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Original article

Existence result and approximate solutions for quadratic integro-differential equations of fractional order

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ABSTRACT

This research outlines a reliable strategy for finding a solutions of nonlinear quadratic integro-differential equation of fractional order (FQIDEs). Local and global existence theorems of solutions of the FQIDEs have been obtained by using Schauder's and Tychonoff fixed point theorems. The fractional derivative is described in the Caputo sense. The Laplace decomposition method (LDM) and modified Adomian decomposition method (MADM) are described to be fast and accurate. Illustrative examples are included to demonstrate the efficiency and reliability of presented techniques.

1. Introduction

The fractional calculus deals with derivatives and integrals to an arbitrary orders (real or complex order). The fractional calculus is applied to model the frequency-dependent damping behaviour of many viscoelastic materials (Bagley and Torvik, 1983), continuum and statistical mechanics (Mainardi, 2012), control theory (Bohannan, 2008), bioengineering (Magin, 2004).

This is one of the reason of why fractional calculus has become more and more popular, and it is described as something which is realistic.

Some of the problems that exist in the real world are being modelled by using fractional derivative and integral terms and such equations are known as the fractional integro-differential equations (FIDEs). FIDEs are found in the fields of signal prossing (Diethelm, 2010), mechanics (Rossikhin and Shitikova, 1997), econometrics (Baillie, 1996), fluid dynamics (Kilbas et al., 2006), unclear reactor dynamics, a coustic waves (Oldham and Spanier, 1974) and electromagnetics (Tarasov, 2009) etc.

There are many authors who have investigated the analytic results on existence and uniqueness of problems solutions to FIDEs such as Momani (2000) has got the local and global existence and uniqueness solution of the integro-differential equation. Karthikeyan and Trujillo (2012) studied the existence and uniqueness of FIDEs with boundary value conditions. Most of nonlinear FIDEs has not had the exact analytic solution. So approximation and numerical techniques must be used. Recently, several numerical methods are applied for solving fractional differential equations and fractional integro-differential equations. Momani and Noor (2006) applied Adomian polynomials to solve FIDEs. The authors in Pandey et al. (2009), obtained the approximate solution of Abel's integral equations by using homotopy perturbation method (HPM) and it's modification, also, by using Adomian decomposition method and it's modification. Zhang et al. (2011) produced the homotopy analysis method for higher-order FIDEs. Collocation method is introduced in Eslahchi et al. (2014), Saadatmandi and Dehghan (2011), Zhao et al. (2014) for solving the FIDEs with weakly singular kernels and linear and nonlinear integro-differential equations of fractional orders with Volterra type.

In Mohammed (2014), Mohammed investigated the numerical solution of linear FIDEs by least squares method shifted chebyshev polynomials. More recently in 2017, Kumar and co-authors (Kumar et al., 2017) presented a comparative study three numerical schemes such as linear, Quadratic and Quadratic-Linear for the FIDEs. Wang and Zhu (2017) used the wavelet numerical method to solve nonlinear Volterra FIDEs.

In this paper we negotiate the local and global existence of the solution for the following FQIDEs

$$D^q x(t) = f(t, x(t)) + x(t)\int_{t_0}^{t} K(t,s,x(s))ds, \quad x(t_0) = x_0, \quad 0 < q < 1. \tag{1}$$

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Where D^q is a Caputo's fractional derivative and q is a parameter describing the order of the fractional derivative and $K(t,s,x(s))$ is a nonlinear function, and $x(t)$ is the unknown function to be determined.

We present a comparative study between two methods LDM and the MADM for solving FQIDEs. Several analytical and numerical methods such as LDM were used to solve nonlinear ordinary, partial and integral equation. Yang and Hou (Yang and Hou, 2013) used this method to solve nonlinear FIDEs. The reliable modification of ADM has been done by Wazwaz (1999), this computational method leads to find the analytical solutions and has certain advantages over standard numerical methods. The previous method shows that there is no sign for the rounding of errors in it. As it does not involve discretization and does not require large computer obtained memory of power. The main purpose of this paper is applying LDM and MADM to solve nonlinear FQIDEs because there is no attempt have been made to solve this kind of equations.

The paper is organized as follows. It is started by introducing some necessary definitions and mathematical preliminaries of the fractional calculus theory which are required for establishing our results. In Section 3, local and global existence of FQIDEs is proved by using Schauder's and Tychonoff fixed point theorems. In Section 4, we extend the application of LDM and MADM to construct our analytical approximate solution to nonlinear FQIDEs. In Section 5, we present two examples that show the efficiency of the methods.

2. Basic information about the fractional calculus

Here, we intend to introduce some basic definitions and properties of fractional calculus theory see (Kilbas et al., 2006;Podlubny, 1998; Samko et al., 1993).

Definition 1. A real function $f(x), x > 0$ is said to be in space $C_\mu, \mu \in R$ if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C(0,\infty)$, and it is said to be in the space C_μ^n if and only if $f^n \in C_\mu, n \in N$.

Definition 2. The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $f \in C_\mu, \mu \geqslant -1$, is defined as

$$\mathcal{J}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \quad \alpha > 0, \qquad (2.1)$$
$$\mathcal{J}^0 f(t) = f(t).$$

Some properties of the operator $\mathcal{J}^\alpha$ can be found in (Miller and Ross, 1993), which are needed here, as follows: for $f \in C_\mu, \mu \geqslant -1, \alpha, \beta \geqslant 0$ and $\gamma \geqslant -1$:

1. $\mathcal{J}^\alpha \mathcal{J}^\beta f(t) = \mathcal{J}^{\alpha+\beta} f(t)$,
2. $\mathcal{J}^\alpha \mathcal{J}^\beta f(t) = \mathcal{J}^\beta \mathcal{J}^\alpha f(t)$,
3. $\mathcal{J}^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$.

Definition 3. The fractional derivative of $f(t)$ in the caputo sense is defined as

$$D^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} f^m(s)ds, \qquad (2.2)$$

for $m-1 < \alpha \leqslant m, m \in N, t > 0, f \in C_{-1}^m$.

Property 1. The Laplace transform of the Caputo derivative is defined as

$$\mathcal{L}[D^\alpha x(t)] = s^\alpha X(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} x^{(k)}(0), \qquad (2.3)$$

for $n-1 < \alpha < n$.The function $X(s)$ of the complex variable s defined by $X(s) = \mathcal{L}\{x(t);s\} = \int_0^\infty e^{-st} x(t)dt$is called the Laplace transform of the function $x(t)$.

Lemma 2.1. *If $m-1 < \alpha \leqslant m, m \in N, f \in C_\mu^m, \mu \geqslant -1$, the following two properties hold:*

1. $D^\alpha \mathcal{J}^\alpha f(t) = f(t)$,
2. $(\mathcal{J}^\alpha D^\alpha) f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0) \frac{t^k}{k!}$.

3. Local and Global Existence Solutions

This section is devoted to the study of the initial value problem (IVP) for FQIDEs of the type

$$D^q x(t) = f(t, x(t)) + x(t) \int_{t_0}^t K(t,s,x(s))ds, \quad x(t_0) = x_0, \quad 0 < q < 1. \qquad (3.1)$$

where $f \in C[J \times \mathbb{R}^n, \mathbb{R}^n], K \in C[J \times J \times \mathbb{R}^n, \mathbb{R}^n]$ and $J = [t_0, t_0 + a]$. It is easy to show that the IVP (3.1) is equivalent to the integral equation.

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} f(s, x(s))ds + \frac{1}{\Gamma(q)} \int_{t_0}^t (t-\tau)^{q-1} \left(x(\tau) \int_{t_0}^\tau K(\tau, s, x(s))ds \right) d\tau, \qquad (3.2)$$

which can be seen by integrating (3.1) from t_0 to t. Since f and K are continuous, on differentiating (3.2), we obtain (3.1). We will begin to prove the following local existence result by applying Schauder's fixed point theorem.

Theorem 3.1 (*Schauder (Zeidler, 1995)*). *If E is a closed, bounded, convex subset of a Banach space B and $T: E \to E$ is completely continuous then T has a fixed point.*

Theorem 3.2. Assume that

(i) $f \in C[J \times \mathbb{R}^n], K \in C[J \times J \times \mathbb{R}^n, \mathbb{R}^n], \int_{t_0}^\tau |K(\tau, s, x(s))| ds \leqslant N$ *for* $t_0 \leqslant s \leq t \leqslant t_0 + a, x \in \Omega = \{x \in C[J, \mathbb{R}^n] : x(t_0) = x_0$ *and* $|x(t) - x_0| \leqslant b\}$
(ii) $\|f(s, x(s)) - f(s, y(s))\| < \frac{1}{2} \frac{\epsilon \Gamma(q+1)}{\alpha^q}$
(iii) $\|K(\sigma, s, x(s)) - K(\sigma, s, y(s))\| \leqslant \frac{1}{4} \frac{\epsilon \Gamma(q+1)}{\|x(t)\| \alpha^{q+1}}$
(iv) $\delta = \frac{1}{4} \frac{\epsilon \Gamma(q+1)}{N \alpha^q}$*then the (IVP) (3.1) has at least one solution $x(t)$ on $t_0 \leqslant t \leqslant t_0 + \alpha$, for some $0 < \alpha < a$.*

Proof. Consider the set $D = \{(t,x) : t \in J$ and $|x - x_0| \leqslant b\}$ and let $|f(t, x(t))| \leqslant M$ on D. Choose $\alpha = \min\left\{a, \left(\frac{b\Gamma(q+1)}{M + N\|x\|}\right)^{\frac{1}{q}}\right\}$ and let $\Omega_0 = \{x \in C[J_0, \mathbb{R}^n] : x(t_0) = x_0$ and $|x - x_0| \leqslant b\}$ where $\|x\| = \max_{t_0 \leqslant t \leqslant t_0 + \alpha} |x(t)|$ and$J_0 = [t_0, t_0 + \alpha]$. Clearly the set Ω_0 is closed, convex and bounded. For any $x \in \Omega_0$ define the operator Tx by

$$Tx(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} f(s, x(s))ds + \frac{1}{\Gamma(q)} \int_{t_0}^t (t-\tau)^{q-1} \left(x(\tau) \int_{t_0}^\tau K(\tau, s, x(s))ds \right) d\tau, \qquad (3.3)$$

we may apply Schauder's fixed point theorem to prove the existence of a fixed point of $T \in \Omega_0$ which is equivalent to solving the IVP clearly $Tx(t_0) = x_0$, and for $t \in J_0$.

$$
\begin{aligned}
|Tx(t)-x_0| &= \left|\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}f(s,x(s))ds\right.\\
&\quad\left.+\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-\tau)^{q-1}\left(x(\tau)\int_{t_0}^{\tau}K(\tau,s,x(s))ds\right)d\tau\right|,\\
&\leqslant \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}\mid f(s,x(s))\mid ds\\
&\quad+\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-\tau)^{q-1}\left(x(\tau)\mid\int_{t_0}^{\tau}K(\tau,s,x(s))\mid ds\right)d\tau,\\
&\leqslant \frac{M}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}ds+\frac{N}{\Gamma(q)}\int_{t_0}^{t}(t-\tau)^{q-1}|x(\tau)|d\tau\\
&\leqslant \left(\frac{M}{\Gamma(q)}+\frac{N|x(t)|}{\Gamma(q)}\right)\frac{(t-t_0)^q}{q},\\
&= \frac{\alpha^q}{\Gamma(q+1)}(M+N\|x\|)\leqslant b.
\end{aligned}
$$

Which implies that $T\Omega_0\subset\Omega_0$. Furthermore, for any $t_1,t_2\in J_0$, such that $t_2>t_1$, we obtain

$$
\begin{aligned}
&|Tx(t_2)-Tx(t_1)|\\
&=\left|\frac{1}{\Gamma(q)}\int_{t_0}^{t_2}(t_2-s)^{q-1}f(s,x(s))ds\right.\\
&\quad+\frac{1}{\Gamma(q)}\int_{t_0}^{t_2}(t_2-\tau)^{q-1}\left(x(\tau)\int_{t_0}^{\tau}K(\tau,s,x(s))ds\right)d\tau\\
&\quad-\frac{1}{\Gamma(q)}\int_{t_0}^{t_1}(t_1-s)^{q-1}f(s,x(s))ds\\
&\quad\left.-\frac{1}{\Gamma(q)}\int_{t_0}^{t_1}(t_1-\tau)^{q-1}\left(x(\tau)\int_{t_0}^{\tau}K(\tau,s,x(s))ds\right)d\tau\right|\\
&=\left|\frac{1}{\Gamma(q)}\int_{t_0}^{t_1}(t_2-s)^{q-1}f(s,x(s))ds\right.\\
&\quad+\frac{1}{\Gamma(q)}\int_{t_0}^{t_1}(t_2-\tau)^{q-1}\left(x(\tau)\int_{t_0}^{\tau}K(\tau,s,x(s))ds\right)d\tau\\
&\quad+\frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}f(s,x(s))ds\\
&\quad+\frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-\tau)^{q-1}\left(x(\tau)\int_{t_0}^{\tau}K(\tau,s,x(s))ds\right)d\tau\\
&\quad-\frac{1}{\Gamma(q)}\int_{t_0}^{t_1}(t_1-s)^{q-1}f(s,x(s))ds\\
&\quad\left.-\frac{1}{\Gamma(q)}\int_{t_0}^{t_1}(t_1-\tau)^{q-1}\left(x(\tau)\int_{t_0}^{\tau}K(\tau,s,x(s))ds\right)d\tau\right|\\
&\leqslant\frac{1}{\Gamma(q)}\int_{t_0}^{t_1}\left[(t_2-s)^{q-1}-(t_1-s)^{q-1}\right]\mid f(s,x(s))\mid ds\\
&\quad+\frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}\mid f(s,x(s))\mid ds\\
&\quad+\frac{1}{\Gamma(q)}\int_{t_0}^{t_1}\left[(t_2-\tau)^{q-1}-(t_1-\tau)^{q-1}\right]\\
&\quad\times\left(\mid x(\tau)\mid\int_{t_0}^{\tau}\mid K(\tau,s,x(s))\mid ds\right)d\tau\\
&\quad+\frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-\tau)^{q-1}\left(\mid x(\tau)\mid\int_{t_0}^{\tau}\mid K(\tau,s,x(s))\mid ds\right)d\tau\\
&\leqslant\frac{M}{\Gamma(q)}\int_{t_0}^{t_1}\left[(t_2-s)^{q-1}-(t_1-s)^{q-1}\right]ds+\frac{M}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}ds\\
&\quad+\frac{N\|x\|}{\Gamma(q)}\int_{t_0}^{t_1}\left[(t_2-\tau)^{q-1}-(t_1-\tau)^{q-1}\right]d\tau+\frac{N\|x\|}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-\tau)^{q-1}d\tau\\
&\leqslant\left(\frac{M+N\|x\|}{\Gamma(q+1)}\right)\left[(t_2-t_1)^q-(t_2-t_0)^q+(t_1-t_0)^q+(t_2-t_1)^q\right]\\
&\leqslant\left(\frac{M+N\|x\|}{\Gamma(q+1)}\right)\left[2(t_2-t_1)^q\right],
\end{aligned}
\tag{3.4}
$$

as $t_1\to t_2$ the right hand side of inequality (3.4) tends to zero. Therefore the operator $T:\Omega_0\to\Omega_0$ is equicontinuous, and consequently the closure of $T(\Omega_0)$ is compact.

To show that T is a continuous map, let us take an $\epsilon>0$ and x,y in Ω_0, it follows, using uniform continuity of f and K that for any $\epsilon>0$ there exists $\delta>0$ such that

$$
\begin{aligned}
&\mid Tx(t)-Ty(t)\mid\\
&=\left|\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}(f(s,x(s))-f(s,y(s)))ds\right.\\
&\quad+\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-\tau)^{q-1}x(\tau)\left(\int_{t_0}^{\tau}K(\tau,s,x(s))ds\right)d\tau\\
&\quad-\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-\tau)^{q-1}y(\tau)\left(\int_{t_0}^{\tau}K(\tau,s,y(s))ds\right)d\tau\\
&\quad+\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-\tau)^{q-1}x(\tau)\left(\int_{t_0}^{\tau}K(\tau,s,y(s))ds\right)d\tau\\
&\quad\left.-\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-\tau)^{q-1}x(\tau)\left(\int_{t_0}^{\tau}K(\tau,s,y(s))ds\right)d\tau\right|\\
&=\left|\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}(f(s,x(s))-f(s,y(s)))ds\right.\\
&\quad+\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-\tau)^{q-1}x(\tau)\left(\int_{t_0}^{\tau}(K(\tau,s,x(s))-K(\tau,s,y(s)))ds\right)d\tau\\
&\quad\left.+\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-\tau)^{q-1}(x(\tau)-y(\tau))\left(\int_{t_0}^{\tau}K(\tau,s,y(s))ds\right)d\tau\right|\\
&\leqslant\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}(\mid f(s,x(s))-f(s,y(s))\mid)ds\\
&\quad+\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-\tau)^{q-1}\mid x(\tau)\mid\left(\int_{t_0}^{\tau}(\mid K(\tau,s,x(s))-K(\tau,s,y(s)))\mid ds\right)d\tau\\
&\quad+\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-\tau)^{q-1}\mid x(\tau)-y(\tau)\mid\left(\int_{t_0}^{\tau}\mid K(\tau,s,x(s))\mid ds\right)d\tau.
\end{aligned}
$$

Since f and K is uniformly continuous for the above $\epsilon>0$, there exist $\delta>0$ such that $\mid x(t)-y(t)\mid<\delta$, by using (ii)-(iv) we have

$$
\begin{aligned}
&\leqslant\frac{1}{\Gamma(q)}\frac{\epsilon\Gamma(q+1)}{2\alpha^q}\int_{t_0}^{t}(t-s)^{q-1}ds\\
&\quad+\frac{\|x\|}{\Gamma(q)}\frac{\epsilon\Gamma(q+1)}{4\|x\|\alpha^{q+1}}\int_{t_0}^{t}(t-\tau)^{q-1}\left(\int_{t_0}^{\tau}ds\right)d\tau\\
&\quad+\frac{\delta N}{\Gamma(q)}\int_{t_0}^{t}(t-\tau)^{q-1}d\tau\\
&\leqslant\frac{1}{2}\epsilon+\frac{\|x\|}{q\Gamma(q)}\frac{1}{4}\frac{\epsilon\Gamma(q+1)\alpha}{\alpha^{q+1}\|x\|}(t-t_0)^q+\frac{\delta N}{\Gamma(q+1)}(t-t_0)^q\leqslant\epsilon.
\end{aligned}
$$

We shall next discuss a global existence result for IVP(3.1) using Tychonoff's fixed point theorem, which we state in the following form.

Theorem 3.3 (*Tychonoff* (Zeidler, 1995)). *Let B be a complete, locally convex, linear space and B_0 a closed convex subset of B. Let the mapping $T:B\to B$ be a continuous and $T(B_0)\subset B_0$if the closure of $T(B_0)$ is compact then Thas a fixed point in B_0.*

Theorem 3.4. *Assume that*

(i) $f\in C[\mathbb{R}_+\times\mathbb{R}^n,\mathbb{R}^n], g\in C[\mathbb{R}_+^2,\mathbb{R}_+], g(t,u)$ *is monotone nondecreasing in u for each $T\in J$ and*

$$\mid f(t,x)\mid\leqslant g(t,\mid x\mid),\quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^n;$$

(ii) $K\in C[\mathbb{R}_+^2\times\mathbb{R}^n,\mathbb{R}^n], G\in C[\mathbb{R}_+^3,\mathbb{R}_+], G(t,s,u)$ *is monotone nondecreasing in u for each $(t,s)\in\mathbb{R}_+^2$ and*

$$\mid K(t,s,x)\mid\leqslant G(t,s,\mid x\mid),\quad (t,s,x)\in\mathbb{R}_+^2\times\mathbb{R}^n;$$

(iii) $\int_s^t|K(\sigma,s,x(s))|d\sigma\leqslant N, for t,s\in\mathbb{R}_+, x\in C[\mathbb{R}_+,\mathbb{R}^n]$

Then the fractional quadratic integro differential equation

$$D^q u(t) = g(t, u(t)) + u(t)\int_{t_0}^{t} G(t, s, u(s))ds, \quad u(t_0) = u_0 \tag{3.5}$$

has a solution $u(t)$ existing for for every $u_0 > 0, t \geqslant t_0$, also, then for every $x_0 \in \mathbb{R}^n$ such that $|x_0| \leqslant u_0$, there exists a solution $x(t)$ of (3.1) for $t \geqslant t_0$ satisfying $|x(t)| \leqslant u(t), t \geqslant t_0$.

Proof. Let us consider the real vector space B of all continuous functions from $[t_0, \infty]$ into $\mathbb{R}^n$, the topology on B being that induced by the family of pseudo-norms $\{V_n(x)\}_{n=1}^{\infty}$ where for $x \in B, V_n(x) = \sup_{t_0 \leqslant t \leqslant n} |x(t)|$. A fundamental system of neighborhoods is the given by $\{S_n\}_{n=1}^{\infty}$, where $S_n = \{x \in B : V_n(x) \leqslant 1\}$ under this topology, B is a complete, locally convex linear space.

Now define a subset B_0 of B as follows:

$$B_0 = \{x \in B : |x(t)| \leqslant u(t), t \geqslant t_0\},$$

where $u(t)$ is a solution of (3.5) existing for $t \geqslant t_0$. It is clear that in the topology of B, B_0 is closed convex and bounded. Consider the integral operator defined by (3.3) whose fixed point corresponds to a solution of (3.1), evidently, the operator T is compact in the topology of B, and hence the closure of $T(B_0)$ is compact. In view of the boundedness of B_0. Now to prove $T(B_0) \subset B_0$, observe that for any $x \in B_0$,

$$\begin{aligned}
\|Tx(t)\| \leqslant & \|x_0\| + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}\|f(s, x(s))\|ds \\
& + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-\tau)^{q-1}\|x(\tau)\|\left(\int_{t_0}^{\tau}\|K(\tau, s, x(s))\|ds\right)d\tau \\
\leqslant & \|x_0\| + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}g(s, \|x(s)\|)ds \\
& + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-\tau)^{q-1}\|x(\tau)\|\left(\int_{t_0}^{\tau}G(\tau, s, \|x(s)\|)ds\right)d\tau \\
\leqslant & u_0 + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}g(s, u(s))ds \\
& + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-\tau)^{q-1}u(\tau)\left(\int_{t_0}^{\tau}G(\tau, s, u(s))ds\right)d\tau = u(t).
\end{aligned}$$

Using the monotonicity of g and G, the definition of B_0, and the fact $u(t)$ is a solution of (3.5), therefore $\|Tx(t)\| \leqslant u(t)$, which implies $T(B_0) \subset B_0$.

Hence by Tychonoff's fixed point theorem, T has a fixed point in B_0, which completes the proof of the theorem.

4. Numerical methods to solve the fractional quadratic integro-differential equations

4.1. Laplace decomposition method

Firstly, we consider the FQIDEs (1) and apply the Laplace transform first on both sides of (1)

$$\mathcal{L}\{D^q x(t)\} = \mathcal{L}\{f(t, x(t))\} + \mathcal{L}\left\{x(t)\int_{t_0}^{t}K(t, s, x(s))ds\right\}, \tag{4.1}$$

using the differentiation property of Laplace transform (2.3), we get

$$s^q\mathcal{L}\{x(t)\} - c = \mathcal{L}\{f(t, x(t))\} + \mathcal{L}\left\{x(t)\int_{t_0}^{t}K(t, s, x(s))ds\right\}, \tag{4.2}$$

where $c = \sum_{k=0}^{m-1} s^{\alpha-k-1}x^{(k)}(0)$. Thus, the given equation is equivalent to

$$\mathcal{L}\{x(t)\} = \frac{c}{s^q} + \frac{1}{s^q}\mathcal{L}\{f(t, x(t))\} + \frac{1}{s^q}\mathcal{L}\left\{x(t)\int_{t_0}^{t}K(t, s, x(s))ds\right\}. \tag{4.3}$$

The second step in Laplace decomposition method is that we represent solution as an infinite series given below

$$x(t) = \sum_{n=0}^{\infty} x_n. \tag{4.4}$$

The nonlinear operator is decomposed as

$$f(t, x(t)) = \sum_{n=0}^{\infty} A_n, \tag{4.5}$$

$$K(t, s, x(t)) = \sum_{n=0}^{\infty} B_n, \tag{4.6}$$

where A_n, B_n are Adomian polynomials of $x_0, x_1, x_2, \ldots, x_n, ..$ are given by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[f(t, \sum_{i=0}^{\infty}\lambda^i x_i)\right]_{\lambda=0}, \tag{4.7}$$

$$B_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[K(t, s, \sum_{i=0}^{\infty}\lambda^i x_i)\right]_{\lambda=0}, \tag{4.8}$$

substituting (4.4), (4.5) and (4.6) into (4.3) in case $0 < q < 1$, we will get

$$\mathcal{L}\left\{\sum_{n=0}^{\infty}x_n\right\} = \frac{x(0)}{s} + \frac{1}{s^q}\mathcal{L}\left\{\sum_{n=0}^{\infty}A_n\right\} + \frac{1}{s^q}\mathcal{L}\left\{\sum_{n=0}^{\infty}x_n(t)\int_{t_0}^{t}\sum_{n=0}^{\infty}B_n ds\right\}. \tag{4.9}$$

Matching both sides of (4.9) yields the following iterative algorithm:

$$\mathcal{L}\{x_0\} = \frac{x(0)}{s}, \tag{4.10}$$

$$\mathcal{L}\{x_1\} = \frac{1}{s^q}\mathcal{L}\{A_0\} + \frac{1}{s^q}\mathcal{L}\left\{x_0(t)\int_{t_0}^{t}B_0 ds\right\} \tag{4.11}$$

$$\mathcal{L}\{x_1\} = \frac{1}{s^q}\mathcal{L}\{A_1\} + \frac{1}{s^q}\mathcal{L}\left\{x_1(t)\int_{t_0}^{t}B_1 ds\right\}, \tag{4.12}$$

and so on, in general the recursive relation is given by

$$\mathcal{L}\{x_{n+1}\} = \frac{1}{s^q}\mathcal{L}\{A_n\} + \frac{1}{s^q}\mathcal{L}\left\{x_n(t)\int_{t_0}^{t}B_n ds\right\}, \quad n \geqslant 0. \tag{4.13}$$

Applying inverse Laplace transform to (4.13), so our required recursive relation is given bellow

$$x_0(t) = H(t), \tag{4.14}$$

$$x_{n+1}(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^q}\mathcal{L}\{A_n\}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^q}\mathcal{L}\left\{x_n(t)\int_{t_0}^{t}B_n ds\right\}\right\}, \tag{4.15}$$

where $H(t)$ is a function that arises from the source term and the prescribed initial conditions. The modified Laplace decomposition method suggests the function $H(t)$ defined above in (4.14) be decomposed into two parts

$$H(t) = H_1(t) + H_2(t). \tag{4.16}$$

Instead of iteration procedure (4.14) and (4.15), we suggest the following modification

$$x_0(t) = H_1(t),$$
$$x_1(t) = H_2(t) + \mathcal{L}^{-1}\left\{\frac{1}{s^q}\mathcal{L}\{A_0\}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^q}\mathcal{L}\left\{x_0(t)\int_{t_0}^{t} B_0 ds\right\}\right\},$$
$$x_{n+1}(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^q}\mathcal{L}\{A_n\}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^q}\mathcal{L}\left\{x_n(t)\int_{t_0}^{t} B_n ds\right\}\right\}.$$

The solution through the modified Laplace decomposition method high depend on the choice of $H_1(t)$ and $H_2(t)$.

4.2. Modified Adomian decomposition method

we consider the FQIDEs (1) and apply the operator J^α, the inverse of the operator D^α to both sides of (1) yields

$$x(t) = \sum_{j=0}^{n-1}\gamma_j\frac{t^j}{j!} + J^\alpha(f(t,x(t))) + J^\alpha\left(x(t)\int_{t_0}^{t} K(t,s,x(s))ds\right). \quad (4.17)$$

The Adomian decomposition method suggests the solution $x(t)$ be decomposed by the infinite series of components

$$x(t) = \sum_{n=0}^{\infty} x_n(t), \quad (4.18)$$

and the nonlinear function in (1) is decomposed as follows

$$f(t,x(t)) = \sum_{n=0}^{\infty} A_n, \quad (4.19)$$

$$K(t,s,x(t)) = \sum_{n=0}^{\infty} B_n, \quad (4.20)$$

where A_n, B_n are the so-called Adomian polynomials, substitution the decomposition series (4.18), (4.19) and (4.20) into both sides of (4.3) gives

$$\sum_{n=0}^{\infty} x_n(t) = \sum_{j=0}^{n-1}\gamma_j\frac{t^j}{j!} + J^\alpha\left(\sum_{n=0}^{\infty} A_n\right) + J^\alpha\left(\sum_{n=0}^{\infty} x_n(t)\int_{t_0}^{t}\sum_{n=0}^{\infty} B_n ds\right). \quad (4.21)$$

From this equation, the iterates are determined by the following recursive way

$$x_0 = \sum_{j=0}^{n-1}\gamma_j\frac{t^j}{j!}, \quad (4.22)$$

$$x_{n+1} = J^\alpha(A_n) + J^\alpha\left(x_n(t)\int_{t_0}^{t} B_n ds\right), \quad n \geqslant 0. \quad (4.23)$$

Where $\gamma_0 = x(0)$ the initial condition and the Adomian polynomials A_n, B_n are given by (Adomian, 1994)

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[f(t, \sum_{i=0}^{\infty}\lambda^i x_i)\right]_{\lambda=0}, \quad (4.24)$$

$$B_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[K(t,s, \sum_{i=0}^{\infty}\lambda^i x_i)\right]_{\lambda=0}. \quad (4.25)$$

The decomposition series solutions are generally converge very rapidly. The convergence of the decomposition series have investigated by several authors (Cherruault and Adomian, 1993). For later numerical computation, let the expression

$$x_N(t) = \sum_{n=0}^{N-1} x_n(t), \quad (4.26)$$

denote the N-term approximation to $x(t)$.

5. Numerical examples

In this section, we present some numerical examples of solutions of the FQIDEs via the LDM and MADM.

Example 5.1.

$$D^\alpha u(t) = \frac{1}{\Gamma(\frac{1}{2})}\left(\frac{8}{3}t^3 - 2t^{\frac{1}{2}}\right)u(t) + \frac{t}{1260} + u(t)\int_0^t t\sin(s)u(s)ds, \quad (5.1)$$

with the initial condition is $u(0) = 0$ and $0 < \alpha < 1$. First, we apply the Laplace transform to both sides of (5.1)

$$\mathcal{L}[D^\alpha u(t)] = \mathcal{L}\left[\frac{1}{\Gamma(\frac{1}{2})}\left(\frac{8}{3}t^3 - 2t^{\frac{1}{2}}\right)u(t)\right] + \mathcal{L}\left[\frac{t}{1260}\right] + \mathcal{L}\left[u(t)\int_0^t t\sin(s)u(s)ds\right],$$

using the property of Laplace transform and the initial condition we get

$$s^\alpha\mathcal{L}[u(t)] = \mathcal{L}\left[\frac{1}{\Gamma(\frac{1}{2})}\left(\frac{8}{3}t^3 - 2t^{\frac{1}{2}}\right)u(t)\right] + \mathcal{L}\left[\frac{t}{1260}\right] + \mathcal{L}\left[u(t)\int_0^t t\sin(s)u(s)ds\right],$$

and

$$\mathcal{L}[u(t)] = \frac{1}{s^\alpha}\left\{\mathcal{L}\left[\frac{1}{\Gamma(\frac{1}{2})}\left(\frac{8}{3}t^3 - 2t^{\frac{1}{2}}\right)u(t)\right] + \mathcal{L}\left[\frac{t}{1260}\right] + \mathcal{L}\left[u(t)\int_0^t t\sin(s)u(s)ds\right]\right\},$$

substituting (4.4), (4.5) and (4.6) into above equation, we have

$$\mathcal{L}\left[\sum_{n=0}^{\infty}u_n(t)\right] = \frac{1}{s^\alpha}\left\{\mathcal{L}\left[\frac{1}{\Gamma(\frac{1}{2})}\left(\frac{8}{3}t^3 - 2t^{\frac{1}{2}}\right)\sum_{n=0}^{\infty}u_n(t)\right] + \mathcal{L}\left[\frac{t}{1260}\right] = 1 + \mathcal{L}\left[\sum_{n=0}^{\infty}u_n(t)\int_0^t t\sin(s)\sum_{n=0}^{\infty}u_n(s)ds\right]\right\}. \quad (5.2)$$

Match both side of (5.2), we have the following relation

$$\mathcal{L}[u_0(t)] = \frac{1}{s^\alpha}\mathcal{L}\left[\frac{t}{1260}\right],$$
$$\mathcal{L}[u_{n+1}(t)] = \frac{1}{s^\alpha}\left\{\mathcal{L}\left[\frac{1}{\Gamma(\frac{1}{2})}\left(\frac{8}{3}t^3 - 2t^{\frac{1}{2}}\right)u_n(t)\right] + \mathcal{L}\left[u_n(t)\int_0^t t\sin(s)u_n(s)ds\right]\right\}.$$

Applying inverse Laplace transform to above equation we get

$$u_0(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^\alpha}\mathcal{L}\left[\frac{t}{1260}\right]\right\},$$
$$u_1(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^\alpha}\left\{\mathcal{L}\left[\frac{1}{\Gamma(\frac{1}{2})}\left(\frac{8}{3}t^3 - 2t^{\frac{1}{2}}\right)u_n(t)\right] + \mathcal{L}\left[u_n(t)\int_0^t t\sin(s)u_n(s)ds\right]\right\}\right\}.$$

Therefor, the solution is obtained to be

$$u(t) = \frac{1}{\Gamma(\alpha+5)}\frac{1}{1260}t^{\alpha+1}(2+\alpha)(3+\alpha)(4+\alpha) + \ldots, \quad (5.3)$$

According to ADM, the recursive ADM is

$$u_0(t) = J^\alpha\left(\frac{t}{1260}\right),$$
$$u_{n+1}(t) = J^\alpha\left(\frac{1}{\Gamma(\frac{1}{2})}\left(\frac{8}{3}t^3 - 2t^{\frac{1}{2}}\right)u_n(t)\right) + J^\alpha\left(u_n(t)\int_{t_0}^t t\sin(s)u_n(s)ds\right).$$

Therefor, the solution is obtained to be

$$u(t) = \frac{1}{3175200}\left(1680t^{2\alpha+4}\Gamma\left(2\alpha+\frac{5}{2}\right)\sqrt{\pi}\Gamma(3\alpha+6)\alpha^7 4^{-\alpha}\right) + \ldots, \tag{5.4}$$

Tables 1, 2 presents the approximate solution for different values of α, we have noticed that the accuracy is improving by computing more terms of the approximate solutions (see Fig. 1).

Example 5.2.

$$D^\alpha u(t) = 2\sqrt{t} + 2t^{\frac{3}{2}} - \left(\sqrt{t} + t^{\frac{3}{2}}\right)\ln(1+t)u(t) + u(t)\int_{t_0}^t e^t s u(s)ds, \tag{5.5}$$

with the initial condition is $u(0) = 0$ and $0 < \alpha < 1$. First, we apply the Laplace transform to both sides of (5.5)

$$\mathcal{L}[D^\alpha u(t)] = \mathcal{L}\left[2\sqrt{t} + 2t^{\frac{3}{2}} - \left(\sqrt{t} + t^{\frac{3}{2}}\right)\ln(1+t)u(t)\right] + \mathcal{L}\left[u(t)\int_{t_0}^t e^t s u(s)ds\right],$$

Table 1
Approximate solution for Eq. (5.1) at different values of α.

t	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 1$
0.10	0.00003306	0.00001760	0.00000854	0.00000393
0.20	0.00006887	0.00004630	0.00002766	0.00001542
0.30	0.00010211	0.00007925	0.00005394	0.00003391
0.40	0.00013495	0.00011479	0.00008569	0.00005879
0.50	0.00017335	0.00015398	0.00012242	0.00008971
0.60	0.00022680	0.00020067	0.00016515	0.00012687
0.70	0.00030859	0.00026180	0.00021683	0.00017138
0.80	0.00043603	0.00034789	0.00028290	0.00022580
0.90	0.00063071	0.00047350	0.00037189	0.00029462
1.00	0.00091880	0.00065781	0.00049606	0.00038493

Table 2
Approximate solution for Eq. (5.1) at different values of α.

t	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 1$
0.10	0.00003306	0.00001760	0.00000854	0.00000393
0.20	0.00006887	0.00004627	0.00002765	0.00001542
0.30	0.00010146	0.00007907	0.00005389	0.00003389
0.40	0.00013261	0.00011402	0.00008545	0.00005872
0.50	0.00016694	0.00015166	0.00012162	0.00008945
0.60	0.00021225	0.00019489	0.00016298	0.00012609
0.70	0.00027948	0.00024930	0.00021177	0.00016944
0.80	0.00038294	0.00032352	0.00027235	0.00022147
0.90	0.00054051	0.00042959	0.00035172	0.00028583
1.00	0.00077391	0.00058345	0.00046007	0.00036840

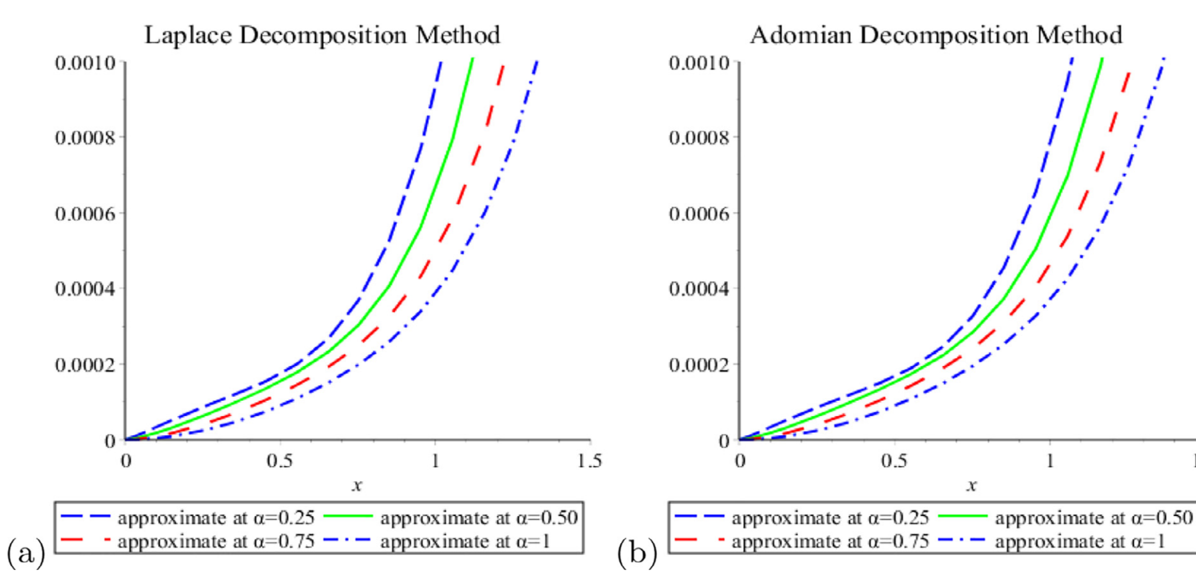

Fig. 1. Approximate solutions by using (LDM) and (ADM).

using the property of Laplace transform and the initial condition, we get

$$\mathcal{L}[u(t)] = \frac{1}{s^{\alpha}}\left\{\mathcal{L}\left[2\sqrt{t}+2t^{\frac{3}{2}}-\left(\sqrt{t}+t^{\frac{3}{2}}\right)\ln(1+t)u(t)\right]\right.$$
$$\left.+\mathcal{L}\left[u(t)\int_{t_0}^{t}e^{t}su(s)ds\right]\right\},$$

substituting (4.4), (4.5) and (4.6) into above equation, we have

$$\mathcal{L}\left[\sum_{n=0}^{\infty}u_n(t)\right] = \frac{1}{s^{\alpha}}\left\{\mathcal{L}\left[2\sqrt{t}+2t^{\frac{3}{2}}-\left(\sqrt{t}+t^{\frac{3}{2}}\right)\ln(1+t)\sum_{n=0}^{\infty}u_n(t)\right]\right.$$
$$\left.= 1+\mathcal{L}\left[\sum_{n=0}^{\infty}u_n(t)\int_{t_0}^{t}e^{t}s\sum_{n=0}^{\infty}u_n(s)ds\right]\right\}. \tag{5.6}$$

Match both side of (5.6), we have the following relation

$$\mathcal{L}[u_0(t)] = \frac{1}{s^{\alpha}}\mathcal{L}\left[2\sqrt{t}+2t^{\frac{3}{2}}\right],$$
$$\mathcal{L}[u_{n+1}(t)] = \frac{1}{s^{\alpha}}\mathcal{L}\left[-\left(\sqrt{t}+t^{\frac{3}{2}}\right)\ln(1+t)u_n(t)\right]$$
$$+\frac{1}{s^{\alpha}}\mathcal{L}\left[u_n(t)\int_{t_0}^{t}e^{t}su_n(s)ds\right].$$

The inverse Laplace transform applied to the previous equations and we obtain

$$u_0(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^{\alpha}}\mathcal{L}\left[2\sqrt{t}+2t^{\frac{3}{2}}\right]\right\},$$
$$u_{n+1}(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^{\alpha}}\mathcal{L}\left[-\left(\sqrt{t}+t^{\frac{3}{2}}\right)\ln(1+t)u_n(t)\right]\right.$$
$$\left.+\frac{1}{s^{\alpha}}\mathcal{L}\left[u_n(t)\int_{t_0}^{t}e^{t}su_n(s)ds\right]\right\}$$

Therefor, the solution is obtained to be

$$u(t) = \frac{-1}{576}\left(64800t^{3+2\alpha}\pi\Gamma(\alpha+\frac{1}{3})\Gamma(\alpha+\frac{2}{3})4^{-\alpha}\right)+\ldots\ldots \tag{5.7}$$

According to ADM, the recursive ADM is

$$u_0(t) = J^{\alpha}(2\sqrt{t}),$$
$$u_{n+1}(t) = J^{\alpha}\left(2t^{\frac{3}{2}}\right)+J^{\alpha}\left(-\left(\sqrt{t}+t^{\frac{3}{2}}\right)\ln(1+t)u_n(t)\right)$$
$$+J^{\alpha}\left(u_n(t)\int_{t_0}^{t}e^{t}su_n(s)ds\right),$$

therefor, the solution is obtained to be

$$u(t) = \frac{1}{1024}\frac{1}{\Gamma(\frac{9}{2}+\alpha)^2\Gamma(6+3\alpha)}49152\sqrt{\pi}\Gamma\left(\frac{9}{2}+\alpha\right)t^{3\alpha+4}\Gamma(2+\alpha)4^{\alpha}\alpha^5+\ldots, \tag{5.8}$$

we have noticed that the result is the same in the previous method, Table 3 presents the approximate solution for the different values of α, we have noticed that the accuracy is improving by computing more terms of the approximate solutions (see Fig. 2).

Table 3
Approximate solution for Eq. (5.5) at different values of α.

t	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 1$
0.10	0.36765557	0.18953458	0.09364270	0.04465906
0.20	0.64992844	0.39965869	0.23491078	0.13300533
0.30	0.91632907	0.62512154	0.40768544	0.25566171
0.40	1.17989872	0.86293000	0.60610126	0.40930681
0.50	1.45649193	1.11417053	0.82680896	0.59157564
0.60	1.77042701	1.38614601	1.06939284	0.80089138
0.70	2.15749580	1.69486410	1.33799321	1.03724849
0.80	2.66751254	2.06790583	1.64365154	1.30364112
0.90	3.36681803	2.54769945	2.00732460	1.60814543
1.00	4.34085479	3.19522247	2.46360149	1.96674225

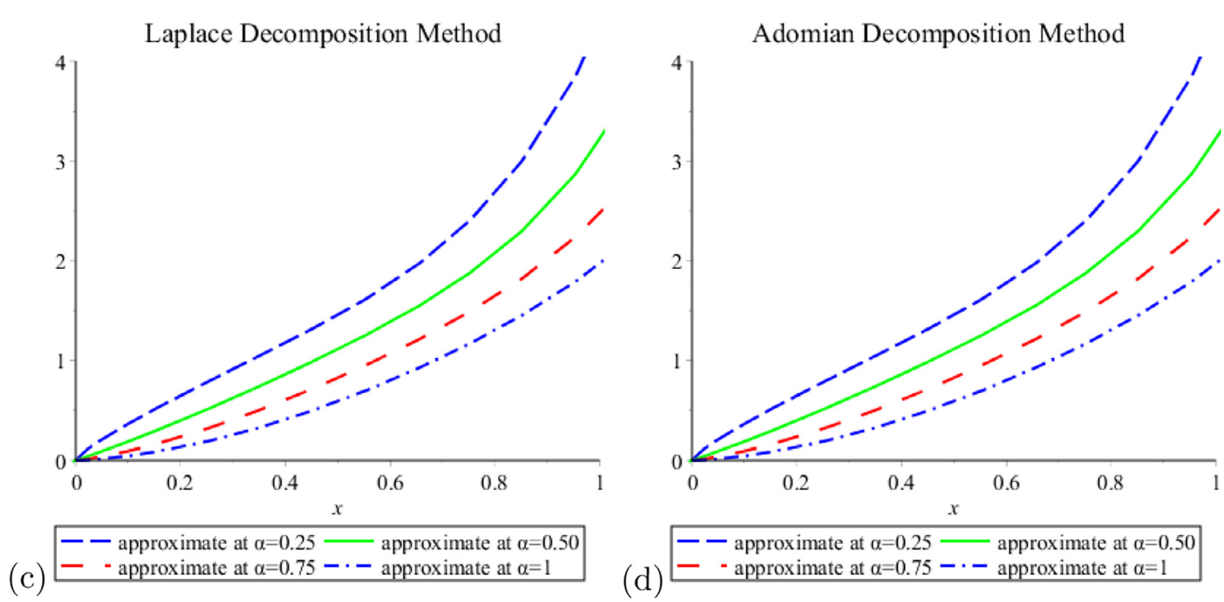

Fig. 2. Approximate solutions by using (LDM) and (ADM).

6. Conclusion

In this paper, we have proved the local and global existence of solutions. Also we have applied the LDM and MADM to find the solution of nonlinear initial value problems of FQIDEs for the first order. Actually, these methods do not require any linearization, perturbation or restrictive assumptions. In our research we have observed that the LDM and MADM is a very effective and powerful tool for finding the solutions for any problems in this field. We use the Maple package (2015) in calculations.

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