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ORIGINAL ARTICLE

HAM solution of some initial value problems arising in heat radiation equations

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Abstract Mathematical modeling of many phenomena, especially in heat transfer, usually leads to a nonlinear equation. Traditional approaches for solving such equations are time consuming and difficult affairs tasks.

In this paper, based on the homotopy analysis method (HAM), a series solution for the problem of unsteady nonlinear convective–radiative equation is obtained.

In HAM, one would be able to control the convergence of approximation series and adjust its convergence region, conveniently. Ability and efficiency of proposed approach are tested via some cases of above mentioned problem.

It is found that homotopy analysis approach provides a greatly accelerated convergence series solution for problem.

1. Introduction

Liao proposed the homotopy analysis method (HAM) in 1992, to get analytic approximations of highly nonlinear equations (Liao, 1992).

Unlike other existing methods, the HAM:

- Provides us a simple way to ensure the convergence of solution series.
- Provides great freedom to choose proper base functions.

These advantages point out the method as a powerful and flexible tool in mathematics and engineering, which can be readily distinguished from existing numerically and analytically methods.

This paper is arranged as follows; in Section 2 the basic idea of standard HAM and some its recent optimal

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modification are reviewed. In Section 3, the implementation of HAM on problem and some comparison discussions are presented. Finally, conclusions are drawn in Section 4.

2. Standard HAM and some its optimal modifications

Using the concept of homotopy, Liao (1992) introduced the early form of the homotopy analysis method (HAM) for a given nonlinear differential equation

$$N[u(x)] = 0, \tag{1}$$

as

$$(1-p)L[U(x;p) - u_0(x)] = -pN[U(x;p)],\ p \in [0,1]. \tag{2}$$

where L is an auxiliary linear operator and $u_0(x)$ is an initial guess of the solution. It is evident that, at $p = 0$ and $p = 1$, one has $U(x;0) = u_0(x)$ and $U(x;1) = u(x)$, respectively.

In the view of HAM the solution of original equation is assumed to be as the power series in p as

$$u(x) = u_0(x) + \sum_{i=0}^{\infty} u_i(x)p^i. \tag{3}$$

As proved by Liao (2003), whereas (3) be convergent at $p = 1$, its limit must satisfy the original Eq. (1).

Liao introduced more artificial degrees of freedom by using the zeroth-order deformation equation in the following form (Liao, 1997a)

$$(1-p)L[U(x;p) - u_0(x)] - c_0 pN[U(x;p)] = 0,\ p \in [0,1]. \tag{4}$$

Appling recently proposed "mth-order homotopy-derivative operator" (Liao, 2009)

$$D_m(\phi) = \frac{1}{m!}\frac{\partial^m \phi}{\partial p^m}\bigg|_{p=0}.$$

to the both sides of (4), one reads

$$L[u_m(x) - \chi_m u_{m-1}(x)] = c_0 R_{m-1}(x), \tag{5}$$

where

$$R_{m-1}(x) = D_{m-1}(N[u(x;p)]) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[u(x;p)]}{\partial p^{m-1}}\bigg|_{p=0},$$

and

$$\chi_m = \begin{cases} 1, & m > 1, \\ 0, & m \leqslant 1. \end{cases}$$

In this way, the component solutions of $u_m, m \geqslant 1$, are not only dependent upon x but also the auxiliary parameter c_0.

As it is known, to find a proper convergence-control parameter c_0, to get a convergent series solution or to get a faster convergent one, there is a classic way of plotting the so-called "c_0-curves" or "curves for convergence-control parameter". For example, one can consider the convergence of $u'(x)$ and $u''(x)$ of a nonlinear differential equation $N[u(x)] = 0$ to find a region say R_h so that, each $c_0 \in R_h$ gives a convergent series solution of such kind of quantities. Such a region can be found, although approximately, by plotting the curves of these unknown quantities versus c_0.

However, it is a pity that curves for convergence-control parameter (i.e. c_0-curves) give us only a graphically region and cannot tell us which value of gives the fastest convergent series. Recently in Mehmood et al. (2010), a misinterpreted usage of c_0-curves has reported. To find the optimal value for convergence-control parameter c_0, an optimal homotopy analysis approach has been presented in Liao (2010).

3. Appling method and comparison discussions

In this section some numerical experiments are provided to illustrate the validity of HAM approach described in Section 2.

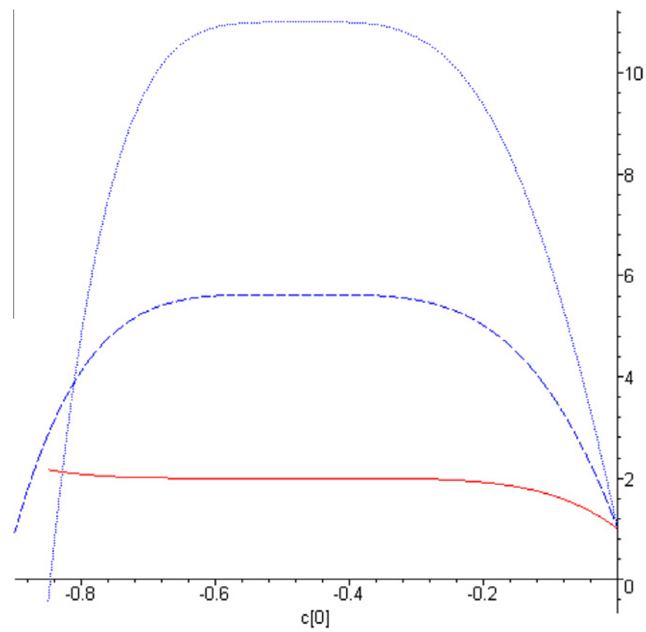

Figure 1 The c_0-curves for 5th-order of HAM approximation of $u''(0)$, for $\varepsilon_1 = 1$ and different values of ε_2. Solid line: $\varepsilon_2 = 1$; dashed line: $\varepsilon_2 = 2$; dotted line: $\varepsilon_2 = 3$.

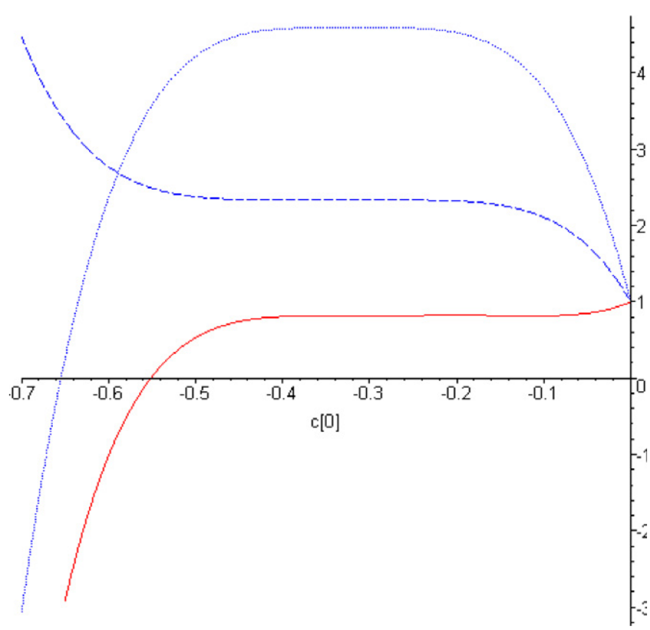

Figure 2 The c_0-curves for 5th-order of HAM approximation of $u''(0)$, for $\varepsilon_1 = 2$ and different values of ε_2. Solid line: $\varepsilon_2 = 1$; dashed line: $\varepsilon_2 = 2$; dotted line: $\varepsilon_2 = 3$.

We consider the problem of unsteady nonlinear convective–radiative equation, which in dimensionless form described as following initial value problem.

$$[1+\varepsilon_1 u(t)]\frac{du(t)}{dt}+u(t)+\varepsilon_2 u^4(t)=0, \quad u(0)=1. \tag{6}$$

Recently, many authors have taken into consideration of different values of $\varepsilon_1, \varepsilon_2$ in (6) using different methods, some are discussed in Liao (1997b), Domairry and Nadim (2008), Ganji et al. (2007), Abbasbandy (2006, 2007), Marinca and Herişanu (2008) and Sajid and Hayat (2008).

Considering Eq. (6), we define

$$N(\phi)=[1+\varepsilon_1\phi]\frac{d\phi}{dt}+\phi+\varepsilon_2\phi^4, \tag{7}$$

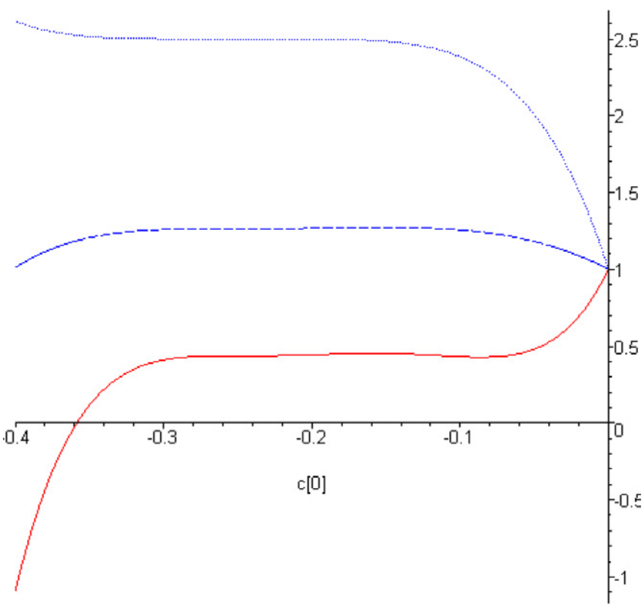

Figure 3 The c_0-curves for 5th-order of HAM approximation of $u''(0)$, for $\varepsilon_1=3$ and different values of ε_2. Solid line: $\varepsilon_2=1$; dashed line: $\varepsilon_2=2$; dotted line: $\varepsilon_2=3$.

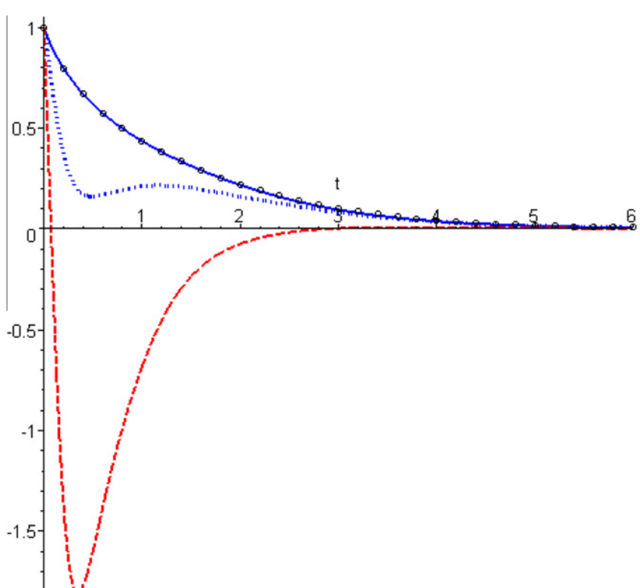

Figure 4 Comparison between the results obtained by the different method for $\varepsilon_1=2, \varepsilon_2=3$. Hollow symbols: numerical solution; solid line: present MHAM for $h=-1/3$; dashed line: HPM; dotted line: HAM for $h=-0.8$.

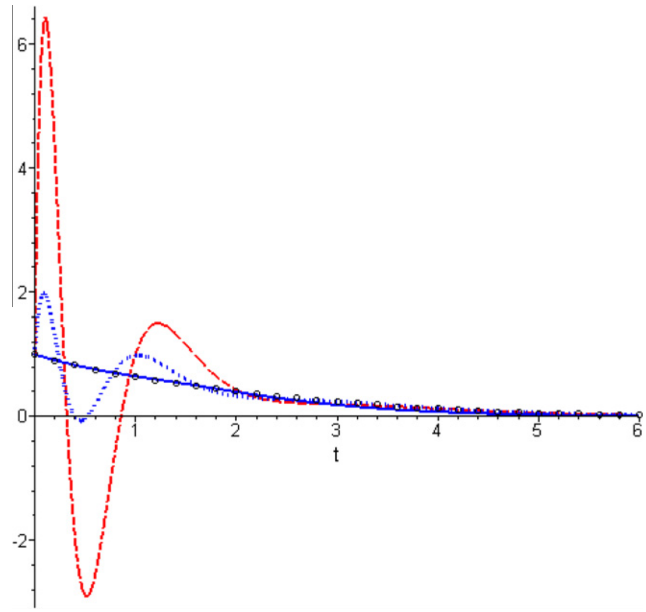

Figure 5 Comparison between the results obtained by the different method for $\varepsilon_1=3, \varepsilon_2=1$. Hollow symbols: numerical solution; solid line: present MHAM for $h=-1/4$; dashed line: HPM; dotted line: HAM for $h=-0.8$.

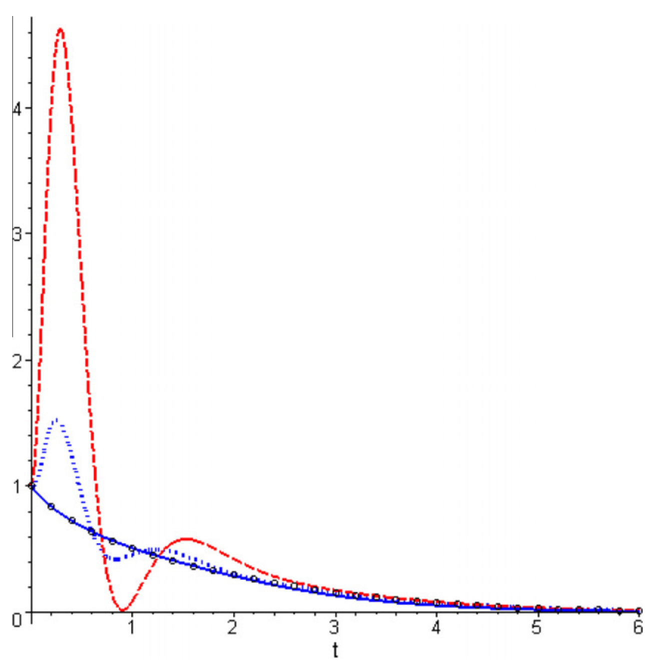

Figure 6 Comparison between the results obtained by the different method for $\varepsilon_1=3, \varepsilon_2=3$. Hollow symbols: numerical solution; solid line: present MHAM for $h=-1/4$; dashed line: HPM; dotted line: HAM for $h=-0.8$.

Table 1 Comparisons of Δ_m of 5th-order solutions of different approaches for $\varepsilon_1 = 1$.

	$\varepsilon_2 = 1$	$\varepsilon_2 = 2$	$\varepsilon_2 = 3$
MHAM for $h = -1/2$	0.00000645794	0.00000225316	0.00185417178
HAM for $h = -0.8$ Abbasbandy (2007)	0.00001294614	0.06332372453	4.39671710926
HAM for $h = -0.9$ Abbasbandy (2007)	0.00026131005	0.45943918392	270.979958125
HPM (HAM for $h = -1$)	0.00278084263	2.08587335265	358312.798216

Table 2 Comparisons of Δ_m of 5th-order solutions of different approaches for $\varepsilon_1 = 2$.

	$\varepsilon_2 = 1$	$\varepsilon_2 = 2$	$\varepsilon_2 = 3$
MHAM for $h = -1/3$	0.0016937936	0.00043080927	0.00018897928
HAM for $h = -0.8$ Abbasbandy (2007)	1.7740609725	0.29707799102	7.73726355555
HAM for $h = -0.9$ Abbasbandy (2007)	13.500883632	2.52627773025	24.9031191524
HPM (HAM for $h = -1$)	86.150467589	18.0667149197	284.907280951

Table 3 Comparisons of Δ_m of 5th-order solutions of different approaches for $\varepsilon_1 = 3$.

	$\varepsilon_2 = 1$	$\varepsilon_2 = 2$	$\varepsilon_2 = 3$
MHAM for $h = -1/4$	0.2251073087	0.007103244505	0.00301374746
HAM for $h = -0.8$ Abbasbandy (2007)	741.25718288	527.1191196254	140.804634925
HAM for $h = -0.9$ Abbasbandy (2007)	9333.2489875	15082.47186259	3983.73926254
HPM (HAM for $h = -1$)	329295.21567	14603605.62548	314587.045985

and

$$L(u) = u' + u \tag{8}$$

as the auxiliary linear and nonlinear operators, respectively.

Subject to the initial condition (6) and linear operator (8), the temperature $u(t)$ can be expressed by the following set of base functions

$$\{\exp(-kt)|k > 0\},$$

as the following series

$$u(t) = \sum_{k=0}^{m} a_{m,k} \exp(-kt).$$

So, it is evident that initial guess should be as $\theta_0(x) = e^{-t}$.

We first construct the zeroth-order deformation equation defined by

$$(1-p)L[u(t,p) - u_0(t)] = phN[u(t,p)]. \tag{9}$$

According to (5) and from initial condition (6), we have

$$L[u_m(t) - \chi_m u_{m-1}(t)] = hR_m(t). \tag{10}$$

subject to initial conditions

$$u_m(0) = 0, \tag{11}$$

where

$$R_m(t) = u'_m(x) + \varepsilon_1 \sum_{i=0}^{m} u'_i(t) u_{m-i}(t) + u_m(t) + \varepsilon_2 \sum_{i=0}^{m-1} \left(\sum_{j=0}^{m-1-i} \left(\sum_{k=0}^{m-1-i-j} u_k(t) u_{m-1-i-j-k}(t) \right) u_j(t) \right) u_i(t).$$

The corresponding mth-order deformation equation reads

$$u_m(t) = u_m^*(t) + Ce^{-t}.$$

where $u_m^*(t)$ is the particular solution of (10) and the constant C are determined by the boundary conditions (11).

To obtain the valid values of c_0, we have plotted the so-called c_0-curve of 5th-order HAM approximation of $u''(0)$ for different values of ε_2 in cases of $\varepsilon_1 = 1$, $\varepsilon_1 = 2$ and $\varepsilon_1 = 3$ in Figs. 1–3, respectively.

Comparisons are made between HAM and the homotopy perturbation method, HPM (HAM with $c_0 = -1$) and numerical results of the fourth order Runge–Kutta method, for some values of $\varepsilon_1, \varepsilon_2$ are plotted in Figs. 4–6. In Tables 1–3, results of comparisons between evaluated square residual error, similar to what was used in Niu and Wang (2010), as:

$$\Delta_m = \int_{\Omega} \left(N\left[\sum_{i=0}^{m} u_i(x) \right] \right)^2 dx, \tag{12}$$

of different approaches are shown.

From Tables 1–3, it seems that proposed approach gives better approximations than the compared approaches and its results are in good agreement with the exact solution.

Since the results of HPM can be obtained as a special case of HAM when $c_0 = -1$, from Figs. 1–3, it is evident that HPM losses its validity for relatively large values of ε_1.

4. Concluding remarks

In this paper a initial value problem in heat transfer has been solved by means of homotopy analysis method are proposed. It is shown that HAM results are in a good agreement with numerical solution of problem. It was shown that homotopy analysis method provides a simple way to control and adjust the convergence regions of solution. It is also pointed out that HPM solutions are not converging in some cases of the understudied problem. Computations are performed by Maple11.

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