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Solving directly third-order ODEs using operational matrices of Bernstein polynomials method with applications to fluid flow equations

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ABSTRACT

In this paper, we adapt for the first time the operational matrices of Bernstein polynomials method for solving directly a class of third-order ordinary differential equations (ODEs). This method gives a numerical solution by converting the equation into a system of algebraic equations which is solved directly. Applications of the present method to the famous Blasius equation describing a boundary layer flow over a flat plate and third-order ODE for thin film flow are presented. Some numerical examples are also given to show the applicability of the method.

1. Introduction

The mathematical formulations of physical phenomena and engineering often lead to ordinary differential equations (ODEs). Finding an approximate analytical solution to an equation has attracted the attention of many researchers (see for example Zhang et al., 2016; Singh et al., 2017; Kumar et al., 2017). There are currently many methods that can provide approximate solutions, and most of these methods first convert the ODEs to a system of lower degree, usually the first-order.

In recent years, several authors have introduced direct methods for solving higher-order ODEs. Majid et al. (2009) used Jocobi iteration and direct methods with a variable step size for solving second-order ODEs. Awoyemi and Idowu (2005) proposed a class of hybrid collocation direct method to solve third-order ODEs. A part from that, power series collocation and interpolation was implemented to derive a 3-step block method for solving ODEs of the third-order by Olabode and Yusuph (2009). Waeleh et al. (2011) developed a block method based on numerical integration and interpolation in order to solve higher-order ODEs where they presented approximation solutions of the fourth- and fifth-orders. Olabode and Alabi (2013) introduced a linear multistep method using interpolation and collocation of power series approximation solution to solve the fourth-order ODEs.

There have been quite a number of papers on applying Bernstein polynomials for solving differential equations. Bernstein polynomials were introduced by Sergi Bernstein in 1912 (see Lorentz, 2012). Yousefi and Behroozifar (2010) employed operational matrices of Bernstein polynomials for solving differential equations. Pandey and Kumar (2012) used Bernstein operational matrices to solve Emden type equations. Alshbool et al. (2015) introduced approximation solutions for singular differential equations using Bernstein polynomials. A numerical solution for the variable order linear cable equation using Bernstein polynomials was presented by Chen et al. (2014). Bellucci (2014) introduced the orthonormal Bernstein polynomials which can be used in a generalized Fourier series to approximate surfaces and curves. Mirkov and Rašuo (2013),Mirkov et al., 2012 used Bernstein polynomials to solve elliptic boundary value problems, in particular, the Poisson and Helmholtz equations on a square domain. Further capabilities of operational matrices of Bernstein polynomials were demonstrated very recently by Asgari and Ezzati (2017) who solved two-dimensional integral equations of fractional order by operational matrix of two-dimensional Bernstein polynomials and Alshbool et al. (2017) who solved fractional-order differential equations. Very recently, Loh et al. (2017) introduced a new operational matrix based on Genocchi polynomials for solving fractional integro-differential equations.

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In this paper, we introduce for the first time an approximate method based on Bernstein polynomials for solving directly third-order ODEs of the form

$$y''' = f(x, y, y', y'') \quad \text{subject to} \quad y(0) = a_0,\ y'(0) = b_0,\ y'' = c_0, \tag{1}$$

where a_0, b_0, c_0 are given constants and $x \in [0, 1]$. While all others methods as introduced in Zhang et al. (2016), Singh et al. (2017), Kumar et al. (2017), Majid et al. (2009), Awoyemi and Idowu (2005)) solved Eq. (1) by reducing it to systems of lower degree of ODEs, the present method converts Eq. (1) to a system of algebraic equations which can be solved easily. The capability of the method shall be tested on a linear and nonlinear third-order ODEs.

2. Description of method

The Bernstein polynomials of degree n are defined as

$$B_{v,n}(x) = \binom{n}{v} x^v (1-x)^{n-v}, \tag{2}$$

where

$$\binom{n}{v} = \frac{n!}{v!(n-v)!}, \quad v = 0, \dots, n. \tag{3}$$

These polynomials form a complete basis for the vector space $\prod_n$of polynomials of degree at most n. For convenience, $B_{v,n}(x) = 0$ if $v < 0, n < v$. These polynomials have many properties which make them important and useful, like continuity and unity partition (see Farouki, 2012). As a result, any polynomial of degree n can be approximated in terms of linear combination of $B_{v,n}(x) = 0$, $(v = 0, \dots, n)$ as given below,

$$y(x) = \sum_{v=1}^{n} C_v B_{v,n} = C^T \Phi(x), \tag{4}$$

where $C^T = [C_0, C_1, \dots, C_n]$ and $\Phi(x) = [B_{0,n}, B_{1,n}, \dots, B_{n,n}]^T$.

Also, we can make the decomposition of the vector $\Phi(x)$ as a product of a square matrix of size $(n+1) \times (n+1)$ and a vector of size $(n+1) \times 1$, i.e. $\Phi(x) = AX$, where

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_n \\ b_0 & b_1 & b_2 & \dots & b_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_0 & k_1 & k_2 & \dots & k_n \end{bmatrix}, \quad X = \begin{bmatrix} 1 \\ x \\ x_2 \\ \vdots \\ x^n \end{bmatrix}. \tag{5}$$

Now, the derivative of the vector $\Phi(x)$ denoted by D' is

$$\frac{d\Phi(x)}{dx} = D'\Phi(x), \tag{6}$$

where D' is the $(n+1) \times (n+1)$ operational matrix of the derivative that given as $D' = A\sigma A^{-1}$, where

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_n \\ b_0 & b_1 & b_2 & \dots & b_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_0 & k_1 & k_2 & \dots & k_n \end{bmatrix}, \quad \sigma = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}. \tag{7}$$

for more illustration, (see Yousefi and Behroozifar, 2010). Eq. (6) can be generalised as follows

$$\frac{d^2\Phi(x)}{dx} = (D')^2\Phi(x), \dots, \frac{d^n\Phi(x)}{dx} = (D')^n\Phi(x). \tag{8}$$

To use this operational matrix for solving an equation, we will approximate $y(x)$ by Bernstein polynomials as $y(x) = C^T\Phi(x)$ and we have

$$y'(x) = C^T D'\Phi(x), \quad y''(x) = C^T (D')^2\Phi(x), \quad y'''(x) = C^T (D')^3\Phi(x). \tag{9}$$

Then Eqn. (1) can be written in the form

$$C^T (D')^3\Phi(x) = f(x, C^T\Phi(x), C^T D'\Phi(x), C^T (D')^2\Phi(x)), \tag{10}$$

$$C^T\Phi(x_0) = a_0,\ C^T D'\Phi(x_0) = b_0,\ C^T (D')^2\Phi(x_0) = c_0. \tag{11}$$

We substitute the collocation nodes $0 \leqslant x_0 < x_1 < \dots < x_m \leqslant 1$ in (10) to get a system of algebraic equations which will be solved by a computer algebra system like Maple. We use the Chebyshev roots as the collocation nodes

$$x_i = \frac{1}{2} + \frac{1}{2\cos((2i+1)\frac{\pi}{2n})}, \quad i = 0, 1, \dots, n-2. \tag{12}$$

A convergent analysis of the method was given by Yousefi et al. (2011) in their lemmas 3 and 4. For more details the readers is referred to Rivlin (2003).

3. Numerical tests

In this section, we present examples of third-order ODEs in order to illustrate the performance and effectiveness of our method. We apply the method with the number of Bernstein terms, $n = 14$. All the algebraic manipulations were done in Maple 8 with the Digits set to 20.

3.1. Example 1

First we consider the following linear problem

$$y''' - y'' + y' - y = 0, \quad y(0) = 1,\ y'(0) = 0,\ y''(0) = -1, \quad 0 \leqslant x \leqslant 1,$$

with the exact solution $y(x) = \cos x$. Applying the method described above, the following 14-term approximate solution is obtained:

$$\begin{aligned} y(x) \approx & -9.64735391 \times 10^{-12}x^{14} - 9.3279316 \times 10^{-12}x^{13} \\ & + 2.111956063 \times 10^{-9}x^{12} - 4.022855 \times 10^{-11}x^{11} \\ & - 2.755270872610^{-7}x^{10} - 3.7973510^{-11}x^9 \\ & + 0.248016101554 \times 10^{-4}x^8 - 1.00854 \times 10^{-11}x^7 \\ & - 0.13888888856464 \times 10^{-2}x^6 - 7.443 \times 10^{-13}x^5 \\ & + 0.4166666666678504 \times 10^{-1}x^4 - 1.264 \times 10^{-14}x^3 \\ & - 0.500000000000000001x^2 \\ & + 0.99999999999999999999. \end{aligned}$$

The absolute errors shown in Table 1 suggest that the present method is very accurate.

Table 1
Absolute errors for Example 1.

x	Exact solution	Bernstein solution	Error
0.1	0.99500416527802576610	0.99500416527802576026	5.84×10^{-18}
0.2	0.98006657784124163112	0.98006657784124160265	2.85×10^{-17}
0.3	0.95533648912560601964	0.95533648912560595103	6.86×10^{-17}
0.4	0.92106099400288508280	0.92106099400288495944	1.23×10^{-16}
0.5	0.87758256189037271612	0.87758256189037253350	1.83×10^{-16}
0.6	0.82533561490967829724	0.82533561490967807361	2.24×10^{-16}
0.7	0.76484218728448842626	0.76484218728448822968	1.97×10^{-16}
0.8	0.69670670934716542092	0.69670670934716545101	3.01×10^{-17}
0.9	0.62160996827066445648	0.62160996827066530979	8.53×10^{-16}
1.0	0.54030230586813971740	0.54030230586814316648	3.45×10^{-15}

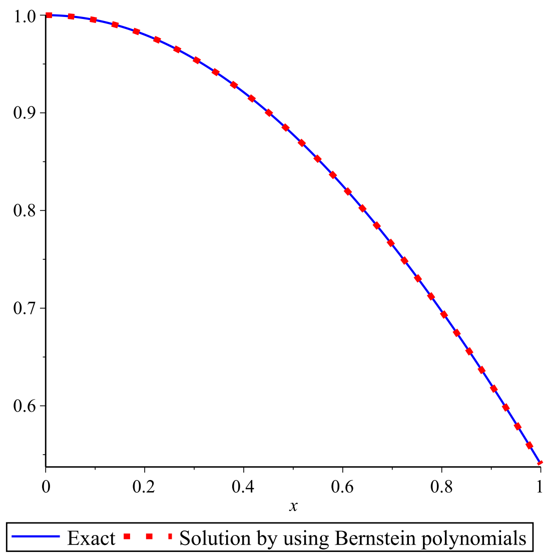

Fig. 1. Approximation solution for example 1 using Bernstein polynomials with exact solution.

3.2. Example 2

Next consider the following problem Fig. 1

$$y''' + 5y'' + 7y' + 3y = 0, \quad y(0) = 1,\ y'(0) = 0,$$
$$y''(0) = -1, \quad 0 \leqslant x \leqslant 1.$$

The exact solution is $y(x) = e^{-x} + xe^{-x}$. The operational matrices of Bernstein polynomials method yields

$$\begin{aligned} y(x) \approx{} & -8.023524662 \times 10^{-11}x^{14} + 1.683945475 \times 10^{-9}x^{13} \\ & - 2.2420566212 \times 10^{-8}x^{12} + 2.4968885757 \times 10^{-7}x^{11} \\ & - 0.247924992485 \times 10^{-5}x^{10} + 0.220451309070 \times 10^{-4}x^{9} \\ & - 0.1736106853770 \times 10^{-3}x^{8} + 0.11904760061469 \\ & \times 10^{-2}x^{7} - 0.69444443862206 \times 10^{-2}x^{6} \\ & + 0.333333333201911 \times 10^{-1}x^{5} \\ & - 0.12499999999796276x^{4} + 0.33333333333314344x^{3} \\ & - 0.500000000000000001x^{2} \\ & + 0.99999999999999999999. \end{aligned}$$

Table 2 clearly shows the accuracy of the present method.

3.3. Example 3

$$y''' = 3\sin x, \quad y(0) = 1,\ y'(0) = 0,\ y''(0) = -2, \quad 0 \leqslant x \leqslant 1.$$

The exact solution of this problem is $y(x) = 3\cos x + (x^2/2) - 2$ and the approximate solution of the present method is

$$\begin{aligned} y(x) \approx{} & -2.374911736 \times 10^{-11}x^{14} - 6.86371074 \times 10^{-11}x^{13} \\ & + 6.478659491 \times 10^{-9}x^{12} - 4.1795110 \times 10^{-10}x^{11} \\ & - 8.2617332888 \times 10^{-7}x^{10} - 5.023725 \times 10^{-10}x^{9} \\ & + 0.744050936033 \times 10^{-4}x^{8} - 1.581103 \times 10^{-10}x^{7} \\ & - 0.41666666125723 \times 10^{-2}x^{6} - 1.30540 \times 10^{-11}x^{5} \\ & + 0.12500000000215427x^{4} - 2.3127 \times 10^{-13}x^{3} \\ & - 1.000000000000000001x^{2} \\ & + 0.99999999999999999999. \end{aligned}$$

Again the errors depicted in Table 3 show the accuracy of the present method. Figs. 2 and 3.

4. Applications to equations of fluid flow

4.1. Boundary layer flow

Now consider the nonlinear boundary layer equation

$$2y''' + yy'' = 0, \quad y(0) = 0,\ y'(0) = 0,\ y''(0) = 1. \qquad (13)$$

This equation is famously known as the Blasius equation. The aim of solving Blasius equation to get the value $y''(0)$ to evaluate the shear stress at the plate. Blasius equation has been solved using different methods like series expansions, Runge Kutta, differential transformation and others. The interested readers can also see Lien-Tsai and Cha'o-Kuang (1998) and Schlichting et al. (1955).

Table 3
Absolute errors for Example 3.

x	Exact solution	Bernstein solution	Error
0.1	0.9900124958340772983	0.9900124958340771930	1.05×10^{-16}
0.2	0.9601997335237248934	0.9601997335237243910 8	5.02×10^{-16}
0.3	0.9110094673768180589	0.9110094673768168 6283	1.20×10^{-15}
0.4	0.8431829820086552484	0.9384480644498940 7304	2.18×10^{-15}
0.5	0.7577476856711181484	0.8431829820086530 6768	3.43×10^{-15}
0.6	0.6560068447290348917	0.6560068447290299 9918	4.89×10^{-15}
0.7	0.5395265618534652788	0.5395265618534588 5671	6.42×10^{-15}
0.8	0.4101201280414962628	0.4101201280414884 8222	7.78×10^{-15}
0.9	0.2698299048119933694	0.2698299048119847 4225	8.63×10^{-15}
1.0	0.1209069176044191522	0.1209069176044104 8523	8.67×10^{-15}

Table 2
Absolute errors for Example 2.

x	Exact solution	Bernstein solution	Error Present method	Error Ref. (Mohammed and Adeniyi, 2014)
0.1	0.9953211598395555 3048	0.9953211598395554 5628	7.42×10^{-17}	1.00×10^{-10}
0.2	0.9824769036935782 3040	0.9824769036935779 2919	3.01×10^{-16}	3.00×10^{-10}
0.3	0.9630636868862332 2589	0.9630636868862326 1523	6.11×10^{-16}	7.00×10^{-10}
0.4	0.9384480644498950 2104	0.9384480644498940 7304	9.48×10^{-16}	7.00×10^{-10}
0.5	0.9097959895689501 3540	0.9097959895689488 7800	1.26×10^{-15}	6.00×10^{-10}
0.6	0.8780986177504422 9221	0.8780986177504408 6917	1.42×10^{-15}	2.00×10^{-10}
0.7	0.8441950164453961 7499	0.8441950164453950 6014	1.11×10^{-15}	9.00×10^{-10}
0.8	0.8087921354109988 6457	0.8087921354109994 4217	5.78×10^{-16}	2.80×10^{-9}
0.9	0.7724823535071383 1257	0.7724823535071442 4660	5.93×10^{-15}	5.40×10^{-9}
1.0	0.7357588823428846 4320	0.7357588823429048 1537	2.02×10^{-14}	3.50×10^{-9}

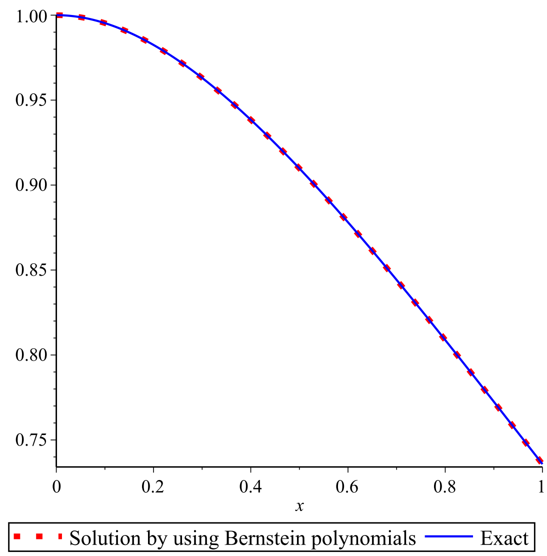

Fig. 2. Approximation solution for example 2 using Bernstein polynomials with exact solution.

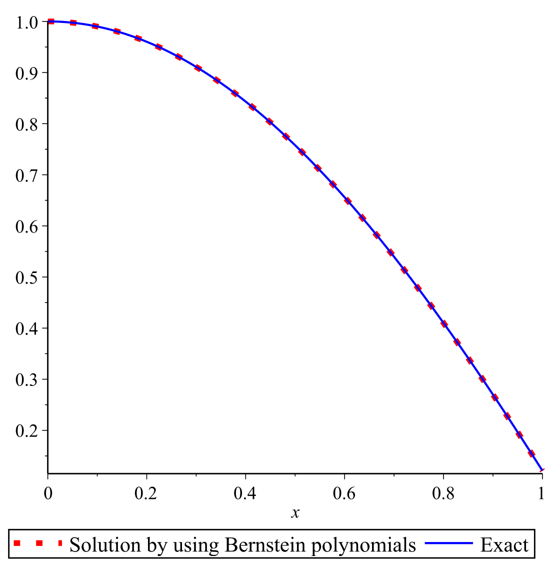

Fig. 3. Approximation solution for example 3 using Bernstein polynomials with exact solution.

$$y(x) \approx 1.12610441 \times 10^{-7}x^{14} - 7.7621579 \times 10^{-7}x^{13} + 0.27970246 \times 10^{-5}x^{12} - 0.7042654 \times 10^{-5}x^{11} + 0.8050211 \times 10^{-5}x^{10} - 0.7633041 \times 10^{-5}x^{9} + 0.73343596 \times 10^{-4}x^{8} - 0.2480836 \times 10^{-5}x^{7} + 8.55631 \times 10^{-7}x^{6} - 0.4166874207 \times 10^{-2}x^{5} + 3.43594 \times 10^{-8}x^{4} - 3.69550 \times 10^{-9}x^{3} + 0.5x^{2}.$$

Comparison with the results from a hybrid block method of Adesanya et al. (2013) shows a good agreement (see Table 4).

4.2. Thin film flow

The motion of the contact line for a thin oil drop spreading on a horizontal surface can be modelled by

Table 4
Comparison with hybrid block method of Adesanya et al. (2013).

x	Ref. (Adesanya et al., 2013)	Present method
0.1	0.00499997916611	0.0049999583341723
0.2	0.01999866666859	0.0199986668419935
0.3	0.044998481293978	0.0449898794745896
0.4	0.079991467388617	0.0799573779857994
0.5	0.124967454367055	0.1248700575229549
0.6	0.179902837409194	0.179677141245484
0.7	0.244755067600357	0.244303616982151
0.8	0.319454500640289	0.318646009310246
0.9	0.403894871267148	0.402568620552525
1.0	0.49792248311043	0.495900382783151

$$y''' = y^{-2}. \tag{14}$$

This equation describes the dynamic balance between surface tension and viscous forces in the absence of gravity in a thin fluid layer. Duffy and Wilson (1997) introduced a parametric representation for the exact solution of Eq. (14) as

$$x = 2^{\frac{1}{3}}\pi \frac{A_i(s)B_i(s_0) - A_i(s_0)B_i(s)}{[aA_i(s_0) + bB_i(s_0)][aA_i(s) + bB_i(s)]}, y = \frac{1}{[aA_i(s) + bB_i(s)]^2},$$

where $A_i(.), B_i(.)$ are the Airy functions, and a, b and s_0 are constants determined by the initial conditions. This equation was also solved by Mechee et al. (2013) and Momoniat and Mahomed (2010). For the initial conditions $y(0) = y'(0) = y''(0) = 1$, Momoniat and Mahomed (2010) obtained

$$a = 0.676482, \quad b = 1.60629, \quad s_0 = -0.39685. \tag{15}$$

We apply our method on this problem with $n = 14$ and Chebyshev roots as a collocation nodes. Applying the present method gives

$$y(x) \approx -0.45984700218196 \times 10^{-5}x^{14} + 0.48584303974767 \times 10^{-4}x^{13} - 0.24134233269265 \times 10^{-3}x^{12} + 0.74592504943532 \times 10^{-3}x^{11} - 0.15918744859666 \times 10^{-2}x^{10} + 0.24661462173050 \times 10^{-2}x^{9} - 0.30219812091711 \times 10^{-2}x^{8} + 0.43032528580877 \times 10^{-2}x^{7} - 0.110923450695375 \times 10^{-1}x^{6} + 0.333291876770568 \times 10^{-1}x^{5} - 0.833326916924882 \times 10^{-1}x^{4} + 0.16666660079020539x^{3} + 0.49999999999999999x^{2} + x + 1.$$

Comparison in Table 5 shows that our present method yields numerical results which are correct to five decimal places.

Table 5
Solutions of the present method as compared with the exact solution and Runge-Kutta type solution.

x	Exact solution Ref. (Duffy and Wilson, 1997)	Ref. (Mechee et al., 2013)	Present method
0.0	1.000000000	1.0000000000	1.0000000000
0.2	1.221211030	1.2212100045	1.2212100043
0.4	1.488834893	1.4888347799	1.4888347792
0.6	1.807361404	1.8073613977	1.8073613962
0.8	2.179819234	2.1798192339	2.1798192314
1	2.608275822	2.6082748676	2.6082748636

5. Conclusion

In this paper, we have modified for the first time the classical operational matrices of Bernstein polynomials method to solve directly a class of third-order ODEs. The method presented in this work avoids the need to transform the ODEs to a system of lower (or first)-order ODEs. The approximate solutions obtained are in the form of series whose terms can be easily computed. The procedure of the method can be programmed symbolically softwares like Maple or Mathematica. The method has also been shown to perform reasonably well for all the test problems.

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