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## Approximate analytical solutions for some obstacle problems

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## ABSTRACT

The paper deals with the variational iteration method to obtain a semi-analytical solution to some obstacle problems. We will focus on applications of a contact problem for a deformed beam with an elastic obstacle. To validate the accuracy this approach, the obtained solution has been compared with the exact solution, in the case where it can be calculated in the closed-form expression. Otherwise, a comparison of variational iteration method with the Adomian decomposition method for some nonlinear examples has been used.

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## 1. Introduction

Obstacle problems are problems well studied in many scientific literatures, including the problems of deformation of an elastic rod in the presence of an elastic or rigid obstacle (Caffarelli, 1998; Caffarelli and Figalli, 2013; Figalli and Serra, 2019). The resolution of the problems is either difficult, or too long to implement like the numerical problems of the contact in an elastic rod with or without friction (Chamekh et al., 2014; Chamekh, 2015; Chamekh et al., 2020a). Thanks to the penalization method, the obstacle problems are transformed into types of differential equations which can be solved by the Adomian decomposition method or the Variational Iteration Method (VIM), see Alderremy et al. (2019), Chamekh et al. (2020b) and Elzaki and Chamekh (2018).

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This work emphasizes studies semi-analytical solutions related to obstacle problem. The term of obstacle problem is usually used for a classic motivating example in the analytical study of differential inequalities and free boundary problems. In the scientific literature, we can find a good number of applications that have a model that is built on the obstacle problem, such as the finances, the engineering problems, Stefan problem etc. For this purpose, for example, we can consult the following references Baxter and Rennie (1996), Brezzi et al. (1978) and Gupta (2003).

In this paper, the original idea had been to find a semi-analytical solution of the deformation of a beam in the presence of an elastic foundation. They answered that surround this engineering application must not prevent to apply these results to other specialties but, on the contrary, must pave the way for more complex obstacle problems. On the alternatives of Lagrange multiplier method, J. H. He has developed the variational iteration method (He, 1999) since the end of the last century. In scientific literature, the VIM is increasingly being used in different research types. For example, we mention the viscoelastic beams (Martin, 2016), behavior fluids (Chamekh and Elzaki, 2018) etc. . .

Amongst the engineering problems leading to an obstacle problem which were classically raised, the bending analysis of clamped-clamped Euler-Bernoulli beams, onto an elastic foundation and subject to an applied load. The goal is to find the deflexion, the maximum deflexion of this beam. This could also be in the case

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of a membrane but with an alternative form occupies a bounded domain of the plane with a rigid obstacle. Vertical displacement of the membrane is represented by the graph of a function  $U$  of  $\Omega$  in  $\mathbb{R}$ . We then look for the equilibrium position taking the membrane subjected to an external force, under the non-penetration constraint  $U \geq \Psi$  on  $\Omega$ .

In the equilibrium state, if the membrane is out of contact with the obstacle, the elasticity produces a balance of the membrane tension and the surface of the membrane is described by  $U$ . If the membrane comes in contact with the obstacle, the function fulfills the curvatures of  $U$ . More generally, in monodimensional case, these discussed problems can induce the type of system

$$\begin{cases} -\frac{\partial^{(k)}U(x)}{\partial x^k} \geq F(U(x), x), & x \in \Omega, \\ U(x) \geq \Psi(x), & x \in \Omega, \\ \left[\frac{\partial^{(k)}U(x)}{\partial x^k} + F(U, x)\right][U(x) - \Psi(x)] = 0, & x \in \Omega, \\ \Psi(x) \leq 0, & x \in \partial\Omega, \end{cases} \quad (1)$$

with  $\Omega = [a, d]$  and subject to  $k$  boundary conditions among the following conditions:

$$\frac{\partial^{(i)}U}{\partial x^i}(a) = \alpha_i \text{ and } \frac{\partial^{(j)}U}{\partial x^j}(d) = \beta_j \text{ for } i + j = 0, 1, \dots, k - 1, \quad (2)$$

where  $\alpha_i$  and  $\beta_j$  are real constants.

The obstacle problem has many extensions well-respected in the literature, particularly in the nonlinearity of the elastic reactions and/or the noncommutative effects are considered of the membrane, because of that, we have obtained the last possible generalized form. The function  $F(U(\cdot), \cdot)$  is continuous on  $\Omega$  and  $\Psi$  is the obstacle function. The system (1–2) results of the mathematical modeling of problems containing the contact, unilateral, obstacle, deformation, and/or free boundary value. In the remainder, some possibilities are discussed concerning the semi-analytical solutions of this type of system using the VIM.

## 2. Main equations

The penalty method is the most commonly used method of constraint handling. The goal is to take advantage of the fact that this penalty can reduce differential inequalities in the form of equalities. In this regard, by using the penalty technique which has been applied by Lewy-Stampacchia (Lewy and Stampacchia, 1969) to give the regularity of the solutions of the variational inequalities, the system (1) results in the equation given by:

$$-\frac{\partial^{(k)}U(x)}{\partial x^k} - [U(x) - \Psi(x)]_+ = F(U(x), x), \quad x \in [a, d], \quad (3)$$

where  $[s]_+ = \max(0, s)$  is the penalty function.

In the following, for the sake of simplicity, we assume that the obstacle function  $\Psi$  is known as follows (see Al-Said et al., 1996):

$$\Psi(x) = \begin{cases} -1 & x \in [a, b] \cup [c, d], \\ 1 & x \in [b, c], \end{cases} \quad (4)$$

where  $a < b < c < d$ .

Upon substitution of (4) into (3) with taking into account the penalty function we finally obtain

$$-\frac{\partial^{(k)}U(x)}{\partial x^k} = \begin{cases} F(U(x), x) + U(x) + 1 & \text{if } x \in [a, b] \cup [c, d], \\ F(U(x), x) & \text{if } x \in [b, c]. \end{cases} \quad (5)$$

As well as taking into consideration the boundary conditions (2) and continuity conditions of  $U, \frac{\partial U}{\partial x}, \dots, \frac{\partial^{(p)}U}{\partial x^p}$  at  $b$  and  $c$ , where  $p < k$ .

## 3. Variational Iteration Method (VIM)

The VIM or the He's method is a simple and however capable method for solving a wide class of nonlinear problems. It successfully has been applied to many problems. For example, He has utilized his method to solve the classical Blasius equation, a few well-known nonlinear problems, and ordinary differential systems. Presently, the VIM has been used in many fields, it is capable of handling a large class of linear or nonlinear differential equations. Adaptability and adjustment given by this method have made it promptly appropriate for cases where the analytical solutions are not available as is often the case in the applied sciences and engineering. The VIM gives effective calculation for analytic approximate solutions and numeric simulations for real-world applications in sciences. This method does not require the utilization of additive computation algorithms regularly utilized to confront the nonlinear terms as the Adomian decomposition method, which would complicate the analytical calculations. The VIM approaches directly the linear and nonlinear problems straightforwardly in the same way.

From He (1999), we recall the basic concept of VIM associated with the problem (5) follows the following iterative formula

$$U_{n+1}(x) = U_n(x) + \begin{cases} \int_a^x \lambda(s) \left[ \frac{\partial^{(k)}U_n(s)}{\partial s^k} + F(U_n(s), s) + U_n(s) + 1 \right] ds, & a \leq x < b, \\ \int_b^x \lambda(s) \left[ \frac{\partial^{(k)}U_n(s)}{\partial s^k} + F(U_n(s), s) \right] ds, & b \leq x < c, \\ \int_c^x \lambda(s) \left[ \frac{\partial^{(k)}U_n(s)}{\partial s^k} + F(U_n(s), s) + U_n(s) + 1 \right] ds, & c \leq x \leq d, \end{cases} \quad (6)$$

with  $\lambda$  is Lagrangian multiplicative. The possible practical form of  $\lambda$  is given by:

$$\lambda(s) = (-1)^k \frac{(s-x)^{k-1}}{(k-1)!}, \quad x \in [a, d]. \quad (7)$$

But we must note that in some special cases, we may first have to face the challenge of finding the optimum value of  $\lambda$  to accelerate the convergence of the solution. So we have to take back the calculation of variations necessary to find  $\lambda$ . Iterative techniques rely on the computation of a recurrence from an initial function  $U_0$  and which, step by step, tends towards the solution. These techniques are effective as long as one starts from a well chosen initial value  $U_0$ . With some initial value, this method may not converge to a desired approximate solution because the starting point is not suitable. To do this, we will try to discuss the choice of the initial value of the different examples treated in the numerical section. Another interesting measure to consider the choice of  $U_0$  is the continuity conditions of the solution and its derivatives within the interior of domain  $\Omega$ .

## 4. Numerical examples

The proposed examples of the solutions of the linear and nonlinear problems show the effectiveness of the studied approach. The classical examples where we can give a closed-form solution that is possible enables us to evaluate the precision of calculated solutions.

### 4.1. Third-order linear obstacle problem

In the areas of oceanography (Al-Said et al., 1996), for example, the modeling can contribute to third-order differential equations. Taking  $F(U, x) = U, k = 3, a = 0, b = \frac{1}{4}, c = \frac{3}{4}$  and  $d = 1$ , with boundary conditions  $\alpha_0 = \alpha_1 = \beta_0 = 0$ , we obtain the following problem

$$-\frac{\partial^{(3)}U(x)}{\partial x^3} = \begin{cases} 2U(x) + 1 & x \in [0, \frac{1}{4} \cup [\frac{3}{4}, 1], \\ U(x) & x \in [\frac{1}{4}, \frac{3}{4}]. \end{cases} \tag{8}$$

Using (7), we obtain  $\lambda(s) = -\frac{(s-x)^2}{2}$ .

From the variational formulation we can easily verify the regularity of solution in Sobolev space  $H_0^2(\Omega)$  of the next problem which is equivalent to problem (8) (see Al-Said et al., 1996):

$$\begin{cases} -\frac{\partial^{(3)}U(x)}{\partial x^3} \geq U(x) & x \in [0, 1], \\ U(x) \geq \Psi(x) & x \in [0, 1], \\ \left[\frac{\partial^{(3)}U(x)}{\partial x^3} + U(x)\right][U(x) - \Psi(x)] = 0 & x \in [0, 1], \end{cases} \tag{9}$$

The exact solution is as follows

$$U(x) = \begin{cases} -\frac{1}{2} + a_0 e^{-\sqrt{3}x} + \left[ a_1 \cos\left(\frac{\sqrt{3}\sqrt{2}x}{2}\right) + a_2 \sin\left(\frac{\sqrt{3}\sqrt{2}x}{2}\right) \right] e^{\frac{\sqrt{3}x}{2}} & x \in [0, \frac{1}{4}], \\ b_0 e^{-x} + \left[ b_1 \cos\left(\frac{\sqrt{3}x}{2}\right) + b_2 \sin\left(\frac{\sqrt{3}x}{2}\right) \right] e^{\frac{x}{2}} & x \in [\frac{1}{4}, \frac{3}{4}], \\ -\frac{1}{2} + c_0 e^{-\sqrt{2}x} + \left[ c_1 \cos\left(\frac{\sqrt{3}\sqrt{2}x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\sqrt{2}x}{2}\right) \right] e^{\frac{\sqrt{3}x}{2}} & x \in [\frac{3}{4}, 1]. \end{cases} \tag{10}$$

where the constants  $a_i, b_i$  and  $c_i$  is given with  $i = 1, \dots, 3$  by applying the continuity conditions on  $U, U', U''$ .

$$a_0 = 0.220300649527402, \quad a_1 = 0.279699350472598, \quad a_2 = 0.0928967833270709,$$

$$b_0 = -0.00973786559844113, \quad b_1 = 0.0208426867753722, \quad b_2 = 0.00712428122870885,$$

$$c_0 = 0.423614465941949, \quad c_1 = 0.145695066676027, \quad c_2 = 0.168780738107810.$$

We can now find the approximate solution of the problem (8) by using VIM. According to the iterative formula (6), we can write the problem (8) in the iterative form

$$U_{n+1}(x) = U_n(x) + \begin{cases} -\frac{1}{2} \int_0^x (s-x)^2 \left[ \frac{\partial^{(3)}U_n(s)}{\partial s^3} + 2U_n(s) + 1 \right] ds, & 0 \leq x < \frac{1}{4}, \\ -\frac{1}{2} \int_{\frac{1}{4}}^x (s-x)^2 \left[ \frac{\partial^{(3)}U_n(s)}{\partial s^3} + U_n(s) \right] ds, & \frac{1}{4} \leq x < \frac{3}{4}, \\ -\frac{1}{2} \int_{\frac{3}{4}}^x (s-x)^2 \left[ \frac{\partial^{(3)}U_n(s)}{\partial s^3} + 2U_n(s) + 1 \right] ds, & \frac{3}{4} \leq x \leq 1. \end{cases} \tag{11}$$

Using the boundary conditions at 0, we obtain a possible parametric expression of  $U_0$  that is given by

$$U_0(x) = \begin{cases} a_2 x^2 + a_1 x + a_0, & x \in [0, \frac{1}{4}], \\ b_2 (x - \frac{1}{4})^2 + b_1 (x - \frac{1}{4}) + b_0 & x \in [\frac{1}{4}, \frac{3}{4}], \\ c_2 (x - \frac{3}{4})^2 + c_1 (x - \frac{3}{4}) + c_0, & x \in [\frac{3}{4}, 1]. \end{cases} \tag{12}$$

Using (12), we obtain a parametric expression of  $U_2$  after two VIM iteration. Then, applying the continuity condition of  $U, U'$  and  $U''$  at the points  $\frac{1}{4}$  and  $\frac{3}{4}$  and the boundary condition at the point 1, we obtain a system of linear equations in seven variables  $a_2, b_0, b_1, b_2, c_0, c_1$  and  $c_2$ . This system get to the following solutions:

$$\begin{aligned} a_2 &= 955/7478, \\ b_0 &= 103/19166, \quad b_1 = 252/7745, \quad b_2 = 52/23575, \\ c_0 &= 184/8365, \quad c_1 = 749/22436, \quad c_2 = -31/25704. \end{aligned}$$

Thus, the expression of  $U_2$  is as follows

$$U_2(x) = \begin{cases} \frac{21x^8}{828766} + \frac{x^6}{360} - \frac{45x^5}{10571} - \frac{x^3}{6} + \frac{955x^2}{7478}, & 0 \leq x < \frac{1}{4}, \\ -\frac{x^8}{9139846} + \frac{3x^7}{481004} - \frac{x^6}{274566} - \frac{7x^5}{176854} - \frac{24x^4}{18373} \\ + \frac{23x^3}{52801} + \frac{37x^2}{15578} + \frac{127x}{4051} - \frac{99}{37879}, & \frac{1}{4} \leq x < \frac{3}{4}, \\ -\frac{x^8}{4178973} + \frac{3x^7}{3107408} + \frac{63x^6}{22850} - \frac{44x^5}{3467} + \frac{154x^4}{7285} \\ - \frac{463x^3}{2442} + \frac{1637x^2}{4151} - \frac{224x}{869} + \frac{397}{5728}, & \frac{3}{4} \leq x \leq 1. \end{cases} \tag{13}$$

Fig. 1 shows that the curve of the exact solution merges with that of the VIM solution. This precise result can be clearer in Fig. 2 where we notice that the order of the relative error is  $10^{-7}$ . After the second iteration of VIM, we obtained satisfactory results, which indicates the effectiveness and high accuracy of this method for an optimal computing cost.

#### 4.2. Third-order nonlinear obstacle problem

In the next, we Take  $F(U, x) = U^2$ ,  $k = 3$ ,  $a = -1$ ,  $b = -\frac{1}{2}$ ,  $c = \frac{1}{2}$  and  $d = 1$ , with boundary conditions  $\alpha_0 = \alpha_1 = \beta_0 = 0$ , under (5) we obtain the following problem

$$-\frac{\partial^{(3)}U(x)}{\partial x^3} = \begin{cases} U^2(x) + 2U(x) + 1 & x \in [-1, -\frac{1}{2} \cup [\frac{1}{2}, 1], \\ U^2(x) + U(x) & x \in [-\frac{1}{2}, \frac{1}{2}]. \end{cases} \tag{14}$$

We can use the iterative formula (6) associated with VIM to find an approximate solution of the previous problem, we write

$$U_{n+1}(x) = U_n(x) + \begin{cases} -\frac{1}{2} \int_0^x (s-x)^2 \left[ \frac{\partial^{(3)}U_n(s)}{\partial s^3} + U_n^2(s) + 2U_n(s) + 1 \right] ds, & -1 \leq x < -\frac{1}{2}, \\ -\frac{1}{2} \int_{\frac{1}{4}}^x (s-x)^2 \left[ \frac{\partial^{(3)}U_n(s)}{\partial s^3} + U_n^2(s) + U_n(s) \right] ds, & -\frac{1}{2} \leq x < \frac{1}{2}, \\ -\frac{1}{2} \int_{\frac{3}{4}}^x (s-x)^2 \left[ \frac{\partial^{(3)}U_n(s)}{\partial s^3} + U_n^2(s) + 2U_n(s) + 1 \right] ds, & \frac{1}{2} \leq x \leq 1. \end{cases} \tag{15}$$

With the same procedures, we consider the following expression of  $U_0$

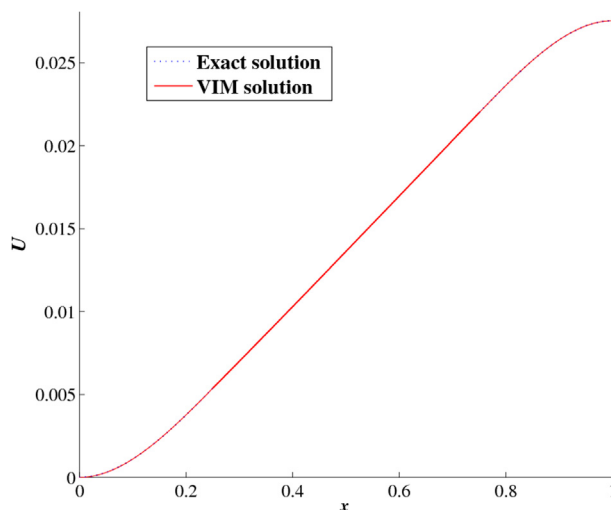


Fig. 1. Comparing the exact solution in blue dashed-line with the two-iteration VIM solution in red straight-line.

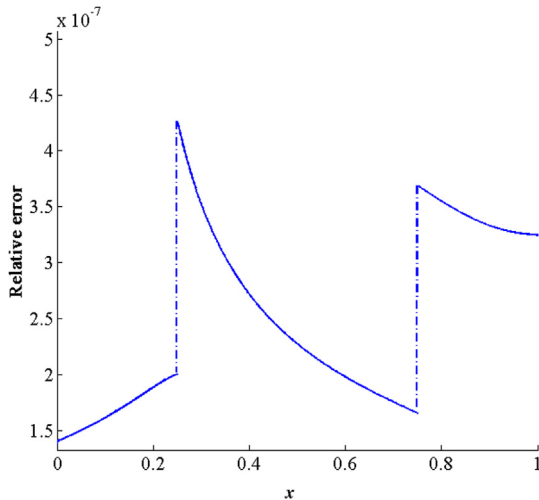


Fig. 2. The curve of relative error.

$$U_0(x) = \begin{cases} a_0 + a_1(x + 1) + a_2(x + 1)^2, & -1 \leq x < -\frac{1}{2}, \\ b_0 + b_1(x + \frac{1}{2}) + b_2(x + \frac{1}{2})^2, & -\frac{1}{2} \leq x < \frac{1}{2}, \\ c_0 + c_1(x - \frac{1}{2}) + c_2(x - \frac{1}{2})^2, & \frac{1}{2} \leq x \leq 1. \end{cases} \quad (16)$$

Applying (15) to  $U_0$ , we obtain  $U_2$  last two VIM iteration. Then, the use of the continuity conditions on  $U, U'$  and  $U''$  at points  $-\frac{1}{2}$  and  $\frac{1}{2}$  and the boundary conditions leads to a nonlinear system in variable  $a_i, b_i$  and  $c_i$ , where  $i = 0, 1, 2$ . This system is solved by using the Newton method. We obtain the following solution

$$\begin{aligned} a_0 &= 321/6281, & a_1 &= -160/463, & a_2 &= -520/2191, \\ b_0 &= 155/3403, & b_1 &= 849/3166, & b_2 &= 93/14320, \\ c_0 &= 311/989, & c_1 &= 721/2197, & c_2 &= -419/1566. \end{aligned}$$

It should be noted that already after two steps of VIM calculation, 14th degree polynomials result. The expression of  $U_2$  is given in the following fourth order and we deny the rest.

$$U_2(x) = \begin{cases} -\frac{242x^4}{19403} - \frac{403x^3}{1799} - \frac{520x^2}{2191} + \frac{323x}{2502} + \frac{243}{1525}, & -1 \leq x < -\frac{1}{2}, \\ -\frac{207x^4}{13696} - \frac{247x^3}{6889} + \frac{113x^2}{2871} + \frac{409x}{1538} + \frac{271}{1489}, & -\frac{1}{2} \leq x < \frac{1}{2}, \\ -\frac{133x^4}{11509} - \frac{299x^3}{1230} + \frac{415x^2}{1166} + \frac{547x}{5163} - \frac{671}{3213}, & \frac{1}{2} \leq x \leq 1. \end{cases} \quad (17)$$

Fig. 3 shows that the curve of the Adomian solution merges with that of the VIM solution. It should be noted, however, that we have used the Adomian method for comparison purposes. This allows us to draw the following conclusion: The VIM and Adomian method give almost the same precision for this nonlinear problem of the obstacle.

### 5. Conclusion

The differential inequalities which describe the obstacle problem are transformed into the differential equations thanks to the penalization method. The equations thus obtained were processed by the variational iteration method. The discussed examples of a beam with an elastic obstacle show some the major benefits: It leaves freedom of choice, on various loads, to an arbitrary point or region on the beam, on the boundary conditions at the ends of the beam; and the VIM is inexpensive and simple to implement and therefore the required programming skills are not necessary.

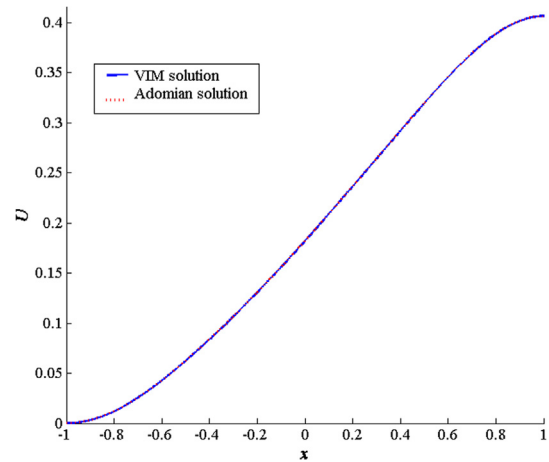


Fig. 3. Comparing the two-iteration VIM solution in straight-line (blue), with the Adomian solution in dashed-line (red).

We conclude the effectiveness and reliability of this method for solving certain type of these problems.

### Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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