -(1+c_1)(c_1+c_1^2)(c_2+c_1c_2)e_n^4 + \cdots. \tag{22}$$

Using Eqs. (10), (15), and (19), one obtains

$$f[y_n,x_n] = c_1 + c_2e_n + \frac{(c_2^2 + c_1c_2^2 + c_1c_3)e_n^2}{c_1} + \cdots, \tag{23}$$

$$f[z_n,x_n] = c_1 + c_2e_n + c_3e_n^2 + \cdots. \tag{24}$$

Substituting Eqs. (10), (13), and (20)–(24) in the third step of scheme (8), we get

$$x_{n+1} = \alpha + \frac{(1+c_1)^4c_2^2((1+2\beta+2(-1+\beta)c_1)c_2^2 - c_1c_3)(c_2^3 - 2c_1c_2c_3 + c_1^2c_4)e_n^8}{c_1^7} + \cdots,$$

which implies that

$$e_{n+1} = \frac{(1+c_1)^4c_2^2((1+2\beta+2(-1+\beta)c_1)c_2^2 - c_1c_3)(c_2^3 - 2c_1c_2c_3 + c_1^2c_4)e_n^8}{c_1^7} + O(e_n^9).$$

This shows that MK_{8b} method proposed in Algorithm 2 has eighth-order of convergence. □

Theorem 6. *Consider the same assumptions of Theorem 4, then the method defined by Algorithm 3 is of eighth-order of convergence.*

Table 1
Comparisons between different methods.

Method	n	x_n	$\|x_n - x_{n-1}\|$	$f(x_n)$	COC
		$f_1(x), x_0 = 0$			
MK_4	4	0.73908513321516064	1.63E−52	−1.75E−209	4
K_4	4	0.73908513321516064	5.30E−18	−9.03E−71	4
YZA_4	4	0.73908513321516064	1.71E−32	1.60E−128	4
$CHMT_4$	4	0.73908513321516064	4.82E−52	−1.49E−207	4
$ZYAJ_4$	4	0.73908513321516064	2.69E−56	−1.02E−224	4
MK_{8a}	3	0.73908513321516064	3.12E−55	−4.94E−441	8
MK_{8b}	3	0.73908513321516064	2.75E−58	5.03E−466	8
YZA_8	3	0.73908513321516064	3.15E−28	9.51E−223	8
$CHMT_8$	3	0.73908513321516064	4.13E−58	5.73E−465	8
$ZYAJ_8$	3	0.73908513321516064	5.79E−57	−5.41E−455	8
		$f_2(x), x_0 = 1$			
MK_4	4	1.4044916482153412	1.76E−44	2.69E−176	4
K_4	5	1.4044916482153412	7.84E−18	−2.19E−68	4
YZA_4	4	1.4044916482153412	2.18E−23	1.90E−90	4
$CHMT_4$	4	1.4044916482153412	1.76E−27	−4.63E−107	4
$ZYAJ_4$	4	1.4044916482153412	2.45E−35	−8.16E−139	4
MK_{8a}	3	1.4044916482153412	3.29E−42	1.44E−333	8
MK_{8b}	3	1.4044916482153412	2.01E−45	−2.42E−359	8
YZA_8	3	1.4044916482153412	8.99E−31	4.00E−240	8
$CHMT_8$	3	1.4044916482153412	4.21E−30	−6.26E−235	8
$ZYAJ_8$	3	1.4044916482153412	5.62E−39	−8.38E−307	8
		$f_3(x), x_0 = 1.5$			
MK_4	3	1	9.64E−16	−4.80E−62	4
K_4	4	1	9.53E−41	5.73E−162	4
YZA_4	4	3.5302670187568383	1.53E−37	−2.30E−147	4
$CHMT_4$	4	1	1.44E−49	4.75E−197	4
$ZYAJ_4$	4	1	1.55E−53	3.22E−213	4
MK_{8a}	3	1	4.29E−54	−3.75E−430	8
MK_{8b}	3	1	7.57E−57	−3.14E−452	8
YZA_8	3	1	8.02E−53	−1.75E−419	8
$CHMT_8$	3	1	6.44E−49	2.04E−388	8
$ZYAJ_8$	3	1	1.28E−49	2.39E−394	8
		$f_4(x), x_0 = 1$			
MK_4	3	0.97416230520054071	2.71E−32	8.46E−128	4
K_4	3	0.97416230520054071	7.45E−31	−1.34E−121	4
YZA_4	3	0.97416230520054071	3.46E−26	4.16E−102	4
$CHMT_4$	3	0.97416230520054071	8.90E−28	−9.65E−109	4
$ZYAJ_4$	3	0.97416230520054071	1.78E−28	−1.11E−111	4
MK_{8a}	3	0.97416230520054071	3.81E−118	1.93E−941	8
MK_{8b}	2	0.97416230520054071	3.81E−16	2.58E−126	8
YZA_8	3	0.97416230520054071	5.03E−107	2.18E−851	8
$CHMT_8$	3	0.97416230520054071	3.78E−112	−1.06E−892	8
$ZYAJ_8$	3	0.97416230520054071	2.32E−111	−2.54E−886	8
		$f_5(x), x_0 = 1.5$			
MK_4	3	1.3961536566409308	6.61E−23	−2.18E−90	4
K_4	3	1.3961536566409308	2.01E−18	−2.16E−71	4
YZA_4	4	1.3961536566409308	1.31E−58	8.75E−232	4
$CHMT_4$	3	1.3961536566409308	1.18E−17	−2.25E−68	4
$ZYAJ_4$	3	1.3961536566409308	4.49E−19	−2.48E−74	4
MK_{8a}	3	1.3961536566409308	3.50E−82	−3.52E−654	8
MK_{8b}	3	1.3961536566409308	9.22E−89	−1.65E−707	8
YZA_8	3	1.3961536566409308	2.76E−64	3.19E−509	8
$CHMT_8$	3	1.3961536566409308	1.28E−70	−1.89E−560	8
$ZYAJ_8$	3	1.3961536566409308	3.01E−75	−5.71E−598	8
		$f_6(x), x_0 = 0.6$			
MK_4	4	1	3.53E−36	−3.09E−142	4
K_4	9	1	3.10E−27	−9.19E−106	4
YZA_4	4	1	5.01E−30	3.79E−117	4
$CHMT_4$	4	1	2.36E−34	−1.25E−134	4
$ZYAJ_4$	4	1	1.69E−42	−1.64E−167	4
MK_{8a}	3	1	2.13E−39	−8.52E−310	8
MK_{8b}	3	1	2.90E−36	−1.01E−284	8
YZA_8	3	1	1.62E−24	6.67E−190	8
$CHMT_8$	3	1	2.41E−35	−9.18E−277	8
$ZYAJ_8$	3	1	1.56E−42	−7.19E−335	8

Table 2
Comparisons of number of iterations needed for different methods with $|x_n - x_{n-1}| < 10^{-200}$.

	$f_1(x)$ $x_0 = 0$	$f_2(x)$ $x_0 = 0$	$f_3(x)$ $x_0 = 1.5$	$f_4(x)$ $x_0 = 1$	$f_5(x)$ $x_0 = 1.5$	$f_6(x)$ $x_0 = 0.6$
MK_4	5	6	5	5	5	6
K_4	6	7	6	5	5	11
YZA_4	6	6	6	5	5	6
$CHMT_4$	5	6	6	5	5	6
$ZYAJ_4$	5	6	5	5	5	6
MK_{8a}	4	4	4	4	4	4
MK_{8b}	4	4	4	4	4	4
YZA_8	4	4	4	4	4	5
$CHMT_8$	4	4	4	4	4	4
$ZYAJ_8$	4	4	4	4	4	4

Proof. Based on the definitions of c_1, c_2, c_3, and c_4 given in the scheme (9), and as the series expansion is too large and can't be expressed in a few lines, we used Mathematica 9 to do the required computations. After some simplification we obtain

$$x_{n+1} = \alpha + \frac{(1+c_1)^4 c_2((1+2\beta+2(-1+\beta)c_1)c_2^2 - c_1c_3)((1+2\beta+2(-1+\beta)c_1)c_2^4 - c_1c_2^2c_3 - c_1^2c_3^2 + c_1^2c_2c_4)e_n^8}{c_1^7} + \cdots.$$

This leads to

$$e_{n+1} = \frac{(1+c_1)^4 c_2((1+2\beta+2(-1+\beta)c_1)c_2^2 - c_1c_3)((1+2\beta+2(-1+\beta)c_1)c_2^4 - c_1c_2^2c_3 - c_1^2c_3^2 + c_1^2c_2c_4)e_n^8}{c_1^7} + O(e_n^9),$$

which proves that MK_{8b} method proposed in Algorithm 3 is of eighth-order of convergence □

4. Numerical examples

To show the efficiency of the new optimal schemes MK_4, MK_{8a} and MK_{8b}, several examples will be tested. We compare the new schemes with the optimal fourth-order technique K_4 presented by King (1973), and with the derivative-free fourth- and eighth-order methods presented by Yasmin et al. (2016), Cordero et al. (2013); and Zafar et al. (2015) denoted respectively as: YZA_4, YZA_8, $CHMT_4$, $CHMT_8$, and $ZYAJ_4$, $ZYAJ_8$. In all examples, we consider that $\alpha = 1$ whenever $w_n = x_n + \alpha f(x_n)$, and that $\beta = 2$ in King's method and our proposed methods.

Six test examples are considered below:

$$\begin{aligned} f_1(x) &= \cos(x) - x, \quad f_2(x) = \sin^2(x) - x^2 + 1, \\ f_3(x) &= \ln(x^2 - x + 1) - 4\sin(x-1), \quad f_4(x) = e^{-x^2} + \cos(x) - x^2, \\ f_5(x) &= \arctan(x) - x^2 + 1, \\ f_6(x) &= \begin{cases} x(x+1), \text{if} x < 0 \\ -2x(x-1), \text{if} x \geqslant 0 \end{cases}. \end{aligned}$$

We take $|x_n - x_{n-1}| < 10^{-15}$ as a stopping criterion of the computer programs. The computations here have been carried out using Mathematica version 9 with 10,000 significant digits.

Table 1 shows the number of iterations n needed so that the stopping criterion is satisfied, the approximate zero x_n, the distance between two successive approximations with $|x_n - x_{n-1}| < 10^{-15}$, the value of $f(x)$ at the approximate zero, and the computational order of convergence (COC) defined by Weerakoon and Fernando (2000), which can be estimated as follows

$$\mathrm{COC} \approx \frac{\ln|(x_{n+1} - x_n)/(x_n - x_{n-1})|}{\ln|(x_n - x_{n-1})/(x_{n-1} - x_{n-2})|}.$$

The second column in Table 1 shows the number of iterations n needed to reach the stopping criterion. It is clear that the new methods need less iterations than the other methods to reach the stopping criterion, or the same number of iterations in some cases. Therefore, the approximate solutions obtained by the proposed techniques are as good as of those obtained by other existing methods of the same order.

Note that, even though the new proposed methods need the same number of iterations to satisfy the stopping criterion as with the other methods, but still they are superior to the other methods as $|x_n - x_{n-1}|$ and $f(x_n)$ are less for the new schemes than the other schemes of the same order.

Table 2 illustrates the number of iterations needed to achieve approximate solution using the stopping criterion $|x_n - x_{n-1}| < 10^{-200}$. Setting the same convergence criterion for all methods, the required number of iterations for the new methods is either less than or equal the needed iterations by the other techniques with identical order.

5. Conclusion

In this work we proposed new optimal three derivative-free root finding schemes for nonlinear equations. These methods are implemented via efficient algorithms. The first method has order four, and derived using King's method with finite difference approximations. The second and the third optimal methods were of order eight. We implement the methods by using the composition technique combined with rational interpolation, and the idea of Padé approximation. The convergence analysis of the proposed optimal methods has been proved, with the convergence order has been established to be of the optimal fourth- and eighth-order, respectively. Six examples were tested, showing the capability of the new techniques. Overall, the new methods are comparable to other well-known schemes with same order of convergence.

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