



ORIGINAL ARTICLE

# The meshless local Petrov–Galerkin based on moving kriging interpolation for solving fractional Black–Scholes model



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equation;  
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Moving kriging interpolation

**Abstract** In this paper, the fractional Black–Scholes equation in financial problem is solved by using the numerical techniques for the option price of a European call or European put under the Black–Scholes model. The MLPG and implicit finite difference method are used for discretizing the governing equation in option price and time variable, respectively. In MLPG method, the shape function is constructed by a moving kriging approximation. The Dirac delta function is chosen to be the test function. The numerical examples for varieties of variables are also included.

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## 1. Introduction

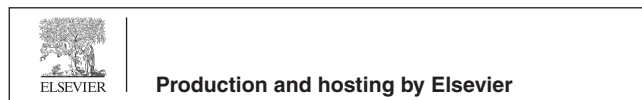
The first idea of fractional calculus is considered to be the Leibniz's letter to L'Hospital in 1695. Fractional calculus is a name for the theory of derivatives and integrals of arbitrary order. The famous definitions of a fractional calculus are the

Riemann–Liouville and Grunwald–Letnikov definition (Ghandehari and Ranjbar, 2014). Caputo reformulated the definition of the Riemann–Liouville in order to use integer order initial conditions to solve fractional differential equation (Ishteva, 2005). The definitions of Riemann–Liouville and the first Caputo version have the weakness for singular kernel. Caputo and Fabrizio proposed a new fractional order derivative without a singular kernel (Atangana and Alkahtani Badr Saad, 2015a,b). Fractional differential equations have attracted much attention during the past few decade. This is the fact that fractional calculus supplies a competent and excellent tool for the description of many important phenomena such as electromagnetic, physics, chemistry, biology, economy and many more.

Black–Scholes equation, which is proposed by Black and Scholes (1973), is the financial model that concern with option. An option is a contract between the seller and the buyer. It

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consists of a call option and a put option. Option valuation depends on the underlying asset price and time. The European option can only be exercised at the expiration date, but the American option can be exercised at any time before expiration date. The solution of Black–Scholes equation provides an option pricing formula for European option. The analytic solution is used in general case with basic assumption but it is not satisfied in some conditions. Some restrictions were appeared in the classical Black–Scholes equation that is the weakness of this model. Original assumptions were relieved by other models such as models with transaction cost (Barles and Soner, 1998; Davis et al., 1993), Jump-diffusion model (Merton, 1976), Stochastic volatility model (Hull and White, 1987) and fractional Black–Scholes model (Bjork and Hult, 2005; Wang, 2010).

Fractional Black–Scholes model is derived by many researchers. Some restrictions were appeared in the classical Black–Scholes equation that is the weakness of this model (Song and Wang, 2013). The fractional Black–Scholes models are derived by substituting the standard Brownian motion with fractional Brownian motion.

In this paper, we propose a numerical method based on meshless local Petrov–Galerkin (MLPG) method to solve a fractional Black–Scholes equation. The MLPG is a truly meshless method, which involves not only a meshless interpolation for the trial functions, but also a meshless integration of the weak-form, (Atluri and Shen, 2002). MLPG2 is chosen for this research so the Kronecker delta is the test function. This method will avoid the domain integral in the weak-form.

## 2. Problem formulation

The Black–Scholes equation is the outstanding financial equation that solves the European option pricing without a transaction cost. Moreover, underlying asset price distributed on the lognormal random walk, risk-free interest rate, no dividend and no arbitrage opportunity are fundamental assumption. Fractional calculus is used in financial market for description the probability of log-price, which is a benefit to specify the variability in prices. Fractional Black–Scholes models are derived by substitution standard Brownian motion with fractional Brownian motion. Some restrictions of original Black–Scholes equation are improved by many methods. The fractional Black–Scholes equation is a choice for reinforcement of original equation. The fractional Black–Scholes equation is following

$$\frac{\partial^\alpha u}{\partial \tau^\alpha} + r(\tau)s \frac{\partial u}{\partial s} + \frac{1}{2} \sigma^2(s, \tau) s^2 \frac{\partial^2 u}{\partial s^2} - r(\tau)u = 0, \quad (s, \tau) \in \mathbb{R}^+ \times [0, T], \quad (2.1)$$

with the terminal and boundary condition

$$u(s, T) = \max(s - E, 0), \quad s \in \mathbb{R}^+, \quad u(0, \tau) = 0, \quad \tau \in [0, T],$$

where  $u(s, \tau)$  is the value of European call option at underlying asset price  $s$  at time  $\tau$ ,  $T$  is the expiration date,  $r$  is the risk-free interest rate,  $\sigma$  is the volatility of underlying asset price and  $E$  is the strike price.

From Eq. (2.1), when  $s$  goes to zero then degenerating will occur in approximation. We transform the Black–Scholes equation into a nondegenerate partial differential equation by using a logarithmic transformation  $x = \ln s$ ,  $t = T - \tau$ ,

and define the computational domain for convenient in numerical experiments by  $\Omega = [x_{\min}, x_{\max}] \times [0, T]$ , where  $x_{\min} = -\ln(4E)$ ,  $x_{\max} = \ln(4E)$ , Haug and Cen, 2014.

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{1}{2} \sigma^2 \frac{\partial^2 u}{\partial x^2} + \left(r - \frac{1}{2} \sigma^2\right) \frac{\partial u}{\partial x} - ru = 0, \quad (2.2)$$

where  $u(x, 0) = \max(e^x - E, 0)$ ,  $x \in (x_{\min}, x_{\max})$ ,

$$u(x_{\min}, t) = 0, \quad u(x_{\max}, t) = e^{x_{\max}} - Ee^{-\int_0^t r(s)ds}, \quad t \in [0, T].$$

The fundamental definition of fractional calculus as following

**Definition 1.** The Riemann–Liouville fractional integral operator of order  $\alpha > 0$ , of a function  $f(t) \in C_\mu, \mu \geq -1$  is defined as (Podlubny, 1999),

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau) d\tau, \quad (\alpha > 0),$$

$$J^0 f(t) = f(t).$$

For the Riemann–Liouville fractional integral we have:

$$J^\alpha t^\gamma = \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + \alpha + 1)} t^{\alpha+\gamma}$$

**Definition 2.** The fractional derivative of  $f(t)$  in the Caputo sense is defined as (Caputo, 1969),

$$D_\tau^\alpha f(t) = J^{m-\alpha} D^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau,$$

for  $m-1 < \alpha \leq m$ ,  $m \in \mathbb{N}$ ,  $t > 0$ .

For the Riemann–Liouville fractional integral and the Caputo fractional derivative, we have the following relation  $J_t^\alpha D_t^\alpha f(t) = f(t) - \sum_{k=0}^{m-1} J^{k-\alpha} f^{(k)}(0_+)$ .

**Definition 3.** The Mittag–Leffler is defined as (Mittag, 1903)

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha \in \mathbb{C}, \quad \text{Re}(\alpha) > 0.$$

## 3. Spatial discretization

The MLPG method are used for spatial discretization. We create the local weak form over local subdomain, which is a small region taken for each node in global domain. Multiplying test function  $v_i$  into Eq. (2.2) and then integrate over subdomain  $\Omega_s^i$  yields the following expression

$$\int_{\Omega_s^i} \left( \frac{\partial^\alpha u}{\partial t^\alpha} + \frac{1}{2} \sigma^2 \frac{\partial^2 u}{\partial x^2} + \left(r - \frac{1}{2} \sigma^2\right) \frac{\partial u}{\partial x} - ru \right) v_i d\Omega = 0, \quad (3.1)$$

where  $v_i$  is a test function that make significant for each nodes. Rearrange Eq. (3.1), we have

$$\int_{\Omega_s^i} \frac{\partial^\alpha u}{\partial t^\alpha} v_i d\Omega + \frac{1}{2} \sigma^2 \int_{\Omega_s^i} u_{,xx} v_i d\Omega + \int_{\Omega_s^i} \left(r - \frac{1}{2} \sigma^2\right) u_{,x} v_i d\Omega - \int_{\Omega_s^i} ru v_i d\Omega = 0, \quad (3.2)$$

where  $u_{,xxx} = \frac{\partial^3 u}{\partial x^3}$ ,  $u_{,xx} = \frac{\partial^2 u}{\partial x^2}$ . Substituting trial function  $u^h(x, t) = \sum_{j=1}^N \phi_j(x) \hat{u}_j(t)$  into  $u$  in Eq. (3.2)

$$\begin{aligned} & \int_{\Omega_j} \sum_{j=1}^N \phi_j(x) v_i(x) \frac{\partial^2 \hat{u}_j(t)}{\partial t^2} d\Omega + \frac{1}{2} \int_{\Omega_j} \sum_{j=1}^N \phi_{j,xx}(x) v_i(x) \hat{u}_j(t) d\Omega \\ & + \int_{\Omega_j} \sum_{j=1}^N \phi_{j,x}(x) \left( r - \frac{1}{2} \sigma^2 \right) v_i(x) \hat{u}_j(t) d\Omega \\ & - r \int_{\Omega_j} \sum_{j=1}^N \phi_j(x) v_i(x) \hat{u}_j(t) d\Omega = 0, \end{aligned} \quad (3.3)$$

where  $N$  is the number of nodes surrounding point  $x$  which has the effect on  $u(x)$  and  $\hat{u}_j$  is value of option at time  $t$ . The shape function,  $\phi_j$ , is constructed by moving kriging interpolation which has the Kronecker delta property, thereby enhancing the arrangement nodal shape construction accuracy. Rearrange Eq. (3.3) yields the following result

$$\begin{aligned} & \sum_{j=1}^N \int_{\Omega_j} \phi_j(x) v_i(x) \left( \frac{\partial^2 \hat{u}_j(t)}{\partial t^2} \right) d\Omega + \frac{1}{2} \sum_{j=1}^N \int_{\Omega_j} \phi_{j,xx}(x) v_i(x) \hat{u}_j(t) d\Omega \\ & + \int_{\Omega_j} \phi_{j,x}(x) \left( r - \frac{1}{2} \sigma^2 \right) v_i(x) \hat{u}_j(t) d\Omega - r \sum_{j=1}^N \int_{\Omega_j} \phi_j(x) v_i(x) \hat{u}_j(t) d\Omega = 0. \end{aligned} \quad (3.4)$$

This research use MLPG2 then the test function is chosen by Kronecker delta function

$$v_i(x) = \begin{cases} 0, & x \neq x_i \\ 1, & x = x_i \end{cases}, \quad i = 1, 2, \dots, N.$$

The test function will define significance for each node in subdomain. In this case, substituting test function  $v_i(x)$  to Eq. (3.4) and then integrate over subdomain  $\Omega_j^i$  yields the following result

$$\begin{aligned} & \sum_{j=1}^N \phi_j(x_i) \frac{d^2 \hat{u}_j(t)}{dt^2} + \sum_{j=1}^N \left[ \frac{1}{2} \phi_{j,xx}(x_i) + \left( r - \frac{1}{2} \sigma^2 \right) \phi_{j,x}(x_i) - r \phi_j(x_i) \right] \\ & \hat{u}_j(t) = 0. \end{aligned} \quad (3.5)$$

Eq. (3.5) can be written in the matrix form as following

$$A \frac{d^2 U}{dt^2} + BU = 0, \quad (3.6)$$

$$\text{where } A = [A_{ij}]_{N \times N}, A_{ij} = \phi_j(x_i) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases},$$

$$B = [B_{ij}]_{N \times N}, B_{ij} = -\frac{1}{2} \phi_{j,xx}(x_i) - \left( r - \frac{1}{2} \sigma^2 \right) \phi_{j,x}(x_i) + r \phi_j(x_i),$$

$$U = [\hat{u}_1 \ \hat{u}_2 \ \hat{u}_3 \ \dots \ \hat{u}_N]^T.$$

Since the shape function that is constructed by the moving kriging interpolation satisfy the Kronecker delta property,  $A$  is the identity matrix. Therefore, Eq. (3.6) can be written as

$$\frac{d^2 U}{dt^2} + BU = 0. \quad (3.7)$$

#### 4. Temporal discretization

The numerical solution of European option use the implicit finite difference method. By a finite approximation made for the time fractional derivative with notation  $\frac{\partial^2 u(x_i, t_n)}{\partial t^2}$  that approximates the exact solution  $u(x_i, t_n)$  at time level  $n$ , we

restrict attention to the finite space domain  $x_{\min} < x < x_{\max}$  with  $0 < \alpha < 1$ . The time fractional derivative use the implicit finite difference (Murio, 2008), defined by

$$\frac{d^\alpha U}{dt^\alpha} = \sigma_{x,\Delta t} \sum_{j=1}^n (U^{n-j+1} - U^{n-j}) + O(\Delta t), \quad (4.1)$$

where  $\sigma_{x,\Delta t} = \frac{1}{\Gamma(1-\alpha)} \frac{1}{1-\alpha} \frac{1}{(\Delta t)^\alpha}$ .

Hence,  $\frac{d^\alpha U}{dt^\alpha} = D_t^{(\alpha)} U_i^m + O(\Delta t)$ .

The first-order approximation method for the computation of Caputo's fractional derivative is given by

$$D_t^{(\alpha)} U^m = \sigma_{x,\Delta t} \sum_{j=1}^n \omega_j^{(\alpha)} (U^{m-j+1} - U^{m-j}), \quad (4.2)$$

where  $\omega_j^{(\alpha)} = j^{1-\alpha} - (j-1)^{1-\alpha}$ .

Consider the Eq. (4.2) and substitute time fractional derivative that following

$$\sigma_{x,\Delta t} \sum_{j=1}^n \omega_j^{(\alpha)} (U^{m-j+1} - U^{m-j}) + BU^m = 0,$$

$$\sigma_{x,\Delta t} \sum_{j=1}^n \omega_j^{(\alpha)} U^{m-j+1} - \sigma_{x,\Delta t} \sum_{j=1}^n \omega_j^{(\alpha)} U^{m-j} + BU^m = 0,$$

$$\sigma_{x,\Delta t} \omega_1^{(\alpha)} (U^m - U^{m-1}) = -\sigma_{x,\Delta t} \sum_{j=2}^n \omega_j^{(\alpha)} (U^{m-j+1} - U^{m-j}) - BU^m,$$

$$\sigma_{x,\Delta t} \omega_1^{(\alpha)} (U^m - U^{m-1}) = -\sigma_{x,\Delta t} \sum_{j=2}^n \omega_j^{(\alpha)} (U^{m-j+1} - U^{m-j}) - BU^m,$$

We consider the first case for  $n = 1$ ,

Case  $n = 1$

$$\sigma_{x,\Delta t} \omega_1^{(\alpha)} U^1 + BU^1 = \sigma_{x,\Delta t} \omega_1^{(\alpha)} U^0,$$

$$(\sigma_{x,\Delta t} \omega_1^{(\alpha)} I + B) U^1 = \sigma_{x,\Delta t} \omega_1^{(\alpha)} U^0.$$

Case  $n \geq 2$

$$(\sigma_{x,\Delta t} \omega_1^{(\alpha)} I + B) U^m = \sigma_{x,\Delta t} \omega_1^{(\alpha)} U^{m-1} - \sigma_{x,\Delta t} \sum_{j=2}^n \omega_j^{(\alpha)} (U^{m-j+1} - U^{m-j}).$$

#### 5. Stability analysis

In this section, we are propose an analysis of the stability of implicit finite difference method and MLPG2 by using the matrix method. A small perturbation at  $n$ th time level is  $e^n = U^n - \tilde{U}^n$ , where  $U^n$  is exact and  $\tilde{U}^n$  is the numerical solution. The equation of the error  $e^n$  can be written as

$$(\sigma_{x,\Delta t} \omega_1^{(\alpha)} I + B) e^n = \sigma_{x,\Delta t} \omega_1^{(\alpha)} e^{n-1} - \sigma_{x,\Delta t} \sum_{j=2}^n \omega_j^{(\alpha)} (e^{n-j+1} - e^{n-j}). \quad (5.1)$$

If  $n = 1$  then  $(\sigma_{x,\Delta t} \omega_1^{(\alpha)} I + B) e^1 = \sigma_{x,\Delta t} \omega_1^{(\alpha)} e^0$ ,

$$e^1 = (\sigma_{x,\Delta t} \omega_1^{(\alpha)} I + B)^{-1} \sigma_{x,\Delta t} \omega_1^{(\alpha)} e^0.$$

where  $G = (\sigma_{x,\Delta t} \omega_1^{(\alpha)} I + B)^{-1} \sigma_{x,\Delta t} \omega_1^{(\alpha)}$ .

It can be seen that the stability is assured if all eigenvalues of the matrix  $G = (\sigma_{x,\Delta t}\omega_1^{(x)}I + B)^{-1}\sigma_{x,\Delta t}\omega_1^{(x)}$  satisfy the following condition:

$$\left| \frac{\sigma_{x,\Delta t}\omega_1^{(x)}}{\sigma_{x,\Delta t}\omega_1^{(x)}I + \lambda} \right| < 1 \tag{5.2}$$

where  $\lambda$  is the eigenvalue of the matrix  $B$ . If  $\rho(G) < 1$  and  $\text{Re}(\lambda)_{\max} > 0$  then  $e^1 < e^0$ . The inequality in Eq. (5.2) is always satisfied  $\text{Re}(\lambda)_{\max} > 0$  provided.

For case  $n > 1$ ,

$$\begin{aligned} (\sigma_{x,\Delta t}\omega_1^{(x)}I + B)e^n &= \sigma_{x,\Delta t}\omega_1^{(x)}e^{n-1} - \sigma_{x,\Delta t}\sum_{j=2}^{n-1}\omega_j^{(x)}(e^{n-j+1} - e^{n-j}) \\ &\quad - \sigma_{x,\Delta t}\omega_n^{(x)}(e^1 - e^0). \end{aligned}$$

$$= \sigma_{x,\Delta t}\omega_1^{(x)}e^{n-1} - \sigma_{x,\Delta t}\sum_{j=2}^{n-1}(\omega_{j+1}^{(x)} - \omega_j^{(x)})e^{n-j} + \sigma_{x,\Delta t}\omega_n^{(x)}e^0.$$

$$\leq \left( \sigma_{x,\Delta t}\omega_1^{(x)} - \sigma_{x,\Delta t}\sum_{j=2}^{n-1}(\omega_{j+1}^{(x)} - \omega_j^{(x)}) + \sigma_{x,\Delta t}\omega_n^{(x)} \right) e^0.$$

$$= \sigma_{x,\Delta t}\omega_1^{(x)}e^0.$$

$$e^n = (\sigma_{x,\Delta t}\omega_1^{(x)}I + B)^{-1}\sigma_{x,\Delta t}\omega_1^{(x)}e^0.$$

$$e^n = Ge^0.$$

where  $\|G\| < 1$ .

$$e^n \leq e^0.$$

From case  $n = 1$  and  $n > 1$ , we conclude that  $e^n \leq e^0$  for all  $n$  where  $\text{Re}(\lambda)_{\max} > 0$ .

The eigenvalues of matrix  $B$  highly depend on the mesh spacing parameter ‘ $h$ ’ ( $h$  is defined to be the minimal distance between any two points in the domain) and the shape parameter  $\varepsilon$ . The present local approximation is free from these complexities. Since it is not possible to find an explicit relationship among the eigenvalues of matrix  $B$ , the number of nodes and the shape parameter  $\varepsilon$  we investigate this dependent numerically and is given in Fig. 1.

Fig. 1 shows maximum eigenvalue  $\text{Re}(\lambda)$  of matrix  $B$  varies as a function of shape parameter  $\varepsilon$ , when mesh spacing parameter  $h$  is constant. Fig. 2 shows the effect of mesh length ‘ $h$ ’ for eigenvalue of matrix  $B$ , when the shape parameter  $\varepsilon$  is constant. Fig. 3 shows that the increasing of volatility trends to decreasing of maximum eigenvalue at the end. In this case, if the shape parameter increases then eigenvalue  $\text{Re}(\lambda)$  will increase. Fig. 4 presents that the risk-free interest rate is stable for all cases of shape parameter and the shape parameter increases then eigenvalue  $\text{Re}(\lambda)$  will increase (see Figs. 5–14).

### 6. Numerical examples

In this section, we are going to present various numerical results to evaluate proposed meshless approaches. Using the MLPG2 method, the resulting problems for European call options are solved via implicit finite difference method.

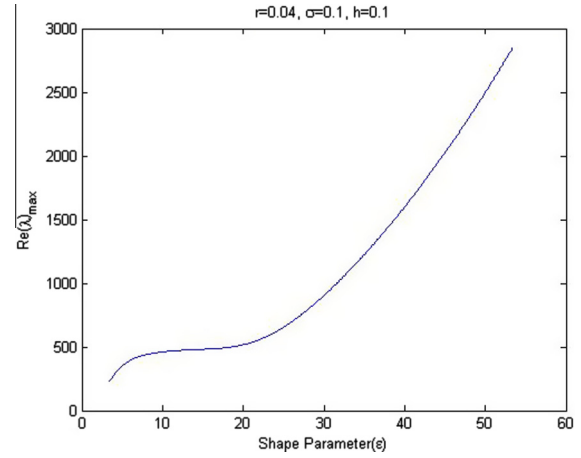


Figure 1 The relation between  $\text{Re}(\lambda)_{\max}$  and shape parameter ( $\varepsilon$ ).

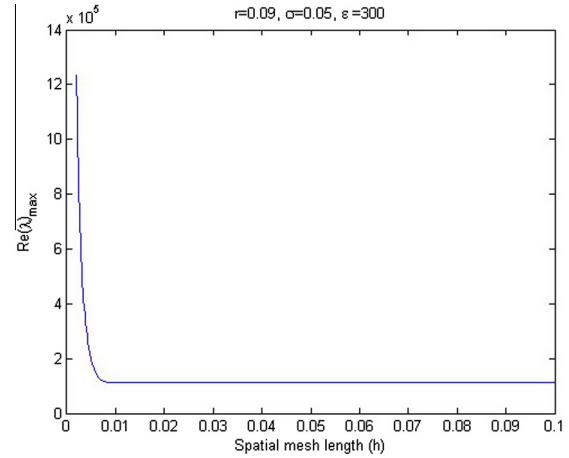


Figure 2 The relation between  $\text{Re}(\lambda)_{\max}$  and spatial mesh length ( $h$ ).

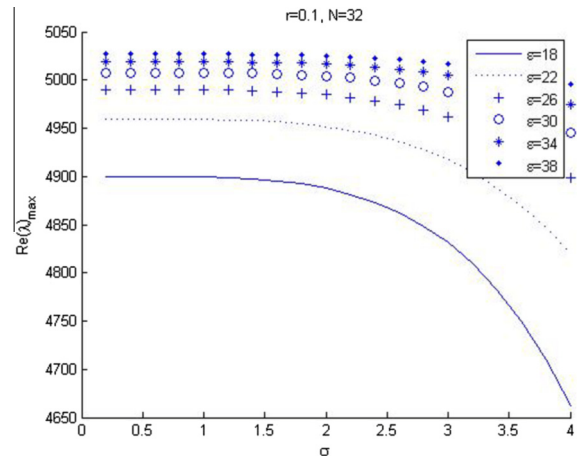
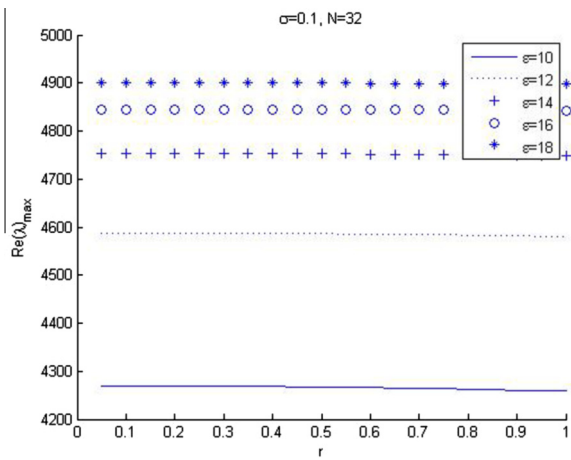


Figure 3 The relation between  $\text{Re}(\lambda)_{\max}$  and the volatility ( $\sigma$ ).



**Figure 4** The relation between  $\text{Re}(\lambda)_{\max}$  and risk free interest rate ( $r$ ).

The European call option can be modeled by Black–Scholes PDE

$$\frac{\partial^\alpha u}{\partial \tau^\alpha} + r(\tau)s \frac{\partial u}{\partial s} + \frac{1}{2}\sigma^2(s, \tau)s^2 \frac{\partial^2 u}{\partial s^2} - r(\tau)u = 0. \quad (6.1)$$

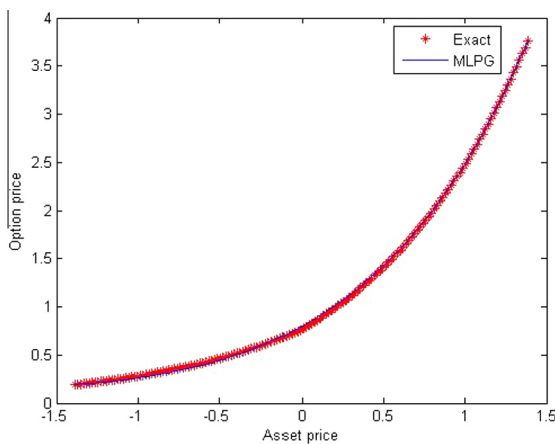
$(s, \tau) \in \mathbb{R}^+ \times [0, T]$  with the terminal and boundary condition.  $u(s, T) = \max(s - E, 0)$ ,  $s \in \mathbb{R}^+$ ,  $u(0, \tau) = 0$ ,  $\tau \in [0, T]$ ,

To illustrate accuracy of proposed method numerical simulation was done for European call option with parameters  $x_{\min} = -\ln(4E)$ ,  $x_{\max} = \ln(4E)$ .

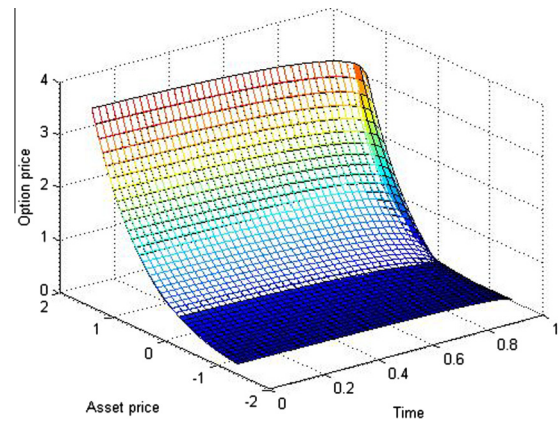
6.1. Example 1

We consider the fractional Black–Scholes equation in Eq. (3.1). The numerical simulation was done for European call option with parameters as following:

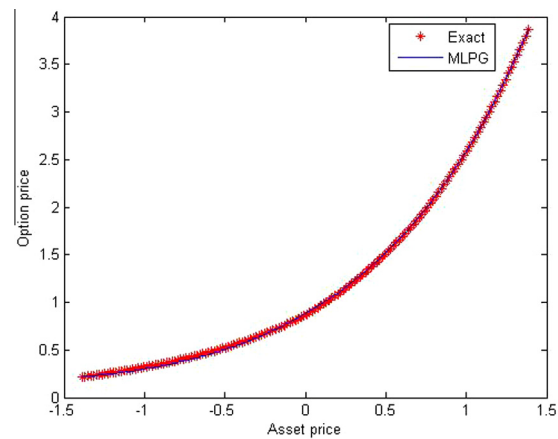
- Case 1. For  $\sigma = 0.2$ ,  $r = 0.04$ ,  $\alpha = 0.5$ ,  $T = 1$ .
- Case 3. For  $\sigma = 0.2$ ,  $r = 0.04$ ,  $\alpha = 0.99$ ,  $T = 1$ .
- Case 4. For  $\sigma = 0.2$ ,  $r = 0.1$ ,  $\alpha = 0.99$ ,  $T = 1$ .



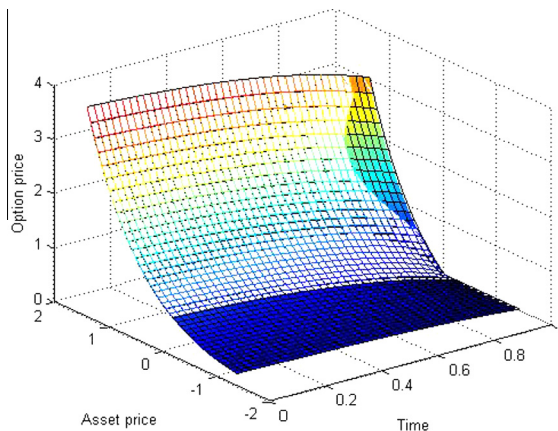
**Figure 5** The approximate solution compare with the exact solution for  $\sigma = 0.2$ ,  $r = 0.04$ ,  $\alpha = 0.5$ ,  $t = T$ .



**Figure 6** The approximate solution compare with the exact solution for  $\sigma = 0.2$ ,  $r = 0.04$ ,  $\alpha = 0.5$ ,  $0 \leq t \leq T$ .



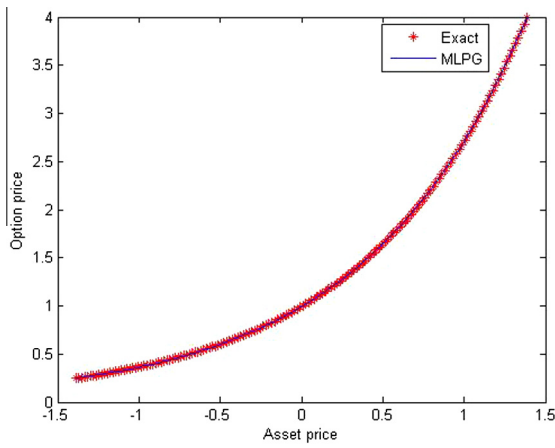
**Figure 7** The approximate solution compare with the exact solution for  $\sigma = 0.2$ ,  $r = 0.04$ ,  $\alpha = 0.99$ ,  $t = T$ .



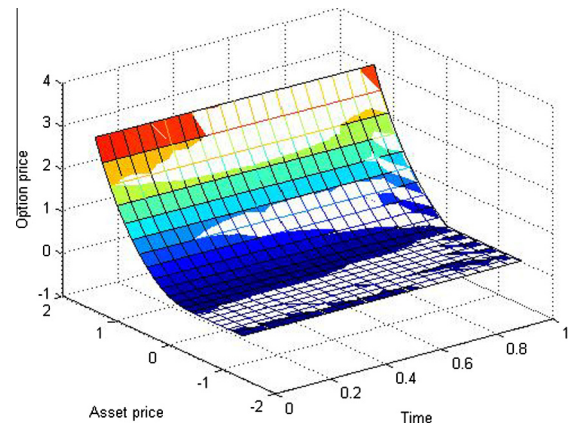
**Figure 8** The approximate solution compare with the exact solution for  $\sigma = 0.2$ ,  $r = 0.04$ ,  $\alpha = 0.99$ ,  $0 \leq t \leq T$ .

6.2. Example 2

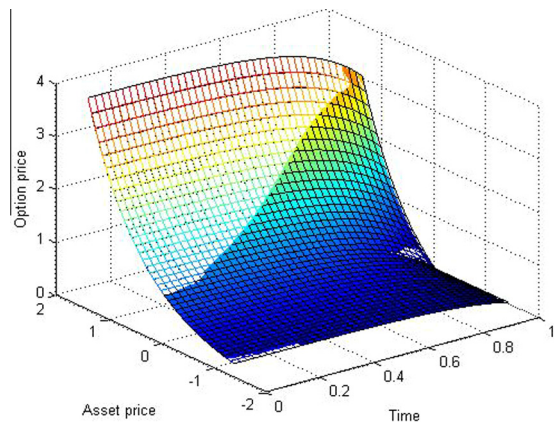
We consider the fractional Black–Scholes equation in Eq. (5.1). The numerical simulation was done for European call option with parameters as following:



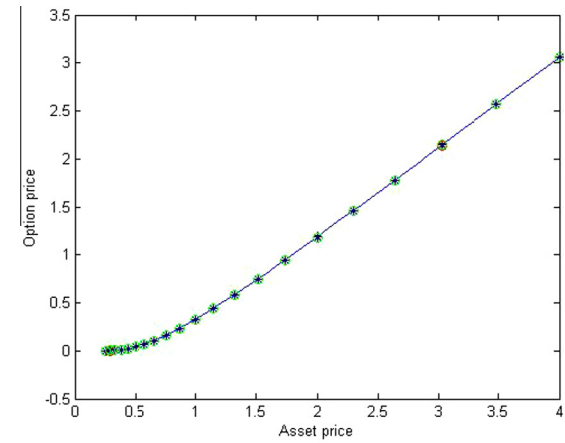
**Figure 9** The approximate solution compare with the exact solution for  $\sigma = 0.2, r = 0.1, \alpha = 0.99, t = T$ .



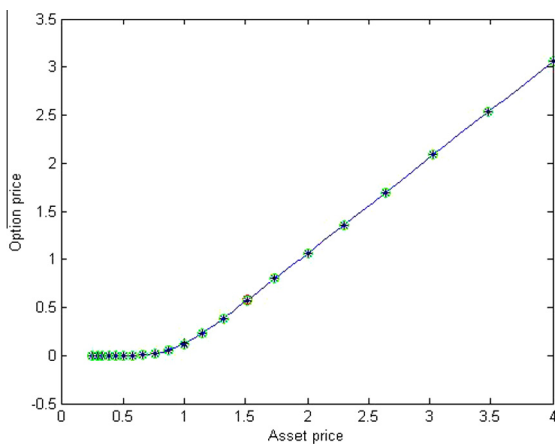
**Figure 12** The comparison of the approximate solutions of the fractional and standard Black-Scholes equation for  $r = 0.06, \alpha = 0.99, 0 \leq t \leq T$ .



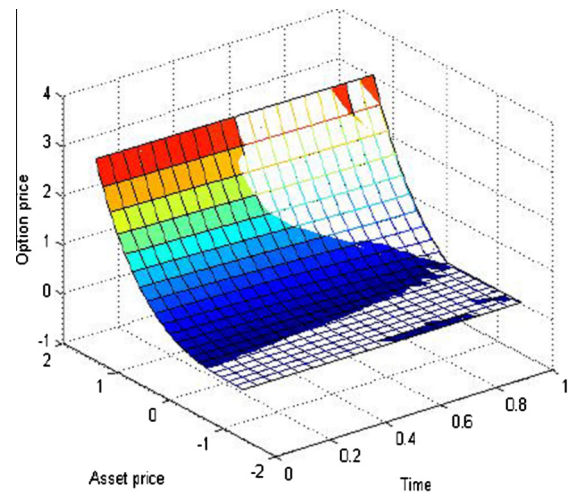
**Figure 10** The approximate solution compare with the exact solution for  $\sigma = 0.2, r = 0.1, \alpha = 0.99, 0 \leq t \leq T$ .



**Figure 13** The comparison of the approximate solutions of the fractional and standard Black-Scholes equation for  $r = 0.06, \alpha = 0.99, t = T$ .



**Figure 11** The comparison of the approximate solutions of the fractional and standard Black-Scholes equation for  $r = 0.06, \alpha = 0.99, t = T$ .



**Figure 14** The comparison of the approximate solutions of the fractional and standard Black-Scholes equation for  $r = 0.06, \alpha = 0.99, 0 \leq t \leq T$ .

Case 1. For  $\sigma = 0.15(0.5 + 2\tau)((s/100 - 1.2)^2/((s/100)^2 + 1.44))$ ,  $r = 0.06$ ,  $\alpha = 0.99$ ,  $T = 1$ .

$x_{\min} = -\ln(4E)$ ,  $x_{\max} = \ln(4E)$ .

In this case, we have unknown the exact solution.

Case 2. For  $\sigma = 0.4(2 + \sin x)$ ,  $r = 0.06$ ,  $\alpha = 0.99$ ,  $T = 1$ .

In this case, we have unknown the exact solution.

## 7. Conclusion

In this paper, the fractional Black–Scholes equation are solved by the implicit finite difference method and MLPG2 for discretizing in time variable and option price, respectively. The stability analysis present relation between maximum eigenvalues of matrix and variety of parameters.

The numerical results are presented in two examples. Example 1 presents numerical results for varieties of parameters in four cases. For case 1 and case 2 present comparison of numerical result for different volatility  $\sigma = 0.1, 0.2$  while variables are fixed,  $\alpha = 0.5, 0.99$  show for case 2 and case 3. Difference of risk free interest rate present for case 3 and case 4, respectively. Example 2 shows the value of option for volatility function in both cases. Volatility function is  $\sigma = 0.15(0.5 + 2\tau)((s/100 - 1.2)^2/((s/100)^2 + 1.44))$  and  $\sigma = 0.4(2 + \sin x)$ , respectively. Moreover, we found that the MLPG give the value option in both volatility constant and volatility function.

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