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On *s*-weakly *gw*-closed sets in *w*-spaces

Won Keun Min

Department of Mathematics, Kangwon National University, Chuncheon, 24341 Republic of Korea

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ABSTRACT

The purpose of this note is to introduce the notion of *s*-weakly *gw*-closed set in *w*-spaces and to study its some basic properties. In particular, the relationships among *wg*-closed sets, *w*-semi-closed sets and *s*-weakly *g*-closed sets are investigated.

1. Introduction

In (Siwiec, 1974), the author introduced the notions of weak neighborhoods and weak base in a topological space. We introduced the weak neighborhood systems defined by using the notion of weak neighborhoods in (Min, 2008). The weak neighborhood system induces a weak neighborhood space which is independent of neighborhood spaces (Kent and Min, 2002) and general topological spaces (Császár, 2002). The notions of weak structure and *w*-space were investigated in (Kim and Min, 2015). In fact, the set of all *g*-closed subsets (Levine, 1970) in a topological space is a kind of weak structure. We introduced the notion of *gw*-closed set in (Min and Kim, 2016a) and some its basic properties. In (Min, 2017), we introduced and studied the notion of weakly *gw*-closed sets for the sake of extending the notion of *gw*-closed sets in *w*-spaces. The purpose of this note is to extend the notion of *gw*-closed sets in *w*-spaces in a different way than the notion of weakly *gw*-closed sets. So, we introduce the new notion of *s*-weakly *gw*-closed sets in weak spaces, and investigate its properties. In particular, the relationships among weakly *wg*-closed sets, *w*-semi-closed sets and *s*-weakly *g*-closed sets are investigated.

E-mail address: wkmin@kangwon.ac.kr

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2. Preliminaries

Let S be a subset of a topological space X. The closure (resp., interior) of S will be denoted by clS (resp., $intS$). A subset S of X is called a *pre-open* (Mashhour et al., 1982) (resp., α-open (Njastad, 1964), *semi-open* (Levine, 1963)) set if $S \subset int(cl(S))$ (resp., $S \subset int(cl(int(S))), S \subset cl(int(S))$). The complement of a pre-open (resp., α-open, *semi-open*) set is called a *pre-closed* (resp., α-closed, *semi-closed*) set. The family of all pre-open (resp., α-open, semi-open) sets in X will be denoted by $PO(X)$ (resp., $\alpha(X), SO(X)$). The δ-interior of a subset A of X is the union of all regular open sets of X contained in A and it is denoted by $\delta - int(A)$ (Velicko, 1968). A subset A is called $\delta - open$ if $A = \delta - int(A)$. The complement of a $\delta - openset$ is called $\delta - closed$. The $\delta - closure$ of a set A in a space (X, τ) is defined by $\{x \in X : A \cap int(cl(B)) \neq , B \in \tau andx \in B\}$ and it is denoted by $\delta - cl(A)$. A subset A of a space (X, δ) is said $a - open$ (Ekici, 2008) if $A \subseteq int(cl(\delta - int(A)))$ and $a - closed$ if $A \subseteq cl(int(\delta - cl(A)))$. And A is said ω^**-open* (Ekici and Jafari, 2010) if for every $x \in V$, there exists an open subset $U \subseteq X$ containing x such that $U - \delta - int(A)$ is countable. The family of all a-open (resp., ω^*-open) sets in X will be denoted by $aO(X)$ (resp., $\omega^*O(X)$).

A subset A of a topological space (X, τ) is said to be:

(a) *g*-closed (Levine, 1970) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X;
(b) *gp*-closed (Noiri et al., 1998) if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X;
(c) *gs*-closed (Arya and Nori, 1990) if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X;
(d) $g\alpha$-closed (Maki et al., 1994) if $\tau^{\alpha}cl(A) \subseteq U$ whenever $A \subseteq U$ and U is α-open in X where $\tau^{\alpha} = \alpha(X)$;

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And the complement of a g-closed (resp., gp-closed, gs-closed, $g\alpha$-closed) set is called a g-open (resp., gp-open, gs-open, $g\alpha$-open) set. The family of all g-open (resp., gp-open sets, gs-open, $g\alpha$-open) sets in X will be denoted by $GO(X)$ (resp., $GPO(X)$, $GSO(X)$, $G\alpha O(X)$).

Let X be a nonempty set. A subfamily w_X of the power set $P(X)$ is called a *weak structure* (Kim and Min, 2015) on X if it satisfies the following:

(1) $\emptyset \in w_X$ and $X \in w_X$.
(2) For $U_1, U_2 \in w_X$, $U_1 \cap U_2 \in w_X$.

Then the pair (X, w_X) is called a *w-space* on X. Then $V \in w_X$ is called a *w-open* set and the complement of a w-open set is a *w-closed* set.

Then the family τ, $\alpha(X)$, $GO(X)$, $aO(X)$, $\omega^* O(X)$ and $g\alpha O(X)$ are all weak structures on X. But $PO(X)$, $SO(X)$, $GPO(X)$ and $GSO(X)$ are not weak structures on X.

Let (X, w_X) be a w-space. For a subset A of X, the *w-closure* of A and the *w-interior* (Kim and Min, 2015) of A are defined as follows:

(1) $wC(A) = \cap\{F : A \subseteq F,\ X - F \in w_X\}$.
(2) $wI(A) = \cup\{U : U \subseteq A,\ U \in w_X\}$.

Theorem 2.1. [Kim and Min, 2015] *Let (X, w_X) be a w-space and $A \subseteq X$.*

(1) *$x \in wI(A)$ if and only if there exists an element $U \in W(x)$ such that $U \subseteq A$.*
(2) *$x \in wC(A)$ if and only if $A \cap V \neq \emptyset$ for all $V \in W(x)$.*
(3) *If $A \subseteq B$, then $wI(A) \subseteq wI(B)$; $wC(A) \subseteq wC(B)$.*
(4) *$wC(X - A) = X - wI(A)$; $wI(X - A) = X - wC(A)$.*
(5) *If A is w-closed (resp., w-open), then $wC(A) = A$ (resp., $wI(A) = A$).*

Let (X, w_X) be a w-space and $A \subseteq X$. Then A is called *a generalized w-closed set* (simply, gw-closed set) (Min and Kim, 2016a) if $wC(A) \subseteq U$, whenever $A \subseteq U$ and U is w-open. If the w_X-structure is a topology, the generalized w-closed set is exactly a generalized closed set in sense of Levine in (Levine, 1970). Obviously, every w-closed set is generalized w-closed, but in general, the converse is not true.

And A is called *a weakly generalized w-closed set* (simply, weakly gw-closed set) (Min, 2017) if $wC(wI(A)) \subseteq U$ whenever $A \subseteq U$ and U is w-open. Obviously, every gw-closed set is weakly gw-closed. In (Min, 2017), we showed that every w-pre-closed set (Min and Kim, 2016b) is weakly gw-closed.

3. Main results

Now, we introduce an extended notion of gw-closed sets in w-spaces as the following:

Definition 3.1. Let (X, w_X) be a w-space and $A \subseteq X$. Then A is said to be *s-weakly generalized w-closed* (simply, s-weakly gw-closed) if $wI(wC(A)) \subseteq U$ whenever $A \subseteq U$ and U is w-open.

Obviously, the next theorem is obtained:

Theorem 3.2. *Every gw-closed set is s-weakly g-closed.*

Remark 3.3. In general, the converse of the above theorem is not true. Furthermore, there is no any relation between s-weakly gw-closed sets and weakly gw-closed sets as shown in the examples below:

Example 3.4. Let $X = \{a, b, c\}$ and $w = \{\emptyset, \{a\}, \{b\}, X\}$ be a weak structure in X. For a w-open set $A = \{b\}$, note that $wI(A) = A$, $wC(A) = \{b, c\}$ and $wI(wC(A)) = wI(\{b, c\}) = A$. So A is s-weakly gw-closed but not gw-closed. And since $wC(wI(A)) = \{b, c\}, A$ is also not weakly gw-closed.

Example 3.5. For $X = \{a, b, c, d\}$, let $w = \{\emptyset, \{d\}, \{a, b\}, \{a, b, c\}, X\}$ be a structure in X. Consider $A = \{a\}$. Then since $wI(A) = \emptyset$, obviously A is weakly gw-closed. For a w-open set $U = \{a, b\}$ with $A \subseteq U, wI(wC(A)) = wI(\{a, b, c\}) = \{a, b, c\} \neg\subseteq U$. So A is not s-weakly gw-closed.

In general, the intersection as well as the union of two s-weakly gw-closed sets is not s-weakly gw-closed as shown in the next examples:

Example 3.6. For $X = \{a, b, c, d\}$, let $w = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, c\}, \{a, c, d\}, X\}$ be a weak structure in X.

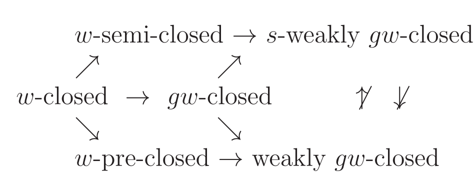
(1) Let us consider $A = \{a\}$ and $B = \{c\}$. Note that $wI(wC(A)) = wI(\{a, d\}) = A, wI(wC(B)) = wI(\{c, d\}) = B$ and $wI(wC(A \cup B)) = wI(\{a, c, d\}) = \{a, c, d\}$. Then we know that A and B are all s-weakly gw-closed sets but the union $A \cup B$ is not s-weakly gw-closed.
(2) Consider two s-weakly gw-closed sets $A = \{a, b, c\}$ and $B = \{a, c, d\}$. Then $A \cap B = \{a, c\}$ is not s-weakly gw-closed in the above (1).

Theorem 3.7. *Let (X, w_X) be a w-space. Then every w-semi-closed set is s-weakly gw-closed.*

Proof. Let A be a w-semi-closed set and U be a w-open set containing A. Since $wI(wC(A)) \subseteq A$, obviously it satisfies $wI(wC(A)) \subseteq U$. It implies that A is s-weakly gw-closed. □

Remark 3.8. In (2) of Example 3.6, the s-weakly gw-closed set $A = \{a, b, c\}$ is not w-semi-closed. So, the converse of the above theorem is not always true.

From the above theorems and examples, the following relations are obtained:



Let X be a nonempty set. Then a family $m(\subseteq P(X))$ of subsets of X is called *a minimal structure* (Maki, 1996) if $\emptyset, X \in m$.

Theorem 3.9. *Let (X, w_X) be a w-space. Then the family of all s-weakly gw-closed sets is a minimal structure in X.*

Lemma 3.10. [Kim and Min, 2015] Let (X, w_τ) be a w-space and $A, B \subseteq X$. Then the following things hold:

(1) $wI(A) \cap wI(B) = wI(A \cap B)$.
(2) $wC(A) \cup wC(B) = wC(A \cup B)$.

Let X be a w-space and $A \subseteq X$. Then A is said to be *w-semi-open* (resp., *w-semi-closed*) (Min and Kim, 2016c) if $A \subseteq wC(wI(A))$ (resp., $wI(wC(A)) \subseteq A$).

Lemma 3.11. *Let (X, w_X) be a w-space. Then for $A \subseteq X, A \cup wI(wC(A))$ is w-semi-closed.*

Proof. From Lemma 3.10 and Theorem 2.1, $wI(wC(A \cup wI(wC(A)))) = wI(wC(A) \cup wC(wI(wC(A))))$
$= wI(wC(A)) \subseteq A \cup wI(wC(A))$.
So, $A \cup wI(wC(A))$ is w-semi-closed. □

Lemma 3.12. *Let (X, w_X) be a w-space and $A \subseteq X$. If F is any w-semi-closed set such that $A \subseteq F$, then $A \cup wI(wC(A)) \subseteq F$.*

Proof. Let F be a w-semi-closed set with $A \subseteq F$. Then $wI(wC(A)) \subseteq wI(wC(F)) \subseteq F$, and so $A \cup wI(wC(A)) \subseteq F$. □

Let (X, w_X) be a w-space. For $A \subseteq X$, the *w-semi-closure* (Min and Kim, 2016c) of A, denoted by $wsC(A)$, is defined as:
$wsC(A) = \cap\{F \subseteq X : A \subseteq F, F \text{ is } w - semi - closed \text{ in } X\}$.

Theorem 3.13. *Let (X, w_X) be a w-space. Then for $A \subseteq X$, $wsC(A) = A \cup wI(wC(A))$.*

Proof. It is obtained from Lemma 3.11 and Lemma 3.12. □

Finally, we have the following theorem:

Theorem 3.14. *Let (X, w_X) be a w-space and $A \subseteq X$. Then A is s-weakly wg-closed if and only if $wsC(A) \subseteq U$ whenever $A \subseteq U$ and U is w-open.*

Proof. Let A be an s-weakly gw-closed subset of X and let U be any w-open set such that $A \subseteq U$. Then $wI(wC(A)) \subseteq U$ and $A \cup wI(wC(A)) \subseteq U$. So, by Theorem 3.13, $wsC(A) \subseteq U$.

For $A \subseteq X$, suppose that $wsC(A) \subseteq U$ whenever $A \subseteq U$ and U is w-open. Let U be any w-open set with $A \subseteq U$. Then from hypothesis and Theorem 3.13, $wI(wC(A)) \subseteq A \cup wI(wC(A)) = wsC(A) \subseteq U$. Hence, A is s-weakly gw-closed. □

Recall that: Let X be a topological space and $A \subseteq X$. Then A is call *a gs-closed set* (Arya and Nori, 1990) if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is open.

Theorem 3.15. *Let (X, w_X) be a w-space and $A \subseteq X$. If w_X is a topology, then the following thing hold: A is gs-closed if and only if $int(cl(A)) \subseteq U$ whenever $A \subseteq U$ and U is open.*

Proof. From $scl(A) = A \cap int(cl(A))$, $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is open if and only if $int(cl(A)) \subseteq U$ whenever $A \subseteq U$ and U is open. So, this theorem is obtained. □

Theorem 3.16. *Let (X, w_X) be a w-space. Then if A is an s-weakly gw-closed set, then $wI(wC(A)) - A$ contains no any non-empty w-closed set.*

Proof. For an s-weakly gw-closed set A, let F be a w-closed subset such that $F \subseteq wI(wC(A)) - A$. Then $A \subseteq X - F$ and $X - F$ is w-open. Since A is s-weakly gw-closed, $wI(wC(A)) \subseteq X - F$. From the facts, $F \subseteq X - wI(wC(A))$ and $F \subseteq wI(wC(A)) - A$, and so $F = \emptyset$. □

In general, the converse in Theorem 3.16 is not true as shown in the next example.

Example 3.17. Let $X = \{a, b, c, d\}$ and a weak structure $w = \{\emptyset, \{a\}, \{b\}, \{a, c\}, \{a, b, c\}, X\}$ in X. For $A = \{a\}$, $wI(wC(A)) = int(\{a, c, d\}) = \{a, c\}$ and $wI(wC(A)) - A = \{c\}$. So, we know that there is no any nonempty w-closed set contained in $wI(wC(A)) - A$. But A is not s-weakly gw-closed.

Corollary 3.18. *Let (X, w_X) be a w-space. Then if A is an s-weakly gw-closed set, then $wsC(A) - A$ contains no any non-empty w-closed set.*

Proof. Since $wI(wC(A)) - A = (A \cup wI(wC(A))) - A = wsC(A) - A$, by Theorem 3.16, the statement is satisfied. □

Theorem 3.19. *Let (X, w_X) be a w-space. Then if A is an s-weakly gw-closed set and $A \subseteq B \subseteq wsC(A)$, then B is s-weakly gw-closed.*

Proof. Let U be any w-open set such that $B \subseteq U$. By hypothesis, obviously $wsC(B) = wsC(A)$. Since A is s-weakly gw-closed and $A \subseteq U$, $wsC(B) = wsC(A) \subseteq U$. So B is s-weakly gw-closed. □

Corollary 3.20. *Let (X, w_X) be a w-space. Then if A is an s-weakly gw-closed set and $A \subseteq B \subseteq wI(wC(A))$, then B is s-weakly gw-closed.*

Proof. From $A \subseteq B \subseteq wI(wC(A))$, $A \subseteq B \subseteq A \cup wI(wC(A)) = wsC(A)$. By Theorem 3.19, the corollary is obtained. □

From now on, we introduce the notion of s-weakly gw-open sets and study its basic properties.

Definition 3.21. Let (X, w_X) be a w-space and $A \subseteq X$. Then A is called *an s-weakly generalized open set* (simply, s-weakly gw-open set) if $X - A$ is s-weakly gw-closed.

Theorem 3.22. *Let (X, w_X) be a w-space and $A \subseteq X$. Then A is s-weakly gw-open if and only if $F \subseteq wC(wI(A))$ whenever $F \subseteq A$ and F is w-closed.*

Proof. Obvious. □

From Theorem 3.13, the following is easily obtained:

Theorem 3.23. *Let (X, w_X) be a w-space. Then for $A \subseteq X$, $wsI(A) = A \cap wC(wI(A))$.*

Theorem 3.24. *Let (X, w_X) be a w-space and $A \subseteq X$. Then A is s-weakly gw-open if and only if $F \subseteq wsI(A)$ whenever $F \subseteq A$ and F is w-closed.*

Proof. For an s-weakly gw-open subset A of X, let F be a w-closed set such that $F \subseteq A$. Then $F \subseteq wC(wI(A))$. Since $F \subseteq A \cap wC(wI(A))$, by Theorem 3.23, $F \subseteq wsI(A)$.

For $A \subseteq X$, suppose that $F \subseteq wsI(A)$ whenever $F \subseteq A$ and F is w-closed. If F is any w-closed set and $F \subseteq A$, then by hypothesis and Theorem 3.23, $F \subseteq wsI(A) = A \cap wC(wI(A))$, and so $F \subseteq wC(wI(A))$. Hence, A is s-weakly gw-open. □

Theorem 3.25. *Let (X, w_X) be a w-space and $A \subseteq X$. Then if A is s-weakly gw-open, then $U = X$, whenever $wC(wI(A)) \cup (X - A) \subseteq U$ and U is w-open.*

Proof. Let U be any w-open set and $wC(wI(A)) \cup (X - A) \subseteq U$. Then $X - U \subseteq (X - wC(wI(A))) \cap A$ $= wI(wC(X - A)) \cap A = wI(wC(X - A)) - (X - A)$. Since $X - A$ is s-weakly gw-closed, by Theorem 3.16, the w-closed set $X - U$ must be empty. Hence, $U = X$. □

Corollary 3.26. *Let (X, w_X) be a w-space and $A \subseteq X$. Then if A is s-weakly gw-open, then $U = X$, whenever $wsI(A) \cup (X - A) \subseteq U$ and U is w-open.*

Proof. Since $wsI(A) \cup (X - A) = (A \cap wC(wI(A))) \cup (X - A)$, by the above theorem, it is obtained. □

Theorem 3.27. *Let (X, w_X) be a w-space. Then if A is an s-weakly gw-open set and $wC(wI(A)) \subseteq B \subseteq A$, then B is s-weakly gw-open.*

Proof. It is similar to the proof of Theorem 3.19 and Corollary 3.20. □

Theorem 3.28. *Let (X, w_X) be a w-space. Then if A is an s-weakly gw-closed set, then $wI(wC(A)) - A$ is s-weakly gw-open.*

Proof. If A is an s-weakly gw-closed set, then by Theorem 3.12, $\emptyset$ is the only one w-closed subset of $wI(wC(A)) - A$. So, $\emptyset \subseteq wC(wI(wI(wC(A)) - A))$. Hence, $wI(wC(A)) - A$ is s-weakly gw-open. □

Corollary 3.29. *Let (X, w_X) be a w-space. Then if A is an s-weakly gw-closed set, then $wsC(A) - A$ is s-weakly gw-open.*

Proof. From $wsC(A) - A = (A \cup wI(wC(A))) - A = wI(wC(A)) - A$, it is obtained. □

Theorem 3.30. *Let (X, w_X) be a w-space. Then if A is an s-weakly gw-open set, then $wC(wI(A)) \cup (X - A)$ is s-weakly gw-closed.*

Proof. If A is an s-weakly gw-open set, then by Theorem 3.25, X is the only one w-open set containing $wC(wI(A)) \cup (X - A)$. So, obviously, $wC(wI(A)) \cup (X - A)$ is s-weakly gw-closed. □

Corollary 3.31. *Let (X, w_X) be a w-space. Then if A is an s-weakly gw-open set, then $wsI(A) \cup (X - A)$ is s-weakly gw-closed.*

Proof. It follows from $wsI(A) \cup (X - A) = (A \cap wC(wI(A))) \cup (X - A) = wC(wI(A)) \cup (X - A)$ and Theorem 3.30. □

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