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A model of angular momentum transport between a planet and accretion disk

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ABSTRACT

This paper presents a simplified model of the viscous transport of angular momentum and sets up a simplified expression of the rate of specific angular momentum transfer from the planet to the disk. Thanks to Whittaker's functions, we have found an analytic solution for the disk surface density.

1. Introduction

The problem of the orbital evolution of planets, due to their interaction with the ambient disk, has been covered in a few recent reviews (Baruteau et al., 2013; Baruteau and Massey, 2013; Kley and Nelson, 2012). It turns out that the change in the orbital elements of the protoplanet (Laughlin et al., 2004) can be predicted as a consequence of the interaction with the disk. The migration of the planet is the most important consequence observed in the overall evolution of the semi major-axis of the planet. Three regimes of migration, depending on the mass of the planet, can be distinguished, with two limiting regimes of migration as the most important. For a mass less than about 50 solar mass, the planet do not open a gap in the disk (Kley and Nelson, 2012; Baruteau and Massey, 2013), so the evolution can be treated in the linear regime: this type of evolution is named an evolution of type I. Analytical formulation of this type of migration has been presented, in the case of adiabatic disks, by (Paardekooper et al., 2010; Paardekooper et al., 2011). For massive planets, the angular momentum deposition in the disk causes an annular gap in the disk at the orbit of the planet. The reduced mass available near the planet causes a slow-down of the migration speed from the linear rate: this kind of evolution is of the type II. At equilibrium the planet is locked in the middle of the gap to maintain the torque equilibrium (Ward, 1982; lin and Papaloizou, 1986; Ward, 1997).

Several numerical calculations of migrating massive planets were done in (Edgar, 2007; Edgar, 2008). In these works a constant kinematic viscosity is considered. In more recent works, (Paardekooper, 2014) analyzed the influence of planetary motion on the type I migration regime. Massive planets migration in an ambient disk has been studied in (Crida and Morbidelli, 2007). In this last model, the planets were locked in a disk with an initial Gaussian shape and a constant kinematic viscosity. The problem of type II migration is still relevant.

In this paper, we present the resolution of this problem and study the migration of planets in slim disks with an approximation of the external torque due to the planet. In the next section, we define the torque. In Sect. 3 and sect.4 we present our analytic solution respecting the boundary conditions. Sect.5 is devoted to concluding remarks.

We do not pretend to solve the problem without resorting to the numerical method, but we have obtained an analytical solution for a simplified model, this solution may be used to build migration models.

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2. The rate of specific angular momentum

In type I migration model, the planets are not massive enough to perturb the disc, within this assumption the total torque is the sum of the contributions from three different resonances: the first is the partial torques from the inner Lindblad resonances, which drive outward migration, the second is outer Lindblad resonances, which drive inward migration whereas the third contribution is due to the co-rotation resonance. However, the partial torques from the inner and the outer Lindblad resonances are of opposing sign but of al most the same magnitude. The prediction of the direction of migration by an analytic calculation is somewhat difficult because a precise calculation of the torque is needed. Moreover, real discs can be turbulent with a dominating fluctuating torques that results from turbulent density fluctuations. We consider a geometrically stationary thin disc, with a viscosity in the form $v = sr^n$, $n < 2$ (Lynden-Bell and Pringle, 1974) The rate of specific angular momentum transfer from the planet, of mass $M_p = qM_*$ to the disk is given by $\Lambda(r)$, in (Armitage et al., 2002; Trilling et al., 1998) this rate of the specific angular momentum is described by the following piecewise continuous function

$$r \le r_p - H, \Lambda(r) = -\frac{q^2 GM_*}{2r}\frac{r^4}{(r-r_p)^4} \tag{1-a}$$

$$r_p - H \le r \le r_p, \Lambda(r) = -\frac{q^2 GM_*}{2r}\frac{r^4}{H^4} \tag{1-b}$$

$$r_p \le r \le r_p + H, \Lambda(r) = \frac{q^2 GM_*}{2r}\frac{{r_p}^4}{H^4} \tag{1-c}$$

$$r_p + H \le r, \Lambda(r) = \frac{q^2 GM_*}{2r}\frac{{r_p}^4}{(r-r_p)^4} \tag{1-d}$$

for a planet of mass qM_*, at radius (semi-major axis) r_p and where H is the disk scale-height.

Protoplanetary disks evolve to viscous transport of angular momentum and photo evaporation by the central star. Planets migrate due to tidal interaction with the disk, and the disk is also subject to tidal torques from planets Treating the evolution equation with $\Lambda(r)$ given by (1) is analytically very complicated. It requires to solve four equations with respect each region. Furthermore, numerical simulations is inevitable. To avoid these difficulties, we will modify the rate of the specific angular momentum function $\Lambda(r)$. This modification allows us to solve only two equations and gives an acceptable result at the vicinity of the planet.

In our model the rate of specific angular momentum is described by two equations, Where $1 < n < 2$ and $n' < 1$:

$$r < r_p, \Lambda^{'(r)} = -\frac{q^2 GM_*}{2r}\left(\frac{r_p}{H}\right)^4\left(\frac{r}{r_p}\right)^{\frac{n+1}{2}} \tag{2-a}$$

$$r_p < r, \Lambda'(r) = \frac{q^2 GM_*}{2r}\left(\frac{r_p}{H}\right)^4\left(\frac{r}{r_p}\right)^{\frac{n'+1}{2}} \tag{2-b}$$

This choice of model is based on two main facts:

The first is that the rate of the specific angular momentum $\Lambda(r)$depends on the tidal dissipation, which in turn, when the viscosity v is very large, is concentrated at $r = r_p$ near the protoplanet orbit (Papaloizou and Lin, 1984). The viscosity expression proposed by (Lynden-Bell and Pringle, 1974) led us to choose this rate, as indicated by the formulas (2-a, b).

The second fact is that near the orbit (where $r = r_p$), the formulas (2-a, b) have approximately the same behavior as the equations (1-b, c). Indeed, by putting$r$ equal to $r_p$in set of equations (1-b, c) and (2-a, b), we find that equation (2-a) coincides with equation (1- b) and equation (2-b) coincides with equation (1-c).

Another feature to note is that this model in the vicinity of the orbit lends itself to an exact analytical treatment.

Indeed Fig. 1 illustrates the approach between the new approximation function $\Lambda'(r)$ and the function $\Lambda(r)$ near the planet orbit for different n, n′ values.

Whence the importance of the viscosity near the planet orbit (Papaloizou and Lin, 1984), we focused the study near the r_p.

For computational convenience, we introduce a set of dimensionless variables such that: $y = \frac{2r_p\Lambda(r)}{q^2GM_*}$, $y' = \frac{2r_p\Lambda'(r)}{q^2GM_*}$, $x = \frac{r}{r_p}$, and for the subsequent application we take $r_p = 2.5H$.

Since the behavior of the rate $\Lambda(r)$ of the specific angular momentum not coincides with $\Lambda'(r)$ for regions far from r_p, the Fig. 1 shows, that a near the plant orbit ($r\ r_p, x\ 1$) for all values of n and n′, $\Lambda'(r)$ and $\Lambda(r)$ have the same shape.

We have motioned this remark when we compared the formulas (2-a, b) and (1-b, c), at $r\ r_p$.

Indeed, for all values of n and n′, near the planet orbit ($r\ r_p$) we have

$$\Lambda^{'}(r_p) = \Lambda(r_p)$$

Our model preserves the physical properties of the rate of specific angular momentum function, which is the fact that the moment is maximum close to the planet and decreases progressively when it departs from this one.

The torque exerted on a planet can be given by the impulse approximation (Papaloizou and Lin, 1984). A loss of angular momentum for the planet is due to the interaction between the external gas and the planet which is overtaken by the planet, while a net gain in angular momentum is due to the gas in the interior part which is overtaken by the planet. The total torque will be the sum of all the torques and depend on the structure of the disk. We took values for n and n′ in such a way that the new function approximates the real function as much as possible.

The purpose of the present paper is to obtain an approximate solution of the evolution equation, given by (Lin and Papaloizou, 1986).

$$\frac{\partial\Sigma(r,t)}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[3\sqrt{r}\frac{\partial}{\partial}\left(v\Sigma(r,t)\sqrt{r}\right) - 2\frac{\Lambda(r)r^{\frac{3}{2}}}{\sqrt{GM_*}}\Sigma(r,t)\right] \tag{3}$$

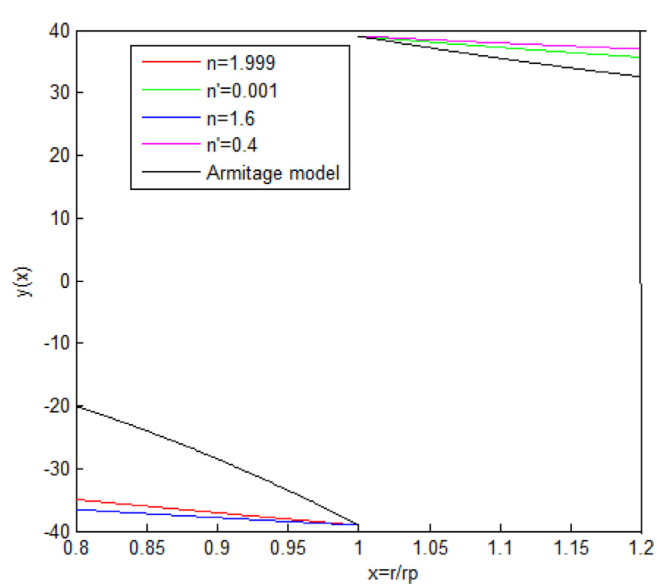

Fig. 1. The rates of specific angular momentum of Armitage model and our model near the orbit of the planet.

Here $\Sigma(\mathrm{r},\mathrm{t})$ is the disk surface density in a cylindrical coordinate density.

where, the torque $\Lambda(\mathrm{r})$ is given by (1)

To deal with the torque term $\Lambda(\mathrm{r})$ we shall use a polynomial-type approximation form, in order to obtain approximate analytical solution of the evolution equation. Therefore, we use an approximation for the torque term by taking (2).

Such an approximation is a good approximation in the neighborhood of the planet orbit $\mathrm{r_p}$, or for thin disk $1 \gg \mathrm{H}$ if $1 < n < 2$ and $\mathrm{n}' < 1$, (see Fig. 1). We do not claim that our approximate analytical solution is accurate everywhere, but this approximation turns to become a good tool for studying evolution of planet orbit.

2.1. Model for planetary migration

We now give our simplified model with an evolution which is little different from the standard form used in literature, by a new approximate function $\Lambda'(\mathrm{r})$which has been obtained in the last section. In our model the evolution of the protoplanetary disks is characterised by the viscous transport of angular momentum. The planets migrate under the influence of the tidal interaction with the disk which is subject to tidal torques from planets. We assume that the shape of the disc have cylindrical symmetry so all equations are expressed in a cylindrical coordinate system. The evolution equation of a protoplanetary disk and a planet is described by the new form given by

$$\frac{\partial\Sigma(r,t)}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[3\sqrt{r}\frac{\partial}{\partial}\left(v\Sigma(r,t)\sqrt{r}\right) - 2\frac{\Lambda'(r)r^{\frac{3}{2}}}{\sqrt{GM_*}}\Sigma(r,t)\right]$$

Here $\Sigma(\mathrm{r},\mathrm{t})$ is the disk surface density in a cylindrical coordinate density, t is time, $v = \mathrm{sr^n}$, $\mathrm{n} < 2$, is the friction coefficient per unit density or kinematic viscosity, and $\mathrm{M_*}$ is the stellar mass. The ordinary viscous evolution of the disk is given by the first term on the right-hand side describes (Lynden-Bell and Pringle,1974; Pringle, 1981), and the second term describes the effect of the planetary torque.

We shall limit ourselves to the stationary regime

$$\frac{\partial\Sigma(r,t)}{\partial t} = -\lambda\Sigma(r,t)$$

Where λ is some constant, so we requires

$$\Sigma(\mathrm{r},\mathrm{t}) = \exp(-\lambda\mathrm{t})\phi(\mathrm{r}) \tag{4}$$

Where $\phi(\mathrm{r})$ is a function depending only on r. Substituting, in (3),$\Lambda'(\mathrm{r})$ by its expression and setting $\alpha = \frac{\mathrm{q^2}\sqrt{\mathrm{GM_*}}}{3\mathrm{s}}\left(\frac{\mathrm{r_p}}{\mathrm{H}}\right)^4$, $\mathrm{N} = 1 - \frac{\mathrm{n}}{2} > 0$ and $\mathrm{N}' = 1 - \frac{\mathrm{n}'}{2} > 0$

we get a homogeneous second order differential equations, for $r < r_p$

$$r^2\phi''(r) + \left[2n + \frac{3}{2} - \alpha r^N\right]r\phi'(r) + \left[n^2 + \frac{n}{2} - \alpha\left(1 + \frac{n}{2}\right)r^N + \frac{\lambda}{3s}r^{2N}\right]\phi(r) = 0 \tag{5}$$

and for $r > r_p$

$$r^2\phi''(r) + \left[2n' + \frac{3}{2} + \alpha r^{N'}\right]r\phi'(r) + \left[n'^2 + \frac{n'}{2} + \alpha\left(1 + \frac{n'}{2}\right)r^{N'} + \frac{\lambda}{3s}r^{2N'}\right]\phi = 0 \tag{6}$$

However, this last equation cannot be immediately solved. Thus in the domain $r < r_p$ we define the new variable $Nx = \alpha r^N$ and after some algebra we have

$$(2-n)x\phi''(x) + [3(n+1) - (2-n)x]\phi'(x) + \left[\frac{\lambda}{3\alpha^2 s}(2-n)x + 2\frac{n(2n+1)}{(2-n)x} - (2+n)\right]\phi(x) = 0 \tag{7}$$

Substituting the function $\phi(x)$ by a new function $u(x)$ as follows

$$\phi(x) = x^b exp\left(-\frac{x}{2}\right)u(x)$$

With:

$$b = \frac{3}{2}\left(1 - \frac{3}{2-n}\right) < -\frac{3}{4}$$

Inserting into, we get the equation:

$$u''(x) + \left[-\beta + \frac{\gamma}{x} + \frac{\delta}{x^2}\right]u(x) = 0 \tag{8}$$

Where:

$$\beta = \frac{3}{4} + \frac{\lambda}{3\alpha^2 s}$$

$$\gamma = \frac{1}{2}\frac{n-1}{2-n}$$

$$\delta = \frac{(n-1)(n-3)}{4(n-2)^2}$$

Make the new variable $z = 2x\sqrt{\beta}$, we obtain the Whittaker equation (Gradshteyn and Ryzhik, 1980)

$$u''(z) + \left[-\frac{1}{4} + \frac{\kappa}{z} + \frac{\frac{1}{4} - \mu^2}{z^2}\right]u(z) = 0 \tag{9}$$

$$\kappa = \frac{\gamma}{2\sqrt{\beta}} = \left(\frac{n-1}{2-n}\right)\frac{1}{2}\left(3 + \frac{4\lambda}{3\alpha^2 s}\right)^{-\frac{1}{2}}$$

$$2\mu = \sqrt{1 - 4\delta} = \frac{1}{2-n}$$

The formal solution of this differential equation is of the form:

$$u(x) = C_1 M_{\kappa,\mu}\left(2x\sqrt{\beta}\right) + C_2 W_{\kappa,\mu}\left(2x\sqrt{\beta}\right) \tag{10}$$

where $M_{\kappa,\mu}(z)$, $W_{\kappa,\mu}(z)$ are the well-known Whittaker functions,C_1 and C_2 are two constants depending on boundary conditions at $r = 0$ and $r = \infty$. In the original function $\phi(r)$ we get for $r < r_p$

$$\Sigma(r,t) = \left(\frac{\alpha r^N}{N}\right)^b exp(-\lambda t)exp\left(-\frac{\alpha r^N}{2N}\right)\left[C_1 M_{\kappa,\mu}\left(\frac{2\alpha}{N}\sqrt{\beta}r^N\right) + C_2 W_{\kappa,\mu}\left(\frac{2\alpha}{N}\sqrt{\beta}r^N\right)\right] \tag{11}$$

And for $r > r_p$

$$\Sigma(r,t) = \left(-\frac{\alpha r^{N'}}{N'}\right)^b exp(-\lambda t)exp\left(\frac{\alpha r^{N'}}{2N'}\right)\left[C'_1 M_{\kappa,\mu}\left(-\frac{2\alpha}{N'}\sqrt{\beta}r^{N'}\right) + C'_2 W_{\kappa,\mu}\left(-\frac{2\alpha}{N'}\sqrt{\beta}r^{N'}\right)\right] \tag{12}$$

where C'_1 and C'_2 are two constants depending on boundary conditions at $r = 0$ and $r = \infty$.

3. Boundary conditions

The dynamics of the accretion disk described by the differential equation, is an initial value problem, and generally the boundary condition has a influence on the global solution. So, the boundary conditions imposed on the accretion disk are important. The outer boundary $r \to \infty$ is a freely expanding surface.

We consider different boundary conditions; zero stress or no accretion at the inner boundary, $r = 0$, and zero mass inflow at the outer boundary $r \to \infty$.

First of all we have the limiting cases as $z \to 0$ (Gradshteyn and Ryzhik, 1980)

$$M_{\kappa,\mu}(z) \sim z^{\frac{1}{2}+\mu} \tag{13}$$

$$W_{\kappa,\mu}(z) \sim \frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2}+\mu-\kappa\right)} z^{\frac{1}{2}-\mu} + \frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2}-\mu+\kappa\right)} z^{\frac{1}{2}+\mu} \tag{14}$$

And as $z \to \infty$

$$M_{\kappa,\mu}(z) \sim \frac{\Gamma(1+2\mu)}{\Gamma\left(\frac{1}{2}+\mu-\kappa\right)} e^{\frac{1}{2}z} z^{-\kappa} \tag{15}$$

$$W_{\kappa,\mu}(z) \sim e^{-\frac{1}{2}z} z^{\kappa} \tag{16}$$

And we remark that:

$$\frac{3}{4} < \frac{1}{2} + \mu < 1$$

$$0 < \frac{1}{2} - \mu < \frac{1}{4}$$

Then because the Whittaker function $W_{\kappa,\mu}(z)$ has an exponential damping in the domain $r > r_p$ we put:

$$\Sigma(r,t) = C'\left(-\frac{\alpha r^{N'}}{N'}\right)^{b} exp(-\lambda t) exp\left(\frac{\alpha r^{N'}}{2N'}\right) W_{\kappa,\mu}\left(-\frac{2\alpha}{N'}\sqrt{\beta} r^{N'}\right) \tag{17}$$

And for $r < r_p$ we have:

$$\Sigma(r,t) = \left(\frac{\alpha r^{N}}{N}\right)^{b} exp(-\lambda t) exp\left(-\frac{\alpha r^{N}}{2N}\right)\left[C_1 M_{\kappa,\mu}\left(\frac{2\alpha}{N}\sqrt{\beta} r^{N}\right) + C_2 W_{\kappa,\mu}\left(\frac{2\alpha}{N}\sqrt{\beta} r^{N}\right)\right]$$

Then for $r \to 0$ we get

$$\Sigma(r,t) \approx \left(\frac{\alpha}{N}\right)^{b} exp(-\lambda t) exp\left(-\frac{\alpha r^{N}}{2N}\right)[C_1 u_1 + C_2 u_2] \tag{18}$$

Where:

$$u_1 = \left(\frac{2\alpha}{N}\sqrt{\beta}\right)^{\frac{1}{2}+\mu} r^{N\left(b+\frac{1}{2}+\mu\right)} \tag{19}$$

$$u_2 = \frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2}+\mu-\kappa\right)}\left(\frac{2\alpha}{N}\sqrt{\beta}\right)^{\frac{1}{2}-\mu} r^{N\left(b+\frac{1}{2}-\mu\right)} + \frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2}-\mu+\kappa\right)}\left(\frac{2\alpha}{N}\sqrt{\beta}\right)^{\frac{1}{2}+\mu} r^{N\left(b+\frac{1}{2}+\mu\right)} \tag{20}$$

But the term $r^{N\left(b+\frac{1}{2}+\mu\right)}$ tend to infinity at $r = 0$we put then $C_2 = 0$.

Finally we have for $r < r_p$:

$$\Sigma(r,t) = C\left(\frac{\alpha r^{N}}{N}\right)^{b} exp(-\lambda t) exp\left(-\frac{\alpha r^{N}}{2N}\right) M_{\kappa,\mu}\left(\frac{2\alpha}{N}\sqrt{\beta} r^{N}\right) \tag{21}$$

And for $r > r_p$ we get

$$\Sigma(r,t) = C'\left(-\frac{\alpha r^{N'}}{N'}\right)^{b} exp(-\lambda t) exp\left(\frac{\alpha r^{N'}}{2N'}\right) W_{\kappa,\mu}\left(-\frac{2\alpha}{N'}\sqrt{\beta} r^{N'}\right) \tag{22}$$

4. Concluding remarks

In this work, we have obtained an analytical solution of the evolution equation of a protoplanetary disk and a planet instead the numerical solution. From this, we have computed the mass density $\sum$ which represented by Whittaker function.

Whence the importance of the second term in the right of the evolution equation, which describes the effect of the planetary torque, specially near the planet orbit, we have suggested an approximate rate $\Lambda'(\mathrm{r})$ of the specific angular momentum which have the same shape of $\Lambda(\mathrm{r})$ near the planet orbit. The choice of $\Lambda'(\mathrm{r})$ is based on the idea of viscosity expression proposed earlier by (Lynden-Bell and Pringle,1974).

Our an analytical solution of the evolution equation allows us to contribute to the study of planetary migration.

In a forthcoming works we will investigate the effect of this approximation on the orbital migration of the planet described by the following equation:

$$\frac{dr_p}{dt} = -Cexp(-\lambda t)\left(\frac{4\pi}{M_p}\right)\sqrt{\frac{r_p}{GM_*}}\omega_1 - C'exp(-\lambda t)\left(\frac{4\pi}{M_p}\right)\sqrt{\frac{r_p}{GM_*}}\omega_2 \tag{23}$$

Where:

$$\omega_1 = \int_0^{r_p} r\Lambda'(r)\left(\frac{\alpha}{N}\right)^{-\frac{3n+1}{22-n}} r^{-\frac{3}{4}(n+1)} exp(-\lambda t) exp\left(-\frac{\alpha r^{N}}{2N}\right) M_{\kappa,\mu}\left(\frac{2\alpha}{N}\sqrt{\beta} r^{N}\right) dr \tag{24}$$

And:

$$\omega_2 = \int_0^{r_p} r\Lambda'(r)\left(-\frac{\alpha}{N'}\right)^{-\frac{3n'+1}{22-n'}} r^{-\frac{3}{4}(n'+1)} exp(-\lambda t) exp\left(\frac{\alpha r^{N'}}{2N'}\right) \times M_{\kappa,\mu}\left(-\frac{2\alpha}{N'}\sqrt{\beta} r^{N'}\right) dr \tag{25}$$

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

Armitage, P.J., Livio, M., Lubow, S.H., Pringle, J.E., 2002. Predictions for the frequency and orbital radii of massive extrasolar planets. MNRAS 248–256, 334.

Baruteau, C., Crida, A., Paardekooper, S.J., et al., 2013.Planet-Disc Interactions and Early Evolution of Planetary Systems.arXiv: 1312.4293.

Baruteau, C., Massey, Q., 2013. In: Tidal effects in Astronomy and Astrophysics. Lecture Notes in Physics. 201, 861 (Springer).

Crida, A., Morbidelli, A., 2007.Cavity opening by a giant planet in a protoplanetary disc and effects on planetary migration.MNRAS.1324-1336, 377.

Edgar, R. G., 2007.Giant Planet Migration in Viscous Power-Law Discs.ApJ.1325-1334, 663.

Edgar, R. G., 2008.Type II Migration: Varying Planet Mass and Disc Viscosity. arXiv:0807.0625.

Gradshteyn, I.S., Ryzhik, I.M., 1980. In: Table of Integrals, Series and Products.

Kley, W., Nelson, R.P., 2012. Planet-Disk Interaction and Orbital Evolution. ARA&A. 211-249, 50.

Laughlin, G., Steinacker, A., Adams, F.C., 2004. Type I Planetary Migration with MHD Turbulence. AJ. 489-496, 608.

Lin, D. N. C., Papaloizou, J., 1986. On the tidal interaction between protoplanets and the protoplanetary disk. III – Orbital migration of protoplanets. ApJ. 846-857,309.

Lynden-Bell, D., Pringle, J.E., 1974. The evolution of viscous discs and the origin of the nebular variables. MNRAS 603–637, 168.

Paardekooper, S.J., 2014.Dynamical corotation torques on low-mass planets. MNRAS. 2031-2042, 444.

Paardekooper, S. J., Baruteau, C., Crida, A., Kley, W., 2010. A torque formula for non-isothermal type I planetary migration – I. Unsaturated horseshoe drag. MNRAS. 1950-1964, 401.
Paardekooper, S.J., Baruteau, C., Kley, W., 2011. A torque formula for non-isothermal Type I planetary migration – II. Effects of diffusion. MNRAS.293-303, 410.
Papaloizou, J., Lin, D.N.C., 1984.On the tidal interaction between protoplanets and the primordial solar nebula. I – Linear calculation of the role of angular momentum exchange.ApJ.818-834,285.
Pringle, J.E., 1981. Accretion discs in astrophysics. ARA&A. 137–162, 19.
Trilling, D.E., Benz, W., Guillot, T., Lunine, J.I., Hubbard, W.B., Burrows, A., 1998. Orbital Evolution and Migration of Giant Planets: Modeling Extrasolar Planets. ApJ.428-439, 500.
Ward, W. R., 1997.Protoplanet Migration by Nebula Tides.ICARUS.261-281,126.
Ward, W. R., 1982. Comments of the long-term stability of the Earth's obliquity. ICARUS. 444–448, 50.