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SHORT COMMUNICATION

Jacobi elliptic function solutions for the modified Korteweg–de Vries equation

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Abstract Formula solutions to the modified Korteweg–de Vries (mKdV) equation with constant coefficients are obtained via the Jacobi elliptic periodic function transform method and symbolic computation. Those periodic solutions degenerate as the corresponding hyperbolic function solutions when the modulus is $m \to 1$ and trigonal solutions with $m \to 0$.

1. Introduction

The investigation of nonlinear partial differential equations plays an important role in the study of nonlinear physical phenomena. A variety of powerful and direct methods have been developed in this direction (Bhrawy et al., 2013; Ebadi et al., 2011, 2012, 2013; Krishnan et al., 2011, 2012; Biswas et al., 2012, 2013). In this Letter, the modified Korteweg–de Vries (mKdV) equation is investigated, which has been mentioned in many branches of nonlinear science field, and given in the form

$$\lambda u_t - \mu u^2 u_x + w u_{xxx} = 0; \quad (1)$$

where the subscripts denote the partial derivates of x and t, λ, μ and w are constant parameters, u represents a real scalar function $u(x,t)$. The equation and equations derived from itself have been widely studied (Trogdon et al., 2012; Bozhkov et al., 2013; Salkuyeh and Bastani, 2013; Ono, 1992; Liang et al., 2011; Anco et al., 2011) because these equations play an important role in many physics contexts, such as anharmonic lattices (Ono, 1992), ion acoustic solitons (Konno and Ichikawa, 1974; Watanabe, 1984; Lonngren, 1998), traffic jam (Komatsu and Sasa, 1995; Nagatani, 1999), thin ocean jets (Cushman-Roisin et al., 1992; Ralph and Pratt, 1994), internal waves (Grimshaw et al., 2004; Grimshaw, 2001), heat pulses in solids (Tappert and Varma, 1970), and so on. With the help of an auxiliary Lamé equation (Fu et al., 2009; Liu, 2009), the Jacobi elliptic function solutions, which are important to label the unidentified spectrum from white dwarf stars (10^2–$10^5 T$) and neutron stars(10^7–$10^9 T$), the periodic solutions for mKdV equations are shown.

2. The Jacobi elliptic function and its properties

Usually, three types of Jacobi elliptic functions (Bhrawy et al., 2013) ($sn(\xi,m)$, $cn(\xi,m)$, $dn(\xi,m)$) have the following properties:

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$$cn(\xi,m)^2 + sn(\xi,m)^2 = 1;$$
$$dn(\xi,m)^2 + m^2 sn(\xi,m)^2 = 1;$$

the derivates of Jacobi elliptic functions can be expressed as

$$\frac{dcn(\xi,m)}{d\xi} = -sn(\xi,m)dn(\xi,m),$$
$$\frac{ddn(\xi,m)}{d\xi} = -m^2 sn(\xi,m)cn(\xi,m), \tag{2}$$
$$\frac{dsn(\xi,m)}{d\xi} = cn(\xi,m)dn(\xi,m);$$

where m is the modulus of Jacobi elliptic functions. The Jacobi elliptic functions will asymptotically go into hyperbolic functions and trigonometric functions when the modulus is $m \to 1$ and $m \to 0$, respectively.

$$m \to 1, sn(\xi,m) \to \tanh(\xi), cn(\xi,m) \to \mathrm{sech}(\xi),$$
$$dn(\xi,m) \to \mathrm{sech}(\xi);$$
$$m \to 0, sn(\xi,m) \to \sin(\xi), cn(\xi,m) \to \cos(\xi), dn(\xi,m) \to 1.$$

The Lamé equation (Fu et al., 2009; Liu, 2009) in terms of $\varphi(\xi)$ can be expressed as

$$\frac{d^2\varphi}{d\xi^2} + (\eta - l(l+1)m^2 sn(\xi,m)^2)\varphi = 0; \tag{3}$$

where η is the eigenvalue of φ and l is a positive parameter. The particular solution of lamé equation can be written as:

when $\eta = 4 + m^2$ and $l = 2$,

$$\varphi = sn(\xi,m)cn(\xi,m); \tag{4}$$

when $\eta = 1 + 4m^2$ and $l = 2$,

$$\varphi = sn(\xi,m)dn(\xi,m); \tag{5}$$

when $\eta = 1 + m^2$ and $l = 2$,

$$\varphi = cn(\xi,m)dn(\xi,m). \tag{6}$$

3. The Jacobi elliptic function solutions for mKdV equation

We will seek the solutions with one travelling wave-like variable ξ for mKdV equation: $\xi = \alpha x + k\, t$, where α and k are constant parameters. Then, the evolution Eq. (1) can be transformed as follows :

$$k\lambda u_\xi - \mu\alpha u^2 u_\xi + w\alpha^3 u_{\xi\xi\xi} = 0; \tag{7}$$

Integrating Eq. (7) once with respect to ξ, we have

$$c + k\lambda u - \frac{\mu\alpha u^3}{3} + w\alpha^3 u_{\xi\xi} = 0, \tag{8}$$

where c is the integral constant.

Usually, the scalar function u in terms of perturbation method can be expressed as

$$u = u_0 + pu_1 + p^2 u_2; \tag{9}$$

and the second derivative of Eq. (9) reads

$$u_{\xi\xi} = u_{0\xi\xi} + pu_{1\xi\xi} + p^2 u_{2\xi\xi}, \tag{10}$$

where $p(0 < p\ 1)$ is a small parameter, u_0, u_1 and u_2 are the zeroth-order, first-order and second-order ansatz solutions, respectively.

Substituting Eqs. (9) and (10) into Eq. (8) and equating the coefficients of p^0, p^1 and p^2 to zero, we obtain the zeroth-order, first-order and second-order solutions of Eq. (7) in the following form:

$$3c + 3k\lambda u_0 - \alpha\mu u_0^3 + 3w\alpha^3 u_{0\xi\xi} = 0; \tag{11}$$
$$ku_1\lambda - \alpha\mu u_0^2 u1 + w\alpha^3 u_{1\xi\xi} = 0; \tag{12}$$
$$ku_2\lambda - \alpha\mu u_0 u_1^2 - \alpha\mu u_0^2 u_2 + w\alpha^3 u_{2\xi\xi} = 0. \tag{13}$$

The zeroth-order ansatz solution u_0 in terms of the Jacobi elliptic sine function reads

$$u_0 = a_0 + a_1 sn(\xi,m) + a_2 sn(\xi,m)^2, \tag{14}$$

where a_0, a_1 and a_2 are constant parameters. By substituting Eq. (14) into Eq. (11) and collecting all terms with the same power of $sn(\xi,m)$ together and equating the coefficients of the polynomials to zero, yield a set of simultaneous algebraic equations. Solving the algebraic equations above yields

$$k\lambda = w\alpha^3(1+m^2); a_0 = 0; a_2 = 0; c = 0; a_1 = \pm\sqrt{\frac{6wm^2\alpha^2}{\mu}}.$$

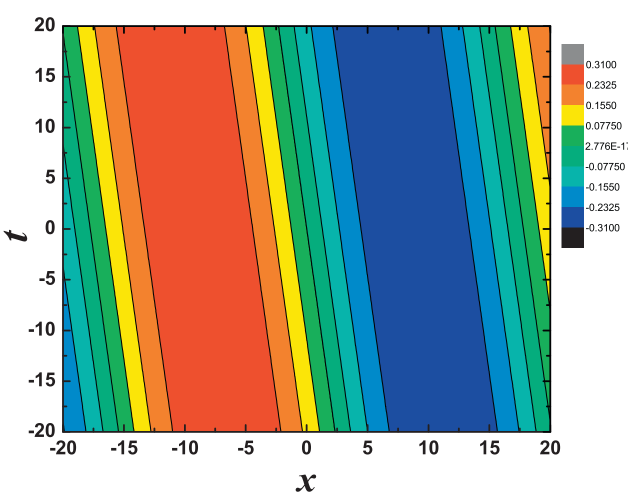

Figure 1 Simulation $u(x,t)$ for Eq. (22) with the modulus $m = 0.4, \alpha = 0.2, w = 2.5, g_0 = 1, a_1 = -\sqrt{6wm^2\alpha^2/\mu}, \mu = 1, \lambda = 1, k = w\alpha^3(1+m^2)$ and $p = 0.001$, the range of x and t are $x \in [-20,20]$ and $t \in [-20,20]$, respectively.

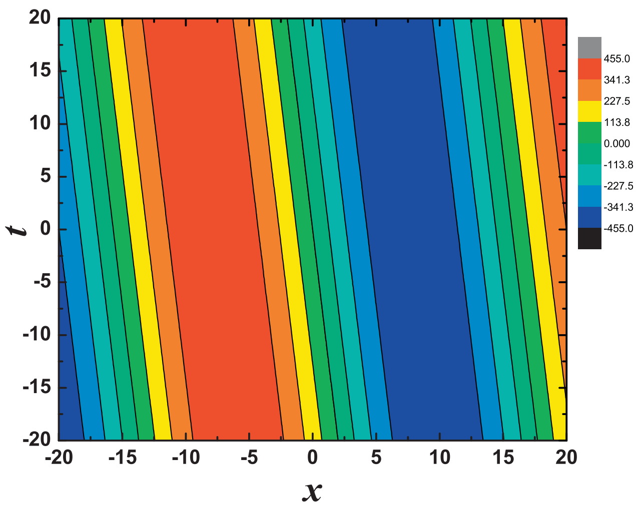

Figure 2 simulation $u(x,t)$ for Eq. (22) with the modulus $m = 0.00000001, \alpha = 0.2, w = 2.5, g_0 = 1, a_1 = -\sqrt{6wm^2\alpha^2/\mu}, \mu = 1, \lambda = 1, k = w\alpha^3(1+m^2)$ and $p = 0.001$, the range of x and t are $x \in [-20,20]$ and $t \in [-20,20]$, respectively.

The zeroth-order solution u_0 in terms of $k\lambda = w\alpha^3\,(1 + m^2)$ is of the form

$$u_0(\xi) = a_1 sn(\xi, m). \tag{15}$$

Substituting Eq. (15) into Eq. (12), we obtain

$$u_{1\xi\xi} + (6dn(\xi,m)^2 - \left(6 - \frac{k\lambda}{w\alpha^3}\right))u_1 = 0. \tag{16}$$

With the help of the Jacobi elliptic properties and Lamé equation Eq. (3), the solution of above equation is expressed as

$$u_1(\xi) = g_0 cn(\xi,m)dn(\xi,m), \tag{17}$$

where g_0 is a constant parameter.

Using Eqs. 15 and 17, Eq. 13 is rewritten as

$$w\alpha^3 u_{2\xi\xi} + (k\lambda - 6wm^2\alpha^3 sn(\xi,m)^2)u_2 - a_1 sn(\xi,m)u_1^2\alpha\mu = 0, \tag{18}$$

The second-order ansatz solution u_2 is supposed to be

$$u_2 = b_0 + b_1 sn(\xi,m) + b_3 sn(\xi,m)^3, \tag{19}$$

where b_0, b_1 and b_2 are constant parameters to be determined.

Substituting Eq. (19) into Eq. (18), we have a set of algebraic equations about $sn(\xi,m)$ and obtain

$$b_0 = 0; b_1 = \frac{-g_0^2\sqrt{\mu}(1+m^2)}{2m\alpha\sqrt{6w}}; b_3 = \frac{mg_0^2\sqrt{\mu}}{\alpha\sqrt{6w}} \tag{20}$$

when

$$a_1 = \sqrt{6wm^2\alpha^2/\mu};$$

and

$$b_0 = 0; b_1 = \frac{-7g_0^2\sqrt{\mu}(1+m^2)}{2m\alpha\sqrt{6w}}; b_3 = \frac{-mg_0^2\sqrt{\mu}}{\alpha\sqrt{6w}}, \tag{21}$$

when

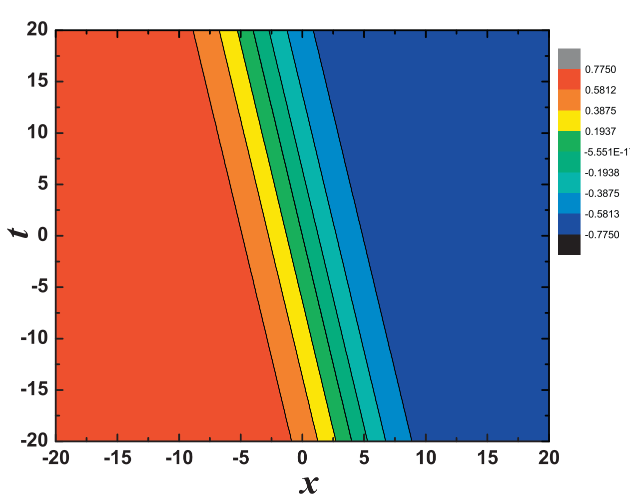

Figure 3 Simulation $u(x,t)$ for Eq. (22) with the modulus $m = 0.99999999, \alpha = 0.2, w = 2.5, g_0 = 1, a_1 = -\sqrt{6wm^2\alpha^2/\mu}, \mu = 1, \lambda = 1, k = w\alpha^3(1+m^2)$ and $p = 0.001$, the range of x and t are $x \in [-20,20]$ and $t \in [-20,20]$, respectively.

$a_1 = -\sqrt{6wm^2\alpha^2/\mu}$.

The solution of Eq. (1) can be written as:

$$u(x,t) = a_1 sn(\xi,m) + pg_0 cn(\xi,m)dn(\xi,m) + p^2 b_1 sn(\xi,m) + p^2 b_3 sn(\xi,m)^3. \quad (22)$$

By using the Jacobi elliptic function properties, when the modulus is $m \to 1$ and $m \to 0$, the above equation is transformed into the following

$$u(x,t) = a_1 \sin\xi + pg_0 \cos\xi + p^2 b_1 \sin\xi + p^2 b_3 \sin\xi^3; \quad (23)$$

$$u(x,t) = a_1 \tanh\xi + pg_0 \mathrm{sech}\xi^2 + p^2 b_1 \tanh\xi + p^2 b_3 \tanh\xi^3, \quad (24)$$

where $\xi = \alpha x + kt$. Eq. (23) is trigonometric solutions of Eq. (1), Eq. (24) is hyperbolic solutions of Eq. (1). The numerical simulations with $a_1 = -\sqrt{6wm^2\alpha^2/\mu}$ are attached, all the figures are obtained in the contour plot form with the aid of Origin software (see Figs. 1–3).

4. Discussions and conclusions

In this paper, we presented the Jacobi elliptic function method for solving the mKdV equations. The Jacobi elliptic function solutions, the trigonometric solutions and hyperbolic solutions are obtained. This present work affirms that the Jacobi elliptic function method is an easy straight forward method to solve nonlinear partial differential equations.

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