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Common fixed point theorems for weakly compatible hybrid mappings

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Abstract The purpose of this paper is to study common fixed point theorems for set-valued and single-valued mappings in fuzzy metric and fuzzy 2-metric spaces. Also, we give an example to support our theorem. Generalizations and extensions of known results are thereby obtained. In particular, theorems by Pathak and Singh (2007), Sharma and Tiwari (2005) and Som (1985).

1. Introduction

In 1965, the concept of fuzzy set introduced by Zadeh (1965), many researchers have defined fuzzy metric spaces in different ways such as Kramosil and Michalek (1975). The concept of compatible mappings has been investigated initially by Jungck (1988), by which the notions of commuting and weakly commuting mappings are generalized. In the last years, the concepts of δ-compatible and weakly compatible mappings were introduced by Jungck and Rhoades (1998). In the last few decades, the common fixed point theorems for compatible mappings have applied to show the existence and uniqueness of the solutions of differential equations, integral equations and many other applied mathematics. Abu-donia et al. (2000) introduced the concept of fuzzy 2-metric spaces and study a fixed point theorems in this space. Sharma (2002) and Sharma and Tiwari (2005) studied unique common fixed point for three mappings in fuzzy 2-metric and fuzzy 3-metric spaces.

The purpose of this paper is to obtain a unique common fixed point for four hybrid mappings in fuzzy metric spaces. We give an example to support our theorem. Also, we prove a unique common fixed point for four hybrid mappings in fuzzy 2-metric spaces.\

2. Basic preliminaries

In this section, we recall some notions and definitions in fuzzy metric, fuzzy 2-metric spaces.

Definition 2.1 (*Sklar and Schweizer (1960)*). A mapping $*: [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if it satisfies the following conditions:

(1) $*$ is associative and commutative,
(2) $*$ is continuous,
(3) $a * 1 = a$ for every $a \in [0,1]$,
(4) $a * b \leqslant c * d$ whenever $a \leqslant c$ and $b \leqslant d$ for each $a, b, c, d \in [0,1]$.

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Definition 2.2 (*Kramosil and Michalek (1975)*). A triple $(X, M, *)$ is a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t norm and M is a fuzzy set on $X \times X \times [0, \infty) \to [0, 1]$ satisfying, $\forall x, y \in X$, the following conditions:

(1) $M(x, y, 0) = 0$,
(2) $M(x, y, t) = 1, \forall t > 0$ iff $x = y$,
(3) $M(x, y, t) = M(y, x, t)$,
(4) $M(x, y, t) * M(y, z, s) \leqslant M(x, z, s + t), s, t \in [0, 1)$,
(5) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous.

Note that $M(y, x, t)$ can be thought of as the degree of nearness between x and y with respect to t.

Definition 2.3 (*Grebiec (1988)*). A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be convergent to a point $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1, \forall t > 0$.

A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is Cauchy sequence if $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1,\ \forall t, p > 0$.

A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 2.4 (*Abu-donia et al. (2000) and Sharma (2002)*). A mapping $* : [0, 1] \times [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if it satisfies the following conditions:

(1) $*$ is associative and commutative,
(2) $*$ is continuous,
(3) $a * 1 = a$ for every $a \in [0, 1]$,
(4) $a_1 * b_1 * c_1 \leqslant a_2 * b_2 * c_2$ whenever $a_1 \leqslant a_2, b_1 \leqslant b_2$ and $c_1 \leqslant c_2$ for each $a_1, b_1, c_1, a_2, b_2, c_2 \in [0, 1]$.

Definition 2.5 (*Abu-donia et al. (2000) and Sharma (2002)*). A triple $(X, M, *)$ is a fuzzy 2-metric space if X is an arbitrary set, $*$ is a continuous t norm and M is a fuzzy set on $X \times X \times X \times [0, \infty)$ satisfying, $\forall x, y, z \in X$, the following conditions:

(1) $M(x, y, z, 0) = 0$,
(2) $M(x, y, z, t) = 1, \forall t > 0$ when at least two of the three point are equal,
(3) $M(x, y, z, t) = M(x, z, y, t) = M(y, z, x, t) = \cdots$ Symmetry about three variables,
(4) $M(x, y, u, t_1) * M(x, u, z, t_1) * M(u, y, z, t_3) \leqslant M(x, y, z, t_1 + t_2 + t_3), t_1, t_2, t_3 \in [0, 1)$,
(5) $M(x, y, z, \cdot) : [0, \infty) \to [0, 1]$ is left continuous.

Definition 2.6 (*Abu-donia et al. (2000) and Sharma (2002)*). A sequence $\{x_n\}$ in a fuzzy 2-metric space $(X, M, *)$ is said to be convergent to a point $x \in X$ if $\lim_{n\to\infty} M(x_n, x, z, t) = 1,\ \forall t > 0, z \in X$.

A sequence $\{x_n\}$ in a fuzzy 2-metric space $(X, M, *)$ is Cauchy sequence if $\lim_{n\to\infty} M(x_{n+p}, x_n, z, t) = 1, \forall z \in X$ and $t, p > 0$.

In the following example, we know that every metric induces a fuzzy metric

Example 2.1 George and Veeramani (1994). Let (X, d) be a metric space. Define $a * b = ab$ and for all $x, y \in X, t > 0$,

$$M(x, y, t) = \frac{t}{t + d(x, y)}.$$

We call M is a fuzzy metric on X induced by metric d.

Definition 2.7 (*Jungck and Rhoades (1998)*). The mappings $I : X \to X$ and $F : X \to B(X)$ are weakly compatible if they commute at coincidence points, i.e., for each point $u \in X$ such that $Fu = \{Iu\}$, we have $FIu = IFu$. Not that the equation $Fu = \{Iu\}$ implies that Fu is singleton.

Definition 2.8 (*Vasuki (1999)*). The mappings $I : X \to X$ and $F : X \to B(X)$ are R-weakly commuting if, for all $R, t > 0, M(FIx, IFx, t) \geqslant M(Fx, Ix, t/R)$, such that $x \in X, IFx \in B(X)$.

R-weakly commuting is weakly compatible but the converse is not true (Pathak and Singh, 2007).

Theorem 2.1 Som (1985). *Let S and T be two continuous self mappings of a complete fuzzy metric space $(X, M, *)$. Let A and B be two self mappings of X satisfying following conditions:*

(1) $A(X) \bigcup B(X) \subset S(X) \bigcap T(X)$,
(2) $\{A, T\}$ and $\{B, S\}$ are R-weakly commuting pairs,
(3) $aM(Tx, Sy, t) + bM(Tx, Ax, t) + cM(Sy, By, t) + \max\{M(Ax, Sy, t), M(By, Tx, t)\} \leqslant qM(Ax, By, t)$,

for all $x, y \in X$, where $a, b, c \geqslant 0, q > 0$ with $q < a + b + c < 1$. Then A, B, S and T have a unique common fixed point.

Pathak and Singh (2007) improved results of Som (1985) as the following:

Theorem 2.2 Pathak and Singh (2007). *Let S and T be two continuous self mappings of a complete fuzzy metric space $(X, M, *)$. Let A and B be two self mappings of X satisfying following conditions:*

(1) $A(X) \bigcup B(X) \subset S(X) \bigcap T(X)$,
(2) $\{A, T\}$ and $\{B, S\}$ are weakly compatible pairs,
(3) $aM(Tx, Sy, t) + bM(Tx, Ax, t) + cM(Sy, By, t) + \max\{M(Ax, Sy, t), M(By, Tx, t)\} \leqslant qM(Ax, By, t)$,

for all $x, y \in X$, where $a, b, c \geqslant 0$ with $0 < q < a + b + c < 1$. Then A, B, S and T have a unique common fixed point.

3. Main results

In this section we generalize, extend and improve the corresponding results given by many authors. In the following we denote the set of all non-empty bounded closed subsets of X by $CB(X)$.

Theorem 3.1. *Let S and T be two self mappings of a fuzzy metric space $(X, M, *)$ and $A, B : X \to CB(X)$ set-valued mappings satisfying following conditions:*

(1) $\bigcup A(X) \subseteq S(X)$ and $\bigcup B(X) \subseteq T(X)$,
(2) $\{A, T\}$ and $\{B, S\}$ are weakly compatible pairs,
(3) $aM(Tx, Sy, t) + bM(Tx, Ax, t) + cM(Sy, By, t) + \max\{M(Ax, Sy, t), M(By, Tx, t)\} \leqslant qM(Ax, By, t)$,

for all $x, y \in X$*, where* $a, b, c \geqslant 0$ *with* $0 < q < a + b + c < 1$ *and if the range of one of the mappings* A, B, S *and* T *is complete subspace of* X*. Then* A, B, S *and* T *have a unique common fixed point.*

Proof. Let x_0 be an arbitrary point in X. From the condition (1), we chose a point x_1 in X such that $Sx_1 \in Ax_0$. For this point x_1 there exist a point x_2 in X such that $Tx_2 \in Bx_1$ and so on. Inductively, we can define a sequence $\{Z_n\}$ in X such that

$$Sx_{2n+1} \in Ax_{2n} = Z_{2n}, \quad Tx_{2n+2} \in Bx_{2n+1} = Z_{2n+1}, \quad \forall n = 0, 1, 2, \ldots.$$

We will prove that $\{Z_n\}$ is Cauchy sequence.

Using inequality (3), we obtain

$$\begin{aligned} qM(Z_{2n}, Z_{2n+1}, t) &= qM(Ax_{2n}, Bx_{2n+1}, t) \\ &\geqslant aM(Tx_{2n}, Sx_{2n+1}, t) + bM(Tx_{2n}, Ax_{2n}, t) \\ &\quad + cM(Sx_{2n+1}, Bx_{2n+1}, t) \\ &\quad + \max\{M(Ax_{2n}, Sx_{2n+1}, t), M(Bx_{2n+1}, Tx_{2n}, t)\} \\ &\geqslant aM(Z_{2n-1}, Z_{2n}, t) + bM(Z_{2n-1}, Z_{2n}, t) \\ &\quad + cM(Z_{2n}, Z_{2n+1}, t) \\ &\quad + \max\{M(Z_{2n}, Z_{2n}, t), M(Z_{2n+1}, Z_{2n-1}, t)\}. \end{aligned}$$

Then $M(Z_{2n}, Z_{2n+1}, t) \geqslant \beta M(Z_{2n-1}, Z_{2n}, t)$, where $\beta = \frac{a+b+1}{q-c} > 1$.

Since $\beta > 1$, we obtain

$$M(Z_{2n+1}, Z_{2n}, t) > M(Z_{2n}, Z_{2n-1}, t).$$

Similarly

$$M(Z_{2n+2}, Z_{2n+1}, t) > M(Z_{2n+1}, Z_{2n}, t).$$

Now for any positive integer p,

$$\begin{aligned} M(Z_n, Z_{n+p}, t) &\geqslant M\left(Z_n, Z_{n+1}, \frac{t}{p}\right) * M\left(Z_{n+1}, Z_{n+2}, \frac{t}{p}\right) * \cdots \\ &\quad * M\left(Z_{n+p-1}, Z_{n+p}, \frac{t}{p}\right). \end{aligned}$$

As $n \to \infty$, we get $M(Z_n, Z_{n+p}, t) \to 1 * 1 * \cdots * 1 \to 1$.

Hence Z_n is a Cauchy sequence. Suppose that SX is complete, therefore by the above, $\{Sx_{2n+1}\}$ is a Cauchy sequence and hence $Sx_{2n+1} \to z = Sv$ for some $v \in X$. Hence, $Z_n \to z$ and the subsequences Tx_{2n+2}, Ax_{2n} and Bx_{2n+1} converge to z.

We shall prove that $z = Sv \in Bv$, by (3), we have

$$\begin{aligned} qM(Ax_{2n}, Bv, t) &\geqslant aM(Tx_{2n}, Sv, t) + bM(Tx_{2n}, Ax_{2n}, t) \\ &\quad + cM(Sv, Bv, t) \\ &\quad + \max\{M(Ax_{2n}, Sv, t), M(Bv, Tx_{2n}, t)\}. \end{aligned}$$

As $n \to \infty$, we obtain

$$\begin{aligned} qM(z, Bv, t) &\geqslant aM(z, z, t) + bM(z, z, t) + cM(z, Bv, t) \\ &\quad + \max\{M(z, z, t), M(Bv, z, t)\}, \end{aligned}$$

$$M(z, Bv, t) \geqslant \left(\frac{a+b+1}{q-c}\right) > 1,$$

which yields $\{z\} = \{Sv\} = Bv$.

Since $\bigcup B(X) \subseteq T(X)$, thus, there exist $u \in X$ such that $\{Tu\} = Bv = \{z\} = \{Sv\}$.

Now if $Au \neq Bv$, we get

$$\begin{aligned} qM(Au, Bv, t) &\geqslant aM(Tu, Sv, t) + bM(Tu, Au, t) + cM(Sv, Bv, t) \\ &\quad + \max\{M(Au, Sv, t), M(Bv, Tu, t)\}, \end{aligned}$$

$$\begin{aligned} qM(Au, z, t) &\geqslant aM(z, z, t) + bM(z, Au, t) + cM(z, z, t) \\ &\quad + \max\{M(Au, z, t), M(z, z, t)\}, \end{aligned}$$

$$M(Au, z, t) \geqslant \left(\frac{a+c+1}{q-b}\right) > 1,$$

which yields $Au = \{z\} = \{Tu\} = \{Sv\} = Bv$.

Since $Au = \{Tu\}$ and $\{A, T\}$ is weakly compatible, $ATv = TAv$ gives $Az = \{Tz\}$.

On using (3), we obtain

$$\begin{aligned} qM(Az, Bv, t) &\geqslant aM(Tz, Sv, t) + bM(Tz, Az, t) + cM(Sv, Bv, t) \\ &\quad + \max\{M(Az, Sv, t), M(Bv, Tz, t)\}, \end{aligned}$$

$$\begin{aligned} qM(Az, z, t) &\geqslant aM(Tz, z, t) + bM(z, Az, t) + cM(z, z, t) \\ &\quad + \max\{M(Az, z, t), M(z, z, t)\}. \end{aligned}$$

Hence, $Az = \{z\} = \{Tz\}$. Similarly, $Bz = \{z\} = \{Sz\}$ where $\{B, S\}$ is weakly compatible. Then, $Az = \{Tz\} = \{z\} = \{Sz\} = Bz$, i.e., z is the common fixed point of A, B, S and T have a unique.

To see z is unique, suppose that $p \neq z$ such that $Ap = \{Tp\} = \{p\} = \{Sp\} = Bp$.

On using (3), we get

$$\begin{aligned} qM(Az, Bp, t) &\geqslant aM(Tz, Sp, t) + bM(Tz, Az, t) \\ &\quad + cM(Sp, Bp, t) \\ &\quad + \max\{M(Az, Sp, t), M(Bp, Tz, t)\}, \end{aligned}$$

$$M(z, p, t) \geqslant \left(\frac{b+c}{q-a-1}\right),$$

which is impossible, $z = p$. Then A, B, S and T have a unique common fixed point. □

In Theorem 3.1, we no used two continuous self mappings condition and replaced a complete fuzzy metric space by one mapping is complete.

Remark 3.1. Theorem 3.1 is a generalization, extension and improvement for results of Pathak and Singh (2007) in fuzzy metric space.

In Theorem 3.1, we replaced a complete fuzzy metric space by one mapping is complete and three self mappings into four mappings,two self mappings and two set-valued mappings.

Remark 3.2. Theorem 3.1 is a generalization, extension and improvement for results of Sharma and Tiwari (2005) in fuzzy metric space.

Now, we give an example to support our theorem.

Example 3.1. Let $X = [0, \infty]$ endowed with the Euclidean metric d and $M(Ax, By, t) = \frac{t}{t+\delta(Ax,By)}$, $\delta(A,B) = \max\{d(a,b) : a \in A, b \in B\}$. Define

$$Ax = \left[0, \frac{x^6}{6}\right], \quad Bx = \left[0, \frac{x^3}{6}\right],$$
$$Sx = \frac{x^{12}}{2} + x^6 + \frac{x^3}{2}, \quad Tx = x^6 + 6x^3,$$

for all $x \in X$. We have $\bigcup A(X) = T(X) = \bigcup B(X) = S(X) = X$.

From the above, we have that $M(A(0), 0, t) = 1$, $M(B(0), 0, t) = 1$, $M(S(0), 0, t) = 1$ and $M(T(0), 0, t) = 1$.

Thus 0 is a common fixed point of A, B, S and T. Also, $\{A, T\}$ and $\{B, S\}$ are weakly compatible pairs, where, $M(AT(0), TA(0), t) = 1$ and $M(BS(0), SB(0), t) = 1$.

For any $x, y \in X, x \neq y$

$$\begin{aligned}\delta(Ax, By) &= \max\left\{\frac{x^6}{6}, \frac{y^3}{6}\right\} = \max\left\{\frac{1}{3}\frac{x^6}{2}, \frac{1}{3}\frac{y^3}{2}\right\} \\ &\leqslant \frac{1}{3}\max\left\{x^6 + 6x^3, \frac{y^{12}}{2} + y^6 + \frac{y^3}{2}\right\} \\ &\leqslant \frac{1}{3}\max\{\delta(By, Tx), \delta(Ax, Sy)\}.\end{aligned}$$

Thus

$$\frac{1}{\delta(Ax, By)} \geqslant 3\max\left\{\frac{1}{\delta(By, Tx)}, \frac{1}{\delta(Ax, Sy)}\right\},$$
$$\frac{t}{t + \delta(Ax, By)} \geqslant 3\max\left\{\frac{t}{t + \delta(By, Tx)}, \frac{t}{t + \delta(Ax, Sy)}\right\},$$
$$M(Ax, By, t) \geqslant 3\max\{M(Ax, Sy, t), M(By, Tx, t)\}.$$

Then

$$\begin{aligned}\frac{2}{3}M(Ax, By, t) \geqslant{}& \frac{1}{4}M(Tx, Sy, t) + \frac{1}{5}M(Tx, Ax, t) + \frac{1}{3}M(Sy, By, t) \\ &+ \max\{M(Ax, Sy, t), M(By, Tx, t)\},\end{aligned}$$

where $q = \frac{2}{3}, a = \frac{1}{4}, b = \frac{1}{5}$ and $c = \frac{1}{3}$.

Theorem 3.2. *Let S and T be two self mappings of a fuzzy 2-metric space $(X, M, *)$ and $A, B : X \to CB(X)$ set-valued mappings satisfying following conditions:*

(1) $\bigcup A(X) \subseteq S(X)$ and $\bigcup B(X) \subseteq T(X)$,
(2) $\{A, T\}$ and $\{B, S\}$ are weakly compatible pairs,
(3) $aM(Tx, Sy, w, t) + bM(Tx, Ax, w, t) + cM(Sy, By, w, t) + \max\{M(Ax, Sy, w, t), M(By, Tx, w, t)\} \leqslant qM(Ax, By, w, t)$,

for all $x, y, w \in X$, where $a, b, c \geqslant 0$ with $0 < q < a + b + c < 1$ and if the range of one of the mappings A, B, S and T is complete subspace of X. Then A, B, S and T have a unique common fixed point.

Proof. We can define a sequence $\{Z_n\}$ in X such that

$$Sx_{2n+1} \in Ax_{2n} = Z_{2n}, \quad Tx_{2n+2} \in Bx_{2n+1} = Z_{2n+1}, \quad \forall n = 0, 1, 2, \ldots.$$

We will prove that $\{Z_n\}$ is Cauchy sequence.
Using inequality (3), we obtain

$$\begin{aligned}qM(Z_{2n}, Z_{2n+1}, w, t) &= qM(Ax_{2n}, Bx_{2n+1}, w, t) \\ &\geqslant aM(Tx_{2n}, Sx_{2n+1}, w, t) + bM(Tx_{2n}, Ax_{2n}, w, t) \\ &\quad + cM(Sx_{2n+1}, Bx_{2n+1}, w, t) \\ &\quad + \max\{M(Ax_{2n}, Sx_{2n+1}, w, t), \\ &\quad M(Bx_{2n+1}, Tx_{2n}, w, t)\} \\ &\geqslant aM(Z_{2n-1}, Z_{2n}, w, t) + bM(Z_{2n-1}, Z_{2n}, w, t) \\ &\quad + cM(Z_{2n}, Z_{2n+1}, w, t) + \max\{M(Z_{2n}, Z_{2n}, w, t), \\ &\quad M(Z_{2n+1}, Z_{2n-1}, w, t)\} \\ &\geqslant (a + b)M(Z_{2n-1}, Z_{2n}, w, t) \\ &\quad + cM(Z_{2n}, Z_{2n+1}, w, t) \\ &\quad + \max\{1, M(Z_{2n}, Z_{2n-1}, w, t) \\ &\quad * M(Z_{2n+1}, Z_{2n}, w, t) \\ &\quad * M(Z_{2n+1}, Z_{2n-1}, Z_{2n}, t)\},\end{aligned}$$

$$M(Z_{2n}, Z_{2n+1}, w, t) \geqslant \beta M(Z_{2n-1}, Z_{2n}, w, t),$$

where $\beta = \dfrac{a + b + 1}{q - c} > 1$.

Since $\beta > 1$, we obtain

$$M(Z_{2n+1}, Z_{2n}, w, t) > M(Z_{2n}, Z_{2n-1}, w, t).$$

Similarly

$$M(Z_{2n+2}, Z_{2n+1}, w, t) > M(Z_{2n+1}, Z_{2n}, w, t).$$

Now for any positive integer p,

$$\begin{aligned}M(Z_n, Z_{n+p}, w, t) \geqslant{}& M\left(Z_n, Z_{n+1}, w, \frac{t}{p}\right) \\ &* M\left(Z_{n+1}, Z_{n+2}, Z_{n+1}, \frac{t}{p}\right) * \cdots \\ &* M\left(Z_{n+p-1}, Z_{n+p}, w, \frac{t}{p}\right).\end{aligned}$$

As $n \to \infty$, we get $M(Z_n, Z_{n+p}, w, t) \to 1$.

Hence Z_n is a Cauchy sequence. Suppose that SX is complete, therefore by the above, $\{Sx_{2n+1}\}$ is a Cauchy sequence and hence $Sx_{2n+1} \to z = Sv$ for some $v \in X$. Hence, $Z_n \to z$ and the subsequences Tx_{2n+2}, Ax_{2n} and Bx_{2n+1} converge to z.

We shall prove that $z = Sv \in Bv$, by (3), we have

$$\begin{aligned}qM(Ax_{2n}, Bv, w, t) \geqslant{}& aM(Tx_{2n}, Sv, w, t) + bM(Tx_{2n}, Ax_{2n}, w, t) \\ &+ cM(Sv, Bv, w, t) \\ &+ \max\{M(Ax_{2n}, Sv, w, t), M(Bv, Tx_{2n}, w, t)\}.\end{aligned}$$

As $n \to \infty$, we obtain

$$\begin{aligned}qM(z, Bv, w, t) \geqslant{}& aM(z, z, w, t) + bM(z, z, w, t) \\ &+ cM(z, Bv, w, t) \\ &+ \max\{M(z, z, w, t), M(Bv, z, w, t)\},\end{aligned}$$

$$M(z, Bv, w, t) \geqslant \left(\frac{a + b + 1}{q - c}\right) > 1,$$

which yields $\{z\} = \{Sv\} = Bv$.

Since $\bigcup B(X) \subseteq T(X)$, thus, there exist $u \in X$ such that $\{Tu\} = Bv = \{z\} = \{Sv\}$. Now if $Au \neq Bv$, we get

$$qM(Au, Bv, w, t) \geqslant aM(Tu, Sv, w, t) + bM(Tu, Au, w, t) + cM(Sv, Bv, w, t) + \max\{M(Au, Sv, w, t), M(Bv, Tu, w, t)\},$$

$$qM(Au, z, w, t) \geqslant aM(z, z, w, t) + bM(z, Au, w, t) + cM(z, z, w, t) + \max\{M(Au, z, w, t), 1\},$$

$$M(Au, z, t) \geqslant \left(\frac{a + c + 1}{q - b}\right) > 1,$$

which yields $Au = \{z\} = \{Tu\} = \{Sv\} = Bv$.

Since $Au = \{Tu\}$ and $\{A, T\}$ is weakly compatible, $ATv = TAv$ gives $Az = \{Tz\}$.

On using (3), we obtain

$$qM(Az, Bv, w, t) \geqslant aM(Tz, Sv, w, t) + bM(Tz, Az, w, t) + cM(Sv, Bv, w, t) + \max\{M(Az, Sv, w, t), M(Bv, Tz, w, t)\},$$

$$qM(Az, z, w, t) \geqslant aM(Tz, z, w, t) + bM(z, Az, w, t) + cM(z, z, w, t) + \max\{M(Az, z, w, t), 1\}.$$

Hence, $Az = \{z\} = \{Tz\}$. Similarly, $Bz = \{z\} = \{Sz\}$ where $\{B, S\}$ is weakly compatible. Then, $Az = \{Tz\} = \{z\} = \{Sz\} = Bz$, i.e., z is the common fixed point of A, B, S and T.

To see z is unique, suppose that $p \neq z$ such that $Ap = \{Tp\} = \{p\} = \{Sp\} = Bp$.

By (3), we get

$$qM(Az, Bp, w, t) \geqslant aM(Tz, Sp, w, t) + bM(Tz, Az, w, t) + cM(Sp, Bp, w, t) + \max\{M(Az, Sp, w, t), M(Bp, Tz, w, t)\},$$

$$M(z, p, w, t) \geqslant \left(\frac{b + c}{q - a - 1}\right),$$

which yields $z = p$. Then A, B, S and T have a unique common fixed point.

In Theorem 3.2, we replaced a complete fuzzy metric space by one mapping is complete and three self mappings into four mappings, two self mappings and two set-valued mappings. □

Remark 3.3. Theorem 3.2 is a generalization, extension and improvement for results of Sharma and Tiwari (2005) in fuzzy 2-metric space.

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