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ORIGINAL ARTICLE

Analytical solution of Volterra's population model

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Abstract In this paper, we used an efficient algorithm to obtain an analytic approximation for Volterra's model for population growth of a species in a closed system. The numerical solutions are obtained by combining homotopy perturbation method (HPM) and Padé technique. The approximate solutions are shown graphically. The results show that HPM–Padé tecnique is an appropriate method in solving the nonlinear equations.
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1. Introduction

The homotopy perturbation method was first proposed by the Chinese mathematician Ji-Huan He (1999, 2003, 2006a). This technique has been employed to solve a large variety of linear and nonlinear problems. The interested reader can see the Refs. (Mohyud-Din et al., 2009; Noor and Mohyud-Din, 2007, 2008; Dehghan and Shakeri, 2007, 2008a,b; Saadatmandi et al., 2009; Yıldırım, 2008a,b, in press; Achouri and Omrani, in press; Ghanmi et al., in press; Shakeri and Dehghan, 2008; He, 2008a,b, 2006b,c) for more applications of the homotopy perturbation method in various problems of physics and engineering. Briefly, in He's homotopy perturbation method; the homotopy with an imbedding parameter $p \in [0, 1]$ is constructed, and the imbedding parameter is considered as a "small parameter", so the method is called homotopy perturbation method and is proceed as the standard perturbation method but taking the full advantage of the traditional perturbation methods and the homotopy techniques. The main merit of the homotopy perturbation method is that the perturbation equation can be easily constructed (therefore is problem dependent) by homotopy in topology and the initial approximation can also be freely selected. One of the most remarkable features of the HPM is that usually just a few perturbation terms are sufficient for obtaining a reasonably accurate solution.

In this study, we extend the homotopy perturbation method to obtain appproximate solutions of the Volterra's model for population growth (Scudo, 1971) of a species within a closed system. The model is characterized by the nonlinear Volterra integro-differential equation

$$\kappa \frac{du}{dt} = u - u^2 - u \int_0^t u(x)dx, \quad u(0) = 0.1, \tag{1}$$

where $u \equiv u(t)$ is the scaled population of identical individuals (Small, 1989) at a time t and κ is a prescribed parameter. The nondimensional parameter $\kappa = c/(ab)$, where $a > 0$ is the birth rate coefficient, $b > 0$ is the crowding coefficient, and $c > 0$ is

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the toxicity coefficient (TeBeest, 1997). The coefficient c indicates the essential behaviour of the population evolution before its level falls to zero in the long run. Volterra introduced this model for a population $u(t)$ of identical individuals which exhibits crowding and sensitivity to the amount of toxins produced. The nonlinear model (1) includes the well-known terms of a logistic equation, and in addition it includes an integral term that characterizes the accumulated toxicity (Small, 1989; TeBeest, 1997) produced since time zero. Recently Wazwaz (Wazwaz, 1999) used Adomian decomposition method for solving the governing problem.

2. Solution of the problem by He's homotopy perturbation method

In this section we consider the population growth model characterized by nonlinear Volterra integro-differential equation

$$\frac{du}{dt} = 10(t) - 10u^2(t) - 10u(t)\int_0^t u(x)dx, \quad u(0) = 0.1, \tag{2}$$

To solve Eq. (2) by homotopy perturbation method, we construct the following homotopy:

$$\frac{du}{dt} = p\left\{10u(t) - 10u^2(t) - 10u(t)\int_0^t u(x)dx\right\}, \tag{3}$$

with the initial condition

$$u(0) = 0.1, \tag{4}$$

Assume the solution of Eq. (3) to be in the form:

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 + \ldots \tag{5}$$

Substituting Eq. (5) into Eq. (3) and collecting terms of the same power of p give

$$p^0 : u_0(t) = 0.1,$$

$$p^1 : u_1(t) = u_0^2(t) + u_0(x)u_0(t)t,$$

$$p^2 : u_2(t) = 2u_0(t)u_1(t) + [u_0(x)u_1(t) + u_1(x)u_0(t)]t,$$

$$p^3 : u_3(t) = 2u_1(t)u_2(t) + 2u_0(t)u_3(t) + [u_0(x)u_3(t) + u_1(x)u_2(t) + u_2(x)u_1(t) + u_3(x)u_0(t)]t\ldots$$

The solution reads

$$u_0(t) = 0.1,$$

$$u_1(t) = 0.9t - 0.05t^2,$$

$$u_2(t) = 3.6t^2 - \frac{7}{12}t^3 + \frac{1}{60}t^4,$$

$$u_3(t) = 6.9t^3 - 3.154166668t^4 + 0.2425t^5 - 0.004722222223t^6, \ldots$$

The components u_4, u_5, u_6, u_7 and u_8 were also determined and will be used, but for brevity not listed. This completes the formal determination of the approximation of $u(t)$ given by

$$u(t) = 0.1 + 0.9t + 3.55t^2 + 6.316666667t^3 - 5.5375t^4 - 63.70916667t^5 - 156.0804167t^6 - 18.47323411t^7 + 1056.288569t^8 + O(t^9) \tag{6}$$

3. Converting to a nonlinear ODE

In this section, it will be useful to convert the Volterra's population model (2) to an equivalent nonlinear ODE. In order to convert (2) to an ODE, we set

$$y(t) = \int_0^t u(x)dx. \tag{7}$$

This transformation readily leads to

$$y'(t) = u(t), \tag{8}$$

$$y''(t) = u'(t). \tag{9}$$

Inserting Eqs. (7)–(9) into Eq. (2) yields the nonlinear differential equation

$$y''(t) = 10y'(t) - 10(y'(t))^2 - 10y(t)y'(t) \tag{10}$$

with initial conditions

$$y(0) = 0, \tag{11}$$

$$y'(0) = 0.1, \tag{12}$$

obtained by using Eqs. (7) and (8) respectively.

$$p^0 : y_0(t) = 0.1t,$$

$$p^1 : y_1(t) = (y_0')^2(t) + y_0(t)y_0'(t)t,$$

$$p^2 : y_2(t) = 2y_0'(t)y_1'(t) + [y_0'(t)y_1(t) + y_1'(t)y_0(t)]t,$$

$$p^3 : u_3(t) = 2y_0'(t)y_2'(t) + (y_1')^2(t) + [y_0'(x)y_2(t) + y_1'(t)y_1(t) + y_2'(t)y_0(t)]t, \ldots$$

The solution reads

$$y_0(t) = 0.1t,$$

$$y_1(t) = 0.45t^2 - \frac{1}{60}t^3,$$

$$y_2(t) = \frac{6}{5}t^3 - \frac{7}{48}t^4 + \frac{1}{300}t^5,$$

$$y_3(t) = \frac{69}{40}t^4 - \frac{757}{1200}t^5 - \frac{97}{2400}t^6 - \frac{17}{25,200}t^7, \ldots$$

The components y_4, y_5, y_6, y_7 and y_8 were also determined and will be used, but for brevity not listed. Recall that

$$u(t) = y'(t).$$

This means that the approximation of the solution $u(t)$ of Eq. (2) in a series form is given by

$$u(t) = 0.1 + 0.9t + 3.55t^2 + 6.316666667t^3 - 5.5375t^4 - 63.70916667t^5 - 156.0804167t^6 - 18.47323411t^7 + 1056.288569t^8 + O(t^9),$$

in a complete agreement with the results previously obtained in the previous sections.

Using the approximation obtained for $u(t)$ in Eq. (6), we find

$$[4/4] = \frac{0.1 + 0.4687931695t + 0.9249573236t^2 + 0.9231293234t^3 + 0.4004233108t^4}{1 - 4.312068305t + 12.55818798t^2 - 13.88064046t^3 + 10.86830522t^4}.$$

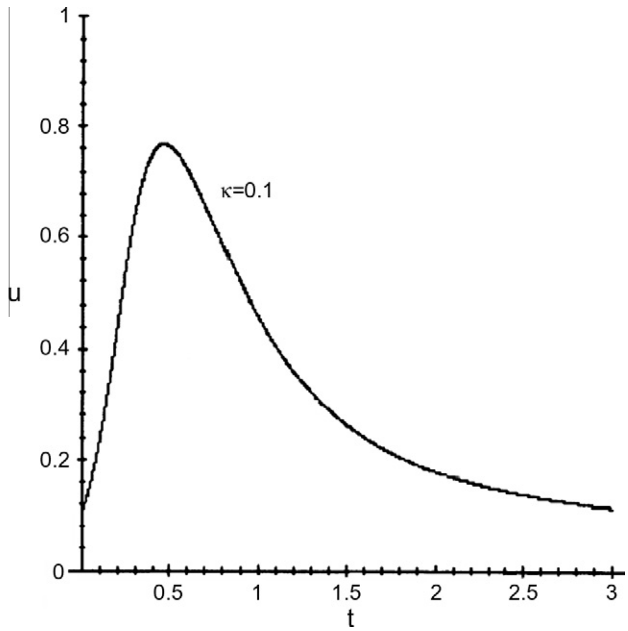

Figure 1 Relation between Pade's approximants [4/4] of $u(t)$ and t for $u(0) = 0.1$, $\kappa = 0.1$.

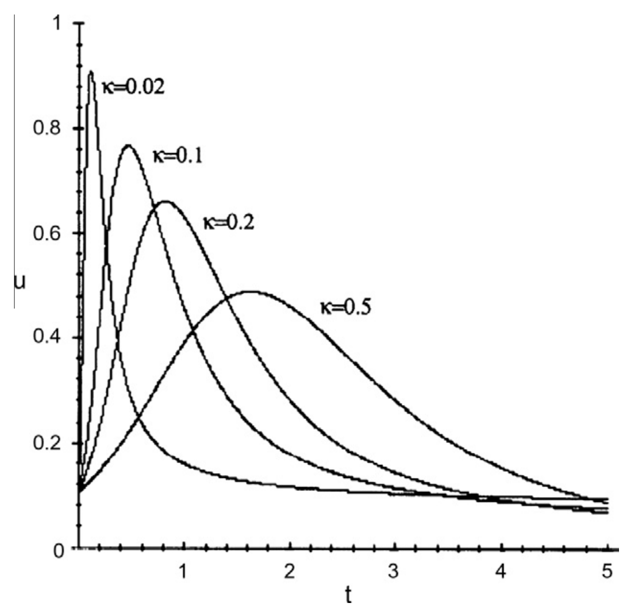

Figure 2 Relation between Pade's approximants [4/4] of $u(t)$ and t for $u(0) = 0.1$, $\kappa = 0.002,\ 0.1,\ 0.2$ and 0.5.

Fig. 1 shows the relation between the Pade approximants [4/4] of $u(t)$ and t. From Fig. 1, we can easily observe that for $u(0) = 0.1$ and $\kappa = 0.1$, we obtain

Table 1 Approximation of u_{max} and exact value of u_{max} for $\kappa = 0.002,\ 0.1,\ 0.2$ and 0.5.

κ	Critical t	Approx. u_{max}	Exact u_{max}
0.02	0.1118454355	0.9038380533	0.923471721
0.04	0.2102464437	0.861240177	0.8737199832
0.1	0.4644767322	0.7651130834	0.76974144907
0.2	0.8168581189	0.6579123080	0.6590503816
0.5	1.6267110031	0.4852823482	0.4851902914

$$u_{max} = 0.7651130834,$$

occurs that

$$t_{critical} = 0.4644767322.$$

Also, Fig. 1 shows the rapid rise along the logistic curve followed by the slow exponential decay after reaching the maximum point.

Fig. 2 shows the Pade approximants [4/4] of $u(t)$ for $u(0) = 0.1$ and for $\kappa = 0.002$ 0.1 0.2 and 0.5. The key finding of this graph is that when κ increases, the amplitude of $u(t)$ decreases, whereas the exponential decay increases.

Table 1 summarizes the relation between κ, u_{max}, and $t_{critical}$. The exact values of u_{max} were evaluated by using

$$u_{max} = 1 + \kappa \ln\left(\frac{\kappa}{1 + \kappa - u_0}\right), \tag{13}$$

obtained by TeBeest (1997).

4. Conclusion

In this paper, an efficient combined method is successfully applied to Volterra's population model. The numerical results show that HPM-Padé technique is an accurate and reliable numerical technique for the solution of the Volterra's population model. This combined method is a very promoting method, which will be certainly found wide applications.

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