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Approximation by Phillips operators via q-Dunkl generalization based on a new parameter

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ABSTRACT

In the present article we study the approximation properties of Phillips operators by q-Dunkl generalization. We construct the operators in a new q-Dunkl form and obtain the approximation properties in weighted function space. We give the rate of convergence in terms of Lipschitz class by initiate the modulus of continuity and finally, we present some direct theorems in Peetre's K -functional.

1. Introduction and auxiliary results

The q-calculus also known as "quantum calculus" began to arise with the interest grown explosively in physic as well as in mathematics both due to the large number of its applications. For instance, some q-analogues of Fourier analysis (see Koornwinder and Swarttouw, 1992), q-harmonic analysis to study q-wavelets and q -wavelets packets (see Bettaibi et al., 2010; Fitouhi and Bettaibi, 2006). The q-calculus plays very important role in the development of approximation process and has led in finding more appropriate generalizations of several classical operators. The q-operators have better rate of convergence than classical ones (see Lupaş, 1987; Phillips, 1997). In the recent years, more development of Szász operators based on Dunkl generalization have been obtained by several mathematicians.

In 1950, the classical Szász (Szász, 1950) operators were defined by

$$S_\rho(h;y) = e^{-\rho y}\sum_{k=0}^{\infty}\frac{(\rho y)^k}{k!}h\left(\frac{k}{\rho}\right), h \in C[0,\infty). \tag{1.1}$$

There are many important research papers on the study of approximation of Szász type operators via Dunkl generalization in q-calculus and in (p,q)-calculus, for instance we refer to Alotaibi (2019), Alotaibi and Mursaleen (2020), İçöz and Çekim (2015), Nasiruzzaman and Mursaleen (2020) and Sucu (2014) etc. Sucu (2014) gave the Dunkl form of classical Szász-operators with an exponential function (see Rosenblum, 1994) and the q-Hermite type polynomials studied by Cheikh et al. (2014).

The basic definition of q-integer and q-factorial is given by:

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$$[\rho]_q = \begin{cases} \frac{1-q^\rho}{1-q}, & q\neq 1,\ \rho\in\mathbb{N} \\ 1, & q=1 \\ 0, & \rho=0 \end{cases}$$

$$[\rho]_q! = \begin{cases} 1, & \rho=0 \\ \prod_{k=1}^{\rho}[k]_q & \rho\in\mathbb{N}. \end{cases} \tag{1.2}$$

Recalling some basic definitions of the exponential functions and their recursion formulas in structure of q-Dunkl, we have

$$e_{\kappa,q}(y) = \sum_{s=0}^{\infty}\frac{y^s}{\gamma_{\kappa,q}(s)}, \quad y\in[0,\infty), \quad E_{\kappa,q}(y) = \sum_{s=0}^{\infty}\frac{q^{\frac{s(s-1)}{2}}y^s}{\gamma_{\kappa,q}(s)}, \quad y\in[0,\infty) \tag{1.3}$$

$$\gamma_{\kappa,q}(s+1) = \left(\frac{1-q^{2\kappa\theta_{s+1}+s+1}}{1-q}\right)\gamma_{\kappa,q}(s), \quad s\in\mathbb{N}, \tag{1.4}$$

$$\theta_s = \begin{cases} 0 & \text{if } s\in 2\mathbb{N}, \\ 1 & \text{if } s\in 2\mathbb{N}+1. \end{cases} \tag{1.5}$$

where

$$\gamma_{\kappa,q}(s) = \frac{(q^{2\kappa+1},q^2)_{[\frac{s+1}{2}]}(q^2,q^2)_{[\frac{s}{2}]}}{(1-q)^s}\gamma_{\kappa,q}(s), \quad s\in\mathbb{N}, \tag{1.6}$$

and some basic calculations for $s = 0,1,2,3,4$, the Dunkl q-integers form are

$$\gamma_{\kappa,q}(0) = 1, \quad \gamma_{\kappa,q}(1) = \frac{1-q^{2\kappa+1}}{1-q},$$

$$\gamma_{\kappa,q}(2) = \left(\frac{1-q^{2\kappa+1}}{1-q}\right)\left(\frac{1-q^2}{1-q}\right),$$

$$\gamma_{\kappa,q}(3) = \left(\frac{1-q^{2\kappa+1}}{1-q}\right)\left(\frac{1-q^2}{1-q}\right)\left(\frac{1-q^{2\kappa+3}}{1-q}\right),$$

$$\gamma_{\kappa,q}(4) = \left(\frac{1-q^{2\kappa+1}}{1-q}\right)\left(\frac{1-q^2}{1-q}\right)\left(\frac{1-q^{2\kappa+3}}{1-q}\right)\left(\frac{1-q^4}{1-q}\right).$$

By using the definition based on q-integrs and their exponential Dunkl generalization the Szász operators obtained by İçöz and Çekim (2015) such as

$$D_{\rho,q}(h;y) = \frac{1}{e_{\kappa,q}([\rho]_q y)}\sum_{s=0}^{\infty}\frac{([\rho]_q y)^s}{\gamma_{\kappa,q}(s)}h\left(\frac{1-q^{2\kappa\theta_s+s}}{1-q^\rho}\right), \tag{1.7}$$

for any $\kappa > \frac{1}{2},\ y \geqslant 0,\ 0<q<1$ and $h\in C[0,\infty)$.

In recent investigation (Nasiruzzaman and Rao, 2018), authors have introduced a new extended form of Phillips operators by implement the new parameter $\kappa \geqslant 0$,

$$\mathscr{P}^*_{\rho,\kappa}(h;y) = \frac{\rho^2}{e_\kappa(\rho y)}\sum_{s=0}^{\infty}\frac{(\rho y)^s}{\gamma_\kappa(s)}\int_0^\infty\frac{e^{-\rho t}\rho^{s+2\kappa\theta_s-1}t^{s+2\kappa\theta_s}}{(s+2\kappa\theta_s)!}f(t)dt,$$

for each $h\in C[0,\infty)$ and $y\in[0,\infty)$, where, $e_\kappa(y) = \sum_{s=0}^{\infty}\frac{y^s}{\gamma_\kappa(s)}$ and $\frac{\gamma_\kappa(s+1)}{(s+1+2\kappa\theta_{s+1})} = \gamma_\kappa(s)$ and θ_s is defined by (5.1).

In this context, we construct the Phillips operators in a newly modified form via q-calculus generated by exponential function and study the approximation properties and its related results. Moreover, we want to calculate the convergence results by applying the modulus of continuity and investigate their order of approximation in Lipschitz space and give some direct theorems. In Nasiruzzaman and Rao (2018), authors have studied a generalized Dunkl type modifications of Phillips operators. The approximation results in the present article enable to give a modified version and have more general results than (Nasiruzzaman and Rao, 2018). Approximation properties of different types of operators are obtained by several authors in Alotaibi and Mursaleen (2020), Kilicman et al. (2020), Milovanovic et al. (2018), Mohiuddine and Özger (2020), Mursaleen et al. (2020), Nasiruzzaman and Mursaleen (2020), Rao et al. (2019) and Srivastava et al. (2019).

Let $\{\mu_{\rho,q}(y)\}_{\rho\geqslant 1}$ be the class of sequence of continuous functions on semi axis $\mathbb{R}^+ = [0,\infty)$ such that

$$\mu_{\rho,q}(y) = \left(y - \frac{1}{2[\rho]_q}\right)_\varrho, \quad \rho\in\mathbb{N}, \tag{1.8}$$

with the notations

$$\lambda_\varrho = \begin{cases} \lambda & \text{if } \lambda \geqslant 0, \\ 0 & \text{if } \lambda < 0. \end{cases} \tag{1.9}$$

Now for every $h\in C_\zeta[0,\infty) = \{h\in C[0,\infty) : h(t) = O(t^\zeta),\}$ when $t\to\infty$, and $y\in[0,\infty), \zeta>\rho, \rho\in\mathbb{N}\cup\{0\}, \kappa \geqslant -\frac{1}{2}$, we define

$$\mathscr{U}^*_{\rho,q}(h;y) = \frac{[\rho]_q}{e_{\kappa,q}([\rho]_q\mu_{\rho,q}(y))}\sum_{p=0}^{\infty}\frac{([\rho]_q\mu_{\rho,q}(y))^p}{\gamma_{\kappa,q}(p)} q^{\frac{(p+2\kappa\theta_p)(p+2\kappa\theta_p+1)}{2}}Q^*_{\kappa,q}\left(h(q^{p+2\kappa\theta_p}t)\right), \tag{1.10}$$

where $\lambda = \frac{1}{[\rho]_q}$ and

$$Q^*_{\kappa,q}\left(h(q^{p+2\kappa\theta_p}t)\right) = \int_0^{\infty/1-q}\frac{e_{\kappa,q}(-[\rho]_q t)[\rho]_q^{p+2\kappa\theta_p}t^{p+2\kappa\theta_p}}{[p+2\kappa\theta_p]_q!}h(q^{p+2\kappa\theta_p}t)d_qt.$$

Lemma 1.1. *For every $h\in C_\zeta[0,\infty) = \{h\in C[0,\infty) : h(t) = O(t^\zeta)\}$ as $t\to\infty$, and $y\in[0,\infty), \zeta>\rho,\ \rho\in\mathbb{N}\cup\{0\},\ \ \kappa\geqslant -\frac{1}{2}$, we have*

$$\int_0^{\infty/1-q} q^{\frac{(p+2\kappa\theta_p)(p+2\kappa\theta_p+1)}{2}} \times\frac{e_{\kappa,q}(-[\rho]_q t)[\rho]_q^{p+2\kappa\theta_p}t^{p+2\kappa\theta_p}}{[p+2\kappa\theta_p]_q!}\left(q^{p+2\kappa\theta_p}t\right)^u d_qt = \frac{1}{[\rho]_q^{u+1}}\frac{[p+2\kappa\theta_p+u]_q!}{[p+2\kappa\theta_p]_q!}\frac{1}{q^{\frac{u(u+1)}{2}}}. \tag{1.11}$$

In order to get the basic estimates we use the appropriate generalized definition of gamma function in q-calculus.

Definition 1.2. The generalized definition of gamma function in q-calculus given by

$$\Gamma_q(t) = \int_0^{1/1-q}y^{t-1}E_q(-qy)d_qy, \ \ t>0, \tag{1.12}$$

$$\gamma_q^F(t) = \int_0^{\infty/F(1-q)}y^{t-1}e_q(-y)d_qy, \ \ t>0, \tag{1.13}$$

where we use the formulas $\Gamma_q(t) = K(F;t)\gamma_q^F(t)$ and $K(F;t) = \frac{1}{1+F}F^t\left(1+\frac{1}{F}\right)_q^t(1+F)_q^{t-1}$. Moreover, in particular for any positive integer ρ, we have $K(F;\rho) = q^{\frac{\rho(\rho-1)}{2}}$ and $\Gamma_q(\rho) = q^{\frac{\rho(\rho-1)}{2}}\gamma_q^F(\rho)$, with the following gamma mathematical expression

$$\Gamma_q(\mu+1) = \begin{cases} [\mu]_q\Gamma_q(\mu) & \text{for } \mu>0 \\ 1 & \text{for } \mu=0. \end{cases} \tag{1.14}$$

$$B_q(t,m) = K(F,t)\int_0^{\infty/F} \frac{y^{t-1}}{(1+y)_q^{t+m}}\, d_q y = \frac{[t-1]_q}{[m]_q} B_q(t-1, m+1), \quad t>1, m>0,$$

with

$$K(F,t+1) = q^t K(F,t),$$

$$K(F,t) = q^{\frac{t(t-1)}{2}},\ K(F,0)=1,$$

and the improper integral of function h for q-integers is calculated as:

$$\int_0^{\infty/F} h(y) d_q y = (1-q)\sum_{\rho\in\mathbb{N}} h\left(\frac{q^\rho}{F}\right)\frac{q^\rho}{F}, F\in\mathbb{R}-\{0\}.$$

For more details we propose to see De Sole and Kac (2005).

In 1954, an inversion formula on the semigroups of positive linear operators was obtained in May (1977), while recently the approximation properties of Phillips operators were studied by applying the Dunkl generalization of an exponential form in Nasiruzzaman and Rao (2018). Most recently, the approximation results of Phillips operators by introducing the Dunkl generalization of q -exponential form were studied in Nasiruzzaman et al. (2019), which include the better generalized approximation properties of Phillips operators for $y\in[0,\infty)$ and function $f\in C[0,\infty)$ rather than (May, 1977; Nasiruzzaman and Rao, 2018). In this paper, our results are primarily concerned with the problem on the domain $0\leqslant y<\frac{1}{2[\rho]_q}$ and $\frac{1}{2[\rho]_q}\leqslant y<\infty$for determining the better generalized approximation results of Phillips operator. Our generalization of Phillips operators gives more appropriate and modified convergence properties in quantum calculus rather than the published article (May, 1977; Nasiruzzaman and Rao, 2018; Nasiruzzaman et al., 2019).

2. Basic Estimates and their moments

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Lemma 2.1. *Suppose* $\mho_j = t^j$ *for* $j=0,1,2,3,4$. *Then for all* $\rho\in\mathbb{N}$ *and* $y\geqslant\frac{1}{2[\rho]_q}$, $0<q_\rho<1$ *the operators* $\mathscr{U}^*_{\rho,q}(\cdot;\cdot)$, *have* $\mathscr{U}^*_{\rho,q}(\mho_0;y)=1$, *and the following properties:*

(1) $\mathscr{U}^*_{\rho,q}(\mho_1;y) = y + \frac{1}{q[2]_q[\rho]_q}\Big([2]_q - q\Big),$

(2)
$$\begin{aligned}\mathscr{U}^*_{\rho,q}(\mho_2;y) \leqslant\ & y^2 + \tfrac{1}{q^2[2]_q[\rho]_q}\Big\{[2]_q^2 + [2]_q q + \Big(-2+[2]_q[1+2\kappa]_q\Big)q^2\Big\}y \\ &+ \tfrac{1}{q^3[2]_q^2[\rho]_q^2}\Big([2]_q^3 - [2]_q^2 q - [2]_q q^2 + \Big(1-[2]_q[1+2\kappa]_q\Big)q^3\Big)\end{aligned}$$

(3)
$$\begin{aligned}\mathscr{U}^*_{\rho,q}(\mho_3;y) \leqslant\ & y^3 + \tfrac{1}{q^4[2]_q[\rho]_q}\Big\{[2]_q^3 q + 2[2]_q q^3 + \Big(-3+3[2]_q[1+2\kappa]_q\Big)q^4\Big\}y^2 \\ &+ \tfrac{1}{q^5[2]_q^2[\rho]_q^2}\Big\{[2]_q^4 + [2]_q^2 q + [2]_q^3\Big([2]_q[1+2\kappa]_q + 2\Big)q^2 \\ &+ [2]_q\Big(2[2]_q[1+2\kappa]_q - 2\Big)q^4 \\ &+ \Big(3 - 3[2]_q[1+2\kappa]_q + [2]_q^2[1+2\kappa]_q^2\Big)q^5\Big\}y \\ &+ \tfrac{1}{q^6[2]_q^3[\rho]_q^3}\Big\{-[2]_q^4 q - [2]_q^2 q^2 + [2]_q^3\Big(-2-[2]_q[1+2\kappa]_q\Big)q^3 \\ &+ 2[2]_q\Big(-[2]_q[1+2\kappa]_q + 1\Big)q^5 \\ &+ \Big(3[2]_q[1+2\kappa]_q - [2]_q^2[1+2\kappa]_q^2 - 1\Big)q^6\Big\}\end{aligned}$$

(4)
$$\begin{aligned}\mathscr{U}^*_{\rho,q}(\mho_4;y) \leqslant\ & \tfrac{[2]_q}{q^{10}[\rho]_q^4}\Big(q + [2]_q + 3[2]_q q^2 + \Big([2]_q + 1\Big)q^4\Big) \\ &+ \tfrac{1}{q^9[\rho]_q^3}\{(1 + 4[2]_q q + 4q^2 + 12[2]_q q^3 + \Big(4[2]_q + 5\Big)q^5 \\ &+ q^2\Big([2]_q + 2q + 7[2]_q q^2 + 2q^3 + 3\Big(2[2]_q + 1\Big)q^4\Big)[1+2\kappa]_q \\ &+ q^5\Big([2]_q + q + 3[2]_q q^2 + q^3\Big)[1+2\kappa]_q^2 + q^9[1+2\kappa]_q^3\} \\ &\Big(y - \tfrac{1}{[2]_q[\rho]_q}\Big) \\ &+ \tfrac{1}{q^8[\rho]_q^2}\Big\{[2]_q q + q^2 + 7[2]_q q^3 + 2q^4 + \Big(6[2]_q + 3\Big)q^5 \\ &+ \Big([2]_q q^4 + q^5 + 3[2]_q q^6 + q^7\Big)[1+2\kappa]_q + 7[1+2\kappa]_q^2 q^8\Big\} \\ &\Big(y - \tfrac{1}{[2]_q[\rho]_q}\Big)^2 \\ &+ \tfrac{1}{q^7[\rho]_q}\Big\{[2]_q q^3 + q^4 + 3[2]_q q^5) + q^6 + 6[1+2\kappa]_q q^7\Big\} \\ &\Big(y - \tfrac{1}{[2]_q[\rho]_q}\Big)^3 + \Big(y - \tfrac{1}{[2]_q[\rho]_q}\Big)^4.\end{aligned}$$

Proof. We take into account Lemma 1.1 and the generalized definition of gamma function in q-integers 1.2. Take $h(t)=\mho_0,\mho_1,\mho_2$, then easily get that

$$\begin{aligned}\mathscr{U}^*_{\rho,q}(\mho_0;y) =\ & \frac{[\rho]_q}{e_{\kappa,q}([\rho]_q\mu_{\rho,q}(y))}\sum_{p=0}^{\infty}\frac{([\rho]_q\mu_{\rho,q}(y))^p}{\gamma_{\kappa,q}(p)}\frac{[p+2\kappa\theta_p]_q!}{[\rho]_q[p+2\kappa\theta_p]_q!} \\ =\ & 1.\end{aligned}$$

For $h(t)=\mho_1$, hence

$$\begin{aligned}\mathscr{U}^*_{\rho,q}(\mho_1;y) =\ & \frac{[\rho]_q}{e_{\kappa,q}([\rho]_q\mu_{\rho,q}(y))}\sum_{p=0}^{\infty}\frac{([\rho]_q\mu_{\rho,q}(y))^p}{\gamma_{\kappa,q}(p)}\frac{[p+2\kappa\theta_p+1]_q!}{q[\rho]_q^2[p+2\kappa\theta_p]_q!} \\ =\ & \frac{1}{q[\rho]_q e_{\kappa,q}([\rho]_q\mu_{\rho,q}(y))}\sum_{p=0}^{\infty}\frac{([\rho]_q\mu_{\rho,q}(y))^p}{\gamma_{\kappa,q}(p)}[p+2\kappa\theta_p+1]_q \\ =\ & \frac{1}{q[\rho]_q e_{\kappa,q}([\rho]_q\mu_{\rho,q}(y))}\sum_{p=0}^{\infty}\frac{([\rho]_q\mu_{\rho,q}(y))^p}{\gamma_{\kappa,q}(p)} \\ &+ \frac{1}{[\rho]_q e_{\kappa,q}([\rho]_q\mu_{\rho,q}(y))}\sum_{p=0}^{\infty}\frac{([\rho]_q\mu_{\rho,q}(y))^p}{\gamma_{\kappa,q}(p)}[p+2\kappa\theta_p]_q \\ =\ & \mu_{\rho,q}(y) + \tfrac{1}{q[\rho]_q}.\end{aligned}$$

For $h(t)=\mho_2$, we have

$$\begin{aligned}\mathscr{U}^*_{\rho,q}(\mho_2;y) =\ & \frac{[\rho]_q}{e_{\kappa,q}([\rho]_q\mu_{\rho,q}(y))}\sum_{p=0}^{\infty}\frac{([\rho]_q\mu_{\rho,q}(y))^p}{\gamma_{\kappa,q}(p)}\frac{[p+2\kappa\theta_p+2]_q!}{q^3[\rho]_q^3[p+2\kappa\theta_p]_q!} = \frac{1}{q^3[\rho]_q^2 e_{\kappa,q}([\rho]_q\mu_{\rho,q}(y))} \\ & \sum_{p=0}^{\infty}\frac{([\rho]_q\mu_{\rho,q}(y))^p}{\gamma_{\kappa,q}(p)}[p+2\kappa\theta_p+2]_q[p+2\kappa\theta_p+1]_q \\ =\ & \frac{1}{q^3[\rho]_q^2 e_{\kappa,q}([\rho]_q\mu_{\rho,q}(y))}\sum_{p=0}^{\infty}\frac{([\rho]_q\mu_{\rho,q}(y))^p}{\gamma_{\kappa,q}(p)} \\ & \times\{[2]_q + q(1+2q)[p+2\kappa\theta_p]_q + q^3[p+2\kappa\theta_p]_q^2\} \\ =\ & \frac{(1+q)}{q^3[\rho]_q^2} + \frac{(1+2q)}{q^2[\rho]_q}\mu_{\rho,q}(y) + \frac{1}{[\rho]_q^2 e_{\kappa,q}([\rho]_q\mu_{\rho,q}(y))}\sum_{p=0}^{\infty}\frac{([\rho]_q\mu_{\rho,q}(y))^p}{\gamma_{\kappa,q}(p)} \\ & [p+2\kappa\theta_p]_q^2. \\ \leqslant\ & \frac{(1+q)}{q^3[\rho]_q^2} + \frac{(1+2q)}{q^2[\rho]_q}\mu_{\rho,q}(y) + \Big(\mu_{\rho,q}(y)\Big)^2 + \frac{[1+2\kappa]_q}{[\rho]_q}\mu_{\rho,q}(y)\end{aligned}$$

Similarly, if $h(t)=\mho_3$ and $h(t)=\mho_4$, then we have

$$\mathscr{U}^*_{\rho,q}(\mho_3;y) = \frac{1}{q^4[\rho]_q^3 e_{\kappa,q}([\rho]_q\mu_{\rho,q}(y))}\sum_{p=0}^{\infty}\frac{([\rho]_q\mu_{\rho,q}(y))^p}{\gamma_{\kappa,q}(p)} [p+2\kappa\theta_p+3]_q[p+2\kappa\theta_p+2]_q[p+2\kappa\theta_p+1]_q$$

and

$$\mathscr{U}^*_{\rho,q}(\mho_4;y) = \frac{1}{q^{10}[\rho]_q^4 e_{\kappa,q}([\rho]_q\mu_{\rho,q}(y))}\sum_{p=0}^{\infty}\frac{([\rho]_q\mu_{\rho,q}(y))^p}{\gamma_{\kappa,q}(p)} \times[p+2\kappa\theta_p+4]_q[p+2\kappa\theta_p+3]_q[p+2\kappa\theta_p+2]_q [p+2\kappa\theta_p+1]_q.$$

Hence for $h(t)=\mho_3$ and $h(t)=\mho_4$ we use results by (1.7) (see İçöz and Çekim, 2015), and the recent results obtained in Nasiruzzaman et al. (2019) we get the required other proofs.

Lemma 2.2. *For $\mho_j = t^j$ for $j = 0,1,2,3,4$. Then for all $\rho\in\mathbb{N}$ and $0<y<\frac{1}{2[\rho]_q}$, $0<q_\rho<1$ then we get*

(2^*) $\mathscr{U}^*_{\rho,q}(\mho_1;y) = \frac{1}{q[\rho]_q}$,

(3^*) $\mathscr{U}^*_{\rho,q}(\mho_2;y) \leqslant \frac{[2]_q}{q^3[\rho]_q^2}$,

(4^*) $\mathscr{U}^*_{\rho,q}(\mho_3;y) \leqslant \frac{1}{q^6[\rho]_q^3}(-[2]_q q - 2q^3)$,

(5^*) $\mathscr{U}^*_{\rho,q}(\mho_4;y) \leqslant \frac{[2]_q}{q^{10}[\rho]_q^4}\left(q+[2]_q+3[2]_q q^2+\left([2]_q+1\right)q^4\right)$.

Lemma 2.3. *For any $i=1,2$, we suppose $\varphi_i = \mathscr{U}^*_{\rho,q}((\mho_1-y)^i;y)$ then*

$$\varphi_i = \begin{cases} \frac{1}{q^2[2]_q[\rho]_q}\{[2]_q^2-[2]_q q+[2]_q[1+2\kappa]_q q^2\}y \\ +\frac{1}{q^3[2]_q^2[\rho]_q^2}\left([2]_q^3-[2]_q^2 q-[2]_q q^2+\left(1-[2]_q[1+2\kappa]_q\right)q^3\right), \\ \quad \text{for } i=2 \\ \frac{1}{q[2]_q[\rho]_q}\left(-q+[2]_q\right), \quad \text{for} i=1 \end{cases}$$

3. Korovkin type theorems in weighted spaces

In mathematics and other branches of sciences the Korovkin's type approximation theory has many applications see Altomare (2010) (see also Alotaibi et al., 2021; Alotaibi and Mursaleen, 2011, 2016; Braha, 2018, 2020, 2021; Mursaleen and Karakaya, 2012). In the present section, we approximate the sequence of our new operators $\{\mathscr{U}^*_{\rho,q}(\cdot;\cdot)\}_{\rho\geqslant1}$ by (1.10) by make use of Korovkin's and weighted Korovkin's theorem. We take $q=q_\rho, (0<q_\rho<1)$ with $\lim_{\rho\to\infty}q_\rho=1$ and $\lim_{\rho\to\infty}q_\rho^\rho=\eta$where $0<\eta<1$. In this context, we suppose $\mathbb{R}^+=[0,\infty)$ and let we symbolize the set of all bounded and continuous functions by $C_B(\mathbb{R}^+)$, and equipped the norm function h on $C_B(\mathbb{R}^+)$ by

$$\|h\|_{C_B} = \sup_{y\geqslant0}|h(y)|.$$

Theorem 3.1. *Let $q=q_\rho$ be the sequences positive numbers satisfying $0<q_\rho<1$. Then, for every $h\in\{\phi:\frac{\phi(y)}{1+y^2}$ is convergent as $y\to\infty\}\cap C[0,\infty)$, we have*

$$\lim_{\rho\to\infty}\mathscr{U}^*_{\rho,q_\rho}(h;y)=h(y).$$

Proof. Clearly, as $\rho\to\infty, \frac{1}{[\rho]_q}\to0$. By using Lemma 2.1 and Korovkin's theorem, for all $j=0,1,2$, we have

$$\lim_{\rho\to\infty}\mathscr{U}^*_{\rho,q_\rho}(\mho_j;y)=y^j$$

is uniformly for every $h\in\{\phi:\frac{\phi(y)}{1+y^2}$ is convergent as $y\to\infty\}\cap C[0,\infty)$. Thus we follows immediately the proof.

Moreover, we recall the following well-known results in weighted spaces:

$$\mathscr{S}_\sigma(\mathbb{R}^+) = \{h:|h(y)|\leqslant M_h\sigma(y)\},$$
$$\mathscr{T}_\sigma(\mathbb{R}^+) = \{h:h\in\mathscr{S}_\sigma(\mathbb{R}^+)\cap C[0,\infty)\},$$
$$\mathscr{T}^p_\sigma(\mathbb{R}^+) = \left\{h:h\in\mathscr{T}_\sigma(\mathbb{R}^+) \text{ and } \lim_{y\to\infty}\frac{h(y)}{1+y^2}=M_h\right\},$$

where M_h depending on h and positive constant. Additionally, we notice here $\mathscr{T}_\sigma(\mathbb{R}^+)$ is a normed space equipped with $\|h\|_\sigma=\sup_{y\geqslant0}\frac{|h(y)|}{1+y^2}$.

Theorem 3.2. *For all $h\in\mathscr{T}^p_\sigma(\mathbb{R}^+)$ and $y\geqslant\frac{1}{2[\rho]_q}$ with $q=q_\rho$, $0<q_\rho<1$ we have*

$$\lim_{\rho\to\infty}\|\mathscr{U}^*_{\rho,q_\rho}(h;y)-h\|_\sigma=0.$$

Proof. To prove this theorem we let $h(t)=\mho_\tau$ for all $\tau=0,1,2$. Then for all $h(t)\in\mathscr{T}^p_\sigma(\mathbb{R}^+)$, the Korovkin's theorem give us $\mathscr{U}^*_{\rho,q_\rho}(t^\tau;y)\to y^\tau$ as $\rho\to\infty$. From the Lemma 2.1, we have $\mathscr{U}^*_{\rho,q_\rho}(\mho_0;y)=1$, then clearly

$$\lim_{\rho\to\infty}\left\|\mathscr{U}^*_{\rho,q_\rho}(\mho_0;y)-1\right\|_\sigma=0. \tag{3.1}$$

For $h(t)=\mho_1$

$$\left\|\mathscr{U}^*_{\rho,q_\rho}(\mho_1;y)-y\right\|_\sigma = \sup_{y\geqslant\frac{1}{2[\rho]_{q_\rho}}}\frac{\left|\mathscr{U}^*_{\rho,q_\rho}(\mho_1;y)-y\right|}{1+y^2} = \frac{1}{q_\rho[2]_{q_\rho}[\rho]_{q_\rho}}\left([2]_{q_\rho}-q_\rho\right)\sup_{y\geqslant\frac{1}{2[\rho]_{q_\rho}}}\frac{1}{1+y^2}$$

$\lim_{\rho\to\infty}\frac{1}{[\rho]_{q_\rho}}=0$ implies that

$$\lim_{\rho\to\infty}\left\|\mathscr{U}^*_{\rho,q_\rho}(\mho_1;y)-y\right\|_\sigma=0. \tag{3.2}$$

Similarly, for $h(t)=\mho_2$

$$\left\|\mathscr{U}^*_{\rho,q_\rho}(\mho_2;y)-y^2\right\|_\sigma = \sup_{y\geqslant\frac{1}{2[\rho]_{q_\rho}}}\frac{\left|\mathscr{U}^*_{\rho,q_\rho}(\mho_2;y)-y^2\right|}{1+y^2} = \frac{1}{q_\rho^2[2]_{q_\rho}[\rho]_{q_\rho}}\{[2]_{q_\rho}^2+[2]_{q_\rho}q_\rho +\left(-2+[2]_{q_\rho}[1+2\kappa]_{q_\rho}\right)q_\rho^2\}\sup_{y\geqslant\frac{1}{2[\rho]_{q_\rho}}}\frac{y}{1+y^2} +\frac{1}{q_\rho^3[2]_{q_\rho}^2[\rho]_{q_\rho}^2}\left([2]_{q_\rho}^3-[2]_{q_\rho}^2q_\rho-[2]_{q_\rho}q_\rho^2 +\left(1-[2]_{q_\rho}[1+2\kappa]_{q_\rho}\right)q_\rho^3\right)\sup_{y\geqslant\frac{1}{2[\rho]_{q_\rho}}}\frac{1}{1+y^2}$$

which implies that

$$\lim_{\rho\to\infty}\left\|\mathscr{U}^*_{\rho,q_\rho}(\mho_2;y)-y^2\right\|_\sigma=0. \tag{3.3}$$

Hence the result.

Corollary 3.3. *For all $0\leqslant y<\frac{1}{2[\rho]_q}$ and $h\in\mathscr{T}^p_\sigma(\mathbb{R}^+)$*

$$\lim_{\rho\to\infty}\|\mathscr{U}^*_{\rho,q_\rho}(h;y)-h\|_\sigma=0.$$

In order to investigate the maximum oscillation of φ, the notion of modulus of smoothness of φ in classical order given by

$$\omega(\varphi;\delta)=\sup_{|\mho_1-y|\leqslant\delta}|\varphi(\mho_1)-\varphi(y)|;\ \mho_1\in[0,\infty),\delta>0,\ \mho_1=t \tag{3.4}$$

for all $y\in[0,\infty)$ and $\varphi\in C_B(\mathbb{R}^+)$, and one has

$$|\varphi(\mho_1)-\varphi(y)|\leqslant\left(\frac{|\mho_1-y|}{\delta}+1\right)\omega(\varphi;\delta). \tag{3.5}$$

Theorem 3.4. *Let $q=q_\rho$ such that $0<q_\rho<1$. Then for every $h\in C_B(\mathbb{R}^+)$ and $y\geqslant\frac{1}{2[\rho]_q}$, operators $\mathscr{U}^*_{\rho,q_\rho}(h;.)$ satisfy*

$$|\mathscr{U}^*_{\rho,q_\rho}(h;y)-h(y)|\leqslant\{1+\sqrt{\Theta_{\rho,q_\rho}(y)}\}\omega\left(h;\frac{1}{\sqrt{[\rho]_{q_\rho}}}\right),$$

where

$$\Theta_{\rho,q_\rho}(y)=\frac{1}{q_\rho^2[2]_{q_\rho}}\{[2]^2_{q_\rho}-[2]_{q_\rho}q_\rho+[2]_{q_\rho}[1+2\kappa]_{q_\rho}q_\rho^2\}y$$
$$+\frac{1}{q_\rho^3[2]^2_{q_\rho}[\rho]_{q_\rho}}\left([2]^3_{q_\rho}-[2]^2_{q_\rho}q_\rho-[2]_{q_\rho}q_\rho^2+\left(1-[2]_{q_\rho}[1+2\kappa]_{q_\rho}\right)q_\rho^3\right).$$

Proof. We use (3.4), (3.5) and apply Cauchy–Schwarz inequality.

$$\mathscr{U}^*_{\rho,q_\rho}(h;y)-h(y)|$$
$$\leqslant\frac{[\rho]_{q_\rho}}{e_{\kappa,q_\rho}([\rho]_{q_\rho}\mu_{\rho,q_\rho}(y))}\sum_{p=0}^{\infty}\frac{([\rho]_{q_\rho}\mu_{\rho,q_\rho}(y))^p}{\gamma_{\kappa,q_\rho}(p)}q_\rho^{\frac{(p+2\kappa\theta_p)(p+2\kappa\theta_p+1)}{2}}$$
$$Q^*_{\kappa,q_\rho}\left(h\left(q_\rho^{p+2\kappa\theta_p}\mho_1-h(y)\right)\right)$$
$$\leqslant\frac{[\rho]_{q_\rho}}{e_{\kappa,q_\rho}([\rho]_{q_\rho}\mu_{\rho,q_\rho}(y))}\sum_{p=0}^{\infty}\frac{([\rho]_{q_\rho}\mu_{\rho,q_\rho}(y))^p}{\gamma_{\kappa,q_\rho}(p)}q_\rho^{\frac{(p+2\kappa\theta_p)(p+2\kappa\theta_p+1)}{2}}$$
$$\times\left(Q^*_{\kappa,q_\rho}(\mho_0)+\tfrac{1}{\delta}Q^*_{\kappa,q_\rho}|q_\rho^{p+2\kappa\theta_k}\mho_1-y|\right)\omega(h;\delta)$$
$$=\tfrac{1}{\delta}\Big(\frac{[\rho]_{q_\rho}}{e_{\kappa,q_\rho}([\rho]_{q_\rho}\mu_{\rho,q_\rho}(y))}\sum_{p=0}^{\infty}\frac{([\rho]_{q_\rho}\mu_{\rho,q_\rho}(y))^p}{\gamma_{\kappa,q_\rho}(p)}q_\rho^{\frac{(p+2\kappa\theta_p)(p+2\kappa\theta_p+1)}{2}}$$
$$Q^*_{\kappa,q_\rho}|q_\rho^{p+2\kappa\theta_k}\mho_1-y|)\omega(h;\delta)+\omega(h;\delta)$$
$$\leqslant\{1+\tfrac{1}{\delta}\Bigg[\frac{[\rho]_{q_\rho}}{e_{\kappa,q_\rho}([\rho]_{q_\rho}\mu_{\rho,q_\rho}(y))}\sum_{p=0}^{\infty}\frac{([\rho]_{q_\rho}\mu_{\rho,q_\rho}(y))^p}{\gamma_{\kappa,q_\rho}(p)}q_\rho^{\frac{(p+2\kappa\theta_p)(p+2\kappa\theta_p+1)}{2}}$$
$$Q^*_{\kappa,q_\rho}\left(q_\rho^{p+2\kappa\theta_k}\mho_1-y\right)^2\Bigg]^{\frac{1}{2}}$$
$$\times\left[\mathscr{U}^*_{\rho,q_\rho}(\mho_0;y)\right]^{\frac{1}{2}}\}\ \omega(h;\delta)$$
$$=\left\{1+\tfrac{1}{\delta}\left(\mathscr{U}^*_{\rho,q_\rho}\left(q_\rho^{p+2\kappa\theta_k}\mho_1-y\right)^2;y\right)^{\frac{1}{2}}\right\}\omega(h;\delta)$$
$$\leqslant\left\{1+\tfrac{1}{\delta}\left(\mathscr{U}^*_{\rho,q_\rho}(\mho_1-y)^2;y\right)^{\frac{1}{2}}\right\}\omega(h;\delta),$$

where clearly, if we take $\delta=\delta_{\rho,q_\rho}(y)=\sqrt{\frac{1}{[\rho]_{q_\rho}}}$, then we get the result.

Corollary 3.5. *Let φ_2 by Lemma 2.3 and suppose $\delta_{\rho,q_\rho}(y)=\sqrt{\varphi_2}$, then*

$$|\mathscr{U}^*_{\rho,q_\rho}(h;y)-h(y)|\leqslant 2\omega\left(h;\delta_{\rho,q_\rho}(y)\right).$$

4. Rate of convergence

In this part of section our purpose is to present the rate of convergence in terms of Lipschitz function for the our newly operators $\mathscr{U}^*_{\rho,q}(h;y)$ (1.10). We suppose the class of Lipschitz functions is defined by

$$Lip_M(\alpha)=\{h:|h(\beta_1)-h(\beta_2)|\leqslant M|\beta_1-\beta_2|^\alpha;\ (\beta_1,\beta_2\in[0,\infty))\} \tag{4.1}$$

for all $h\in C[0,\infty), M>0$ and $0<\alpha\leqslant 1$.

Theorem 4.1. *Suppose there is a numbers $L>0$ and $0<\alpha\leqslant 1$. Then the sequence $q=q_\rho$ for every $h\in Lip_L(\alpha)$ and $y\geqslant\frac{1}{2[\rho]_q}$ have the inequality*

$$|\mathscr{U}^*_{\rho,q_\rho}(h;y)-h(y)|\leqslant L\left(\nabla_{\rho,q_\rho}(y)+\Delta_{\rho,q_\rho}\right)^{\frac{\alpha}{2}}.$$

where

$$\nabla_{\rho,q_\rho}(y)=\frac{1}{q^2[2]_q[\rho]_q}\left([2]^2_{q_\rho}-[2]_{q_\rho}q_\rho+[2]_{q_\rho}[1+2\kappa]_{q_\rho}q_\rho^2\right)y,$$

$$\Delta_{\rho,q_\rho}=\frac{1}{q_\rho^3[2]^2_{q_\rho}[\rho]^2_{q_\rho}}\left([2]^3_{q_\rho}-[2]^2_q q_\rho-[2]_{q_\rho}q_\rho^2+\left(1-[2]_{q_\rho}[1+2\kappa]_{q_\rho}\right)q^3_{q_\rho}\right),$$

and $\nabla_{\rho,q_\rho}(y)+\Delta_{\rho,q_\rho}=\varphi_2$ by Lemma 2.3.

Proof. We demonstrate the proof of this theorem by (4.1) and well-known H ölder's inequality.

$$|\mathscr{U}^*_{\rho,q_\rho}(h;y)-h(y)|$$
$$\leqslant|\mathscr{U}^*_{\rho,q_\rho}(h(\mho_1)-h(y);y)|$$
$$\leqslant\mathscr{U}^*_{\rho,q_\rho}(|h(\mho_1)-h(y)|;y)$$
$$\leqslant L\mathscr{U}^*_{\rho,q_\rho}(|\mho_1-y|^\alpha;y)$$
$$\leqslant L\frac{[\rho]_{q_\rho}}{e_{\kappa,q_\rho}([\rho]_{q_\rho}\mu_{\rho,q_\rho}(y))}\sum_{p=0}^{\infty}\frac{([\rho]_{q_\rho}\mu_{\rho,q_\rho}(y))^p}{\gamma_{\kappa,q_\rho}(p)}q_\rho^{\frac{(p+2\kappa\theta_p)(p+2\kappa\theta_p+1)}{2}}$$
$$\times Q^*_{\kappa,q_\rho}(|\mho_1-y|^\alpha)\leqslant L\frac{[\rho]_{q_\rho}}{e_{\kappa,q_\rho}([\rho]_{q_\rho}\mu_{\rho,q_\rho}(y))}\sum_{p=0}^{\infty}\left(\frac{([\rho]_{q_\rho}\mu_{\rho,q_\rho}(y))^p}{\gamma_{\kappa,q_\rho}(p)}q_\rho^{\frac{(p+2\kappa\theta_p)(p+2\kappa\theta_p+1)}{2}}\right)^{\frac{2-\alpha}{2}}$$
$$\times\left(\frac{([\rho]_{q_\rho}\mu_{\rho,q_\rho}(y))^p}{\gamma_{\kappa,q_\rho}(p)}q_\rho^{\frac{(p+2\kappa\theta_p)(p+2\kappa\theta_p+1)}{2}}\right)^{\frac{\alpha}{2}}Q^*_{\kappa,q_\rho}(|\mho_1-y|^\alpha)$$
$$\leqslant L\left(\frac{[\rho]_{q_\rho}}{e_{\kappa,q_\rho}([\rho]_{q_\rho}\mu_{\rho,q_\rho}(y))}\sum_{p=0}^{\infty}\frac{([\rho]_{q_\rho}\mu_{\rho,q_\rho}(y))^p}{\gamma_{\kappa,q_\rho}(p)}q_\rho^{\frac{(p+2\kappa\theta_p)(p+2\kappa\theta_p+1)}{2}}Q^*_{\kappa,q_\rho}(\mho_0)\right)^{\frac{2-\alpha}{2}}$$
$$\times\Bigg(\frac{[\rho]_{q_\rho}}{e_{\kappa,q_\rho}([\rho]_{q_\rho}\mu_{\rho,q_\rho}(y))}\sum_{p=0}^{\infty}\frac{([\rho]_{q_\rho}\mu_{\rho,q_\rho}(y))^p}{\gamma_{\kappa,q_\rho}(p)}q_\rho^{\frac{(p+2\kappa\theta_p)(p+2\kappa\theta_p+1)}{2}}$$
$$\times Q^*_{\kappa,q_\rho}(|\mho_1-y|)\Bigg)^{\frac{\alpha}{2}}$$
$$=L\mathscr{U}^*_{\rho,q_\rho}\left((\mho_1-y)^2;y\right)^{\frac{\alpha}{2}}.$$

This completes the proof.

Further more we write

$$C^2_B(\mathbb{R}^+)=\{\varphi\in C_B(\mathbb{R}^+):\varphi',\varphi''\in C_B(\mathbb{R}^+)\}, \tag{4.2}$$

with the defined norm by

$$\|\varphi\|_{C^2_B(\mathbb{R}^+)}=\|\varphi\|_{C_B(\mathbb{R}^+)}+\|\varphi'\|_{C_B(\mathbb{R}^+)}+\|\varphi''\|_{C_B(\mathbb{R}^+)}, \tag{4.3}$$

also

$$\|\varphi\|_{C_B(\mathbb{R}^+)}=\sup_{y\in\mathbb{R}^+}|\varphi(y)|. \tag{4.4}$$

Theorem 4.2. *Let $h \in C_B^2(\mathbb{R}^+)$ and $q = q_\rho$ such that $q_\rho \in (0,1)$. Then for $y \geqslant \frac{1}{2[\rho]_q}$, the operators $\mathscr{U}^*_{\rho,q}(\cdot;\cdot)$ defined by (1.10) satisfy*

$$|\mathscr{U}^*_{\rho,q_\rho}(h;y) - h(y)| \leqslant \left(\frac{1}{q_\rho[2]_{q_\rho}[\rho]_{q_\rho}}\left([2]_{q_\rho} - q_\rho\right) + \frac{\nabla_{\rho,q_\rho}(y)}{2}\right)\|h\|_{C_B^2(\mathbb{R}^+)}.$$

Proof. Use the Taylor series formula, we go through the expression

$$2h(\mho_1) = 2h(y) + 2h'(y)(\mho_1 - y) + h''(\chi)(\mho_1 - y)^2,\ \ \chi \in (y,\mho_1).$$

Linearity of $\mathscr{U}^*_{\rho,q_\rho}$ implies that

$$\mathscr{U}^*_{\rho,q_\rho}(h;y) - h(y) = h'(y)\mathscr{U}^*_{\rho,q_\rho}((\mho_1 - y);y) + \frac{h''(\chi)}{2}\mathscr{U}^*_{\rho,q_\rho}\left((\mho_1 - y)^2;y\right).$$

Therefore,

$$|\mathscr{U}^*_{\rho,q_\rho}(h;y) - h(y)| \leqslant \left(\frac{1}{q_\rho[2]_{q_\rho}[\rho]_{q_\rho}}\left([2]_{q_\rho} - q_\rho\right)\right)\|h'\|_{C_B(\mathbb{R}^+)} + (\nabla_{\rho,q_\rho}(y))\frac{\|h''\|_{C_B(\mathbb{R}^+)}}{2}.$$

By (4.3), $\|h'\|_{C_B(\mathbb{R}^+)} \leqslant \|h\|_{C_B^2(\mathbb{R}^+)}$ and $\|h''\|_{C_B(\mathbb{R}^+)} \leqslant \|h\|_{C_B^2(\mathbb{R}^+)}$. Hence

$$|\mathscr{U}^*_{\rho,q_\rho}(h;y) - h(y)| \leqslant \left(\frac{1}{q_\rho[2]_{q_\rho}[\rho]_{q_\rho}}\left([2]_{q_\rho} - q_\rho\right)\right)\|h\|_{C_B^2(\mathbb{R}^+)} + (\nabla_{\rho,q_\rho}(y))\frac{\|h\|_{C_B^2(\mathbb{R}^+)}}{2}.$$

Which completes the proof.

5. Some direct approximation

In 1968, Jaak Peetre introduced K-functional $K_2(h;\delta)$ (Peetre, 1968) as

$$K_2(\varphi;\delta) = \inf\left\{\left(\|\varphi - \chi\|_{C_B(\mathbb{R}^+)} + \delta\|\chi\|_{C_B^2(\mathbb{R}^+)}\right) : \chi \in C_B^2(\mathbb{R}^+)\right\}. \quad (5.1)$$

for a given $\delta > 0$ and $\varphi \in [0,\infty)$.

For an absolute positive real constant C one has $K_2(\varphi,\delta) \leqslant C\omega_2(\varphi;\delta^{\frac{1}{2}})$, where ω_2 indicates modulus of continuity of the function φ by order two such that

$$\omega_2(\varphi;\delta) = \sup_{0<\mu<\delta}\sup_{y\in\mathbb{R}^+}|\varphi(y+2\mu) - 2\varphi(y+\mu) + \varphi(y)|. \quad (5.2)$$

Theorem 5.1. *Let $q = q_\rho$ such that $0 < q_\rho \leqslant 1$. Then for every $h \in C_B(\mathbb{R}^+)$ and $y \geqslant \frac{1}{2[\rho]_q}$ there exits a positive constant $\mathscr{D}$ such that*

$$|\mathscr{U}^*_{\rho,q_\rho}(h;y) - h(y)| \leqslant 2\mathscr{D}\left\{\omega_2\left(h;\sqrt{\frac{1}{2q_\rho[2]_{q_\rho}[\rho]_{q_\rho}}\left([2]_{q_\rho} - q_\rho\right) + \frac{\nabla_{\rho,q_\rho}(y)}{4}}\right) + \min\left(1,\frac{1}{2q_\rho[2]_{q_\rho}[\rho]_{q_\rho}}\left([2]_{q_\rho} - q_\rho\right) + \frac{\nabla_{\rho,q_\rho}(y)}{4}\right)\|h\|_{C_B(\mathbb{R}^+)}\right\}.$$

Proof. Using Theorem 4.2, we have

$$|\mathscr{U}^*_{\rho,q_\rho}(h;y) - h(y)| \leqslant |\mathscr{U}^*_{\rho,q_\rho}(h-\chi;y)| + |\mathscr{U}^*_{\rho,q_\rho}(\chi;y) - \chi(y)| + |h(y) - \chi(y)| \leqslant 2\|h-\chi\|_{C_B(\mathbb{R}^+)} + \left(\frac{1}{q_\rho[2]_{q_\rho}[\rho]_{q_\rho}}\left([2]_{q_\rho} - q_\rho\right) + \frac{\nabla_{\rho,q_\rho}(y)}{2}\right)\|\chi\|_{C_B^2(\mathbb{R}^+)}$$

$$= 2\left(\|h-\chi\|_{C_B(\mathbb{R}^+)} + \frac{\frac{1}{q_\rho[2]_{q_\rho}[\rho]_{q_\rho}}\left([2]_{q_\rho} - q_\rho\right) + \frac{\nabla_{\rho,q_\rho}(y)}{2}}{2}\|\chi\|_{C_B^2(\mathbb{R}^+)}\right).$$

By implement the infimum property over all of $\chi \in C_B^2(\mathbb{R}^+)$ and use (5.1), we get

$$|\mathscr{U}^*_{\rho,q_\rho}(h;y) - h(y)| \leqslant 2K_2\left(h;\frac{\frac{1}{q_\rho[2]_{q_\rho}[\rho]_{q_\rho}}\left([2]_{q_\rho} - q_\rho\right) + \frac{\nabla_{\rho,q_\rho}(y)}{2}}{2}\right).$$

From Ciupa (1995) for an absolute and positive real $\mathscr{D}$, one has introduced the connection with Peetre's K-functional K_2 by

$$K_2(h;\delta) \leqslant \mathscr{D}\{\omega_2(h;\sqrt{\delta}) + \min(1,\delta)\|h\|\}.$$

Which is enable to give the complete prove.

For an arbitrary $h \in \mathscr{T}^p_\sigma(\mathbb{R}^+)$, the recent investigation on weighted modulus of continuity established by Atakut (2002)

$$\Omega(\varphi;\delta) = \sup_{|\mu|\leqslant\delta, y\in\mathbb{R}^+}\frac{|\varphi(y+\mu) - \varphi(y)|}{(1+\mu^2)(1+y^2)}. \quad (5.3)$$

In addition, we have the results $\lim_{\delta\to 0}\Omega(\varphi;\delta) = 0$ and

$$|\varphi(\mho_1) - \varphi(y)| \leqslant 2\left(1 + \frac{|\mho_1 - y|}{\delta}\right)(1+\delta^2)(1+y^2)(1 + (\mho_1 - y)^2)\Omega(\varphi;\delta), \quad (5.4)$$

where $\mho_1, y \in \mathbb{R}^+$.

Theorem 5.2. *For every $h \in \mathscr{T}^p_\sigma(\mathbb{R}^+)$ and $y \geqslant \frac{1}{2[\rho]_q}$ there exits a positive $\mathscr{K}$ such that $\mathscr{K} = 1 + \mathscr{K}_1 + 4\mathscr{K}_1\mathscr{K}_2$, where $\mathscr{K}_1, \mathscr{K}_2 > 0$ such that*

$$\sup_{y\in[\frac{1}{2[\rho]_{q_\rho}},\chi_{\kappa,q_\rho}(\rho))}\frac{\left|\mathscr{U}^*_{\rho,q_\rho}(h;y) - h(y)\right|}{1+y^2} \leqslant \mathscr{K}\left(1 + \Psi_{\kappa,q_\rho}(\rho)\right)\Omega\left(h;\sqrt{\Psi_{\kappa,q_\rho}(\rho)}\right),$$

and

$$\Psi_{\kappa,q_\rho}(\rho) = \max\left\{\frac{1}{q_\rho^2[2]_{q_\rho}[\rho]_{q_\rho}}\left([2]^2_{q_\rho} - [2]_{q_\rho}q_\rho + [2]_{q_\rho}[1+2\kappa]_{q_\rho}q_\rho^2\right), \frac{1}{q_\rho^3[2]^2_{q_\rho}[\rho]^2_{q_\rho}}\left([2]^3_{q_\rho} - [2]^2_{q_\rho}q_\rho - [2]_{q_\rho}q_\rho^2 + \left(1 - [2]_{q_\rho}[1+2\kappa]_{q_\rho}\right)q_\rho^3\right\}.$$

Proof. Using (5.3), (5.4) and then apply the inequality of Cauchy–Schwarz, we get that

$$\left|\mathscr{U}^*_{\rho,q_\rho}(h;y) - h(y)\right| \leqslant 2(1+\delta^2)(1+y^2)\Omega(h;\delta)\left(1 + \mathscr{U}^*_{\rho,q_\rho}\left((\mho_1 - y)^2;y\right) + \mathscr{U}^*_{\rho,q_\rho}\left(\left(1 + (\mho_1 - y)^2\right)\frac{|\mho_1 - y|}{\delta};y\right)\right) \quad (5.5)$$

and

$$\mathscr{U}^*_{\rho,q_\rho}\left(\left(1 + (\mho_1 - y)^2\right)\frac{|\mho_1 - y|}{\delta};y\right) \leqslant 2\left(\mathscr{U}^*_{\rho,q_\rho}\left(1 + (\mho_1 - y)^4;y\right)\right)^{\frac{1}{2}}\left(\mathscr{U}^*_{\rho,q_\rho}\left(\frac{(\mho_1 - y)^2}{\delta^2};y\right)\right)^{\frac{1}{2}}. \quad (5.6)$$

In the light of Lemma 2.3, if we take

$$\Psi_{\kappa,q_\rho}(\rho) = \max\left\{\frac{1}{q_\rho^2[2]_{q_\rho}[\rho]_{q_\rho}}\left([2]^2_{q_\rho} - [2]_{q_\rho}q_\rho + [2]_{q_\rho}[1+2\kappa]_{q_\rho}q_\rho^2\right), \frac{1}{q_\rho^3[2]^2_{q_\rho}[\rho]^2_{q_\rho}}\left([2]^3_{q_\rho} - [2]^2_{q_\rho}q_\rho - [2]_{q_\rho}q_\rho^2 + \left(1 - [2]_{q_\rho}[1+2\kappa]_{q_\rho}\right)q_\rho^3\right\},$$

then easily we see that

$$\mathscr{U}^*_{\rho,q_\rho}\left((\mho_1 - y)^2;y\right) \leqslant \Psi_{\kappa,q_\rho}(\rho)(1+y). \quad (5.7)$$

For a constant $\mathscr{K}_1 > 0$ we have

$$\mathscr{U}^*_{\rho,q_\rho}\left((\mho_1 - y)^2; y\right) \leqslant \mathscr{K}_1(1+y). \tag{5.8}$$

Similarly a small calculation leads us to

$$\mathscr{U}^*_{\rho,q_\rho}((\mho_1 - y)^4; y) \leqslant \left(\rho_{\kappa,q_\rho}(\rho)\right)^2 + 2\rho_{\kappa,q_\rho}(\rho)\vartheta_{\rho,q_\rho}(\rho)y + \left(\vartheta_{\rho,q_\rho}(\rho)\right)^2 y^2 \leqslant \varsigma_{\kappa,q_\rho}(\rho)(1+y+y^2),$$

where

$$\rho_{\kappa,q_\rho}(\rho) = \frac{1}{q_\rho^3 [2]^2_{q_\rho} [\rho]^2_{q_\rho}}\left([2]^3_{q_\rho} - [2]^2_{q_\rho} q_\rho - [2]_{q_\rho} q_\rho^2 + \left(1 - [2]_{q_\rho}[1+2\kappa]_{q_\rho}\right) q_\rho^3\right),$$

$$\vartheta_{\rho,q_\rho}(\rho) = \frac{1}{q_\rho^2 [2]_{q_\rho} [\rho]_{q_\rho}}\left([2]^2_{q_\rho} - [2]_{q_\rho} q_\rho + [2]_{q_\rho}[1+2\kappa]_{q_\rho} q_\rho^2\right)$$

$$\varsigma_{\kappa,q_\rho}(\rho) = \max\{\rho_{\kappa,q_\rho}(\rho), \rho_{\kappa,q_\rho}(\rho)\vartheta_{\rho,q_\rho}(\rho), \vartheta_{\rho,q_\rho}(\rho)\}.$$

Since $\frac{1}{[\rho]^i_{q_\rho}} \to 0$ as $\rho \to \infty$, for all $i = 1, 2, 3$, we have

$$\left(\mathscr{U}^*_{\rho,q_\rho}\left(1 + (\mho_1 - y)^4; y\right)\right)^{\frac{1}{2}} \leqslant \mathscr{K}_2(2 + y + y^2)^{\frac{1}{2}}$$

for a constant $\mathscr{K}_2 > 0$.

Also (5.7) implies that

$$\left(\mathscr{U}^*_{\rho,q_\rho}\left(\frac{(\mho_1 - y)^2}{\delta^2}; y\right)\right)^{\frac{1}{2}} \leqslant \frac{1}{\delta}\left(\Psi_{\kappa,q_\rho}(\rho)\right)^{\frac{1}{2}}(1+y)^{\frac{1}{2}}. \tag{5.9}$$

Hence, by combining (5.6)–(5.9) and (5.5), and finally if we take $\delta = \sqrt{\Psi_{\kappa,q_\rho}(\rho)}$ after taking supremum over all $y \in [\frac{1}{2[\rho]_{q_\rho}}, \Psi_{\kappa,q_\rho}(\rho))$, we get the result.

Authors contributions

All the authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Declaration of Competing Interest

The authors declare that they have no competing interests.

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