Journal of King Saud University – Science 32 (2020) 496–500

Contents lists available at ScienceDirect

Journal of King Saud University – Science

journal homepage: www.sciencedirect.com

Original article

Ostrowski and generalized trapezoid type inequalities on time scales

Adnan Tuna [a], Eze R. Nwaeze [b,*]

[a] Omer Halisdemir University, Art and Science Faculty, Mathematics Department, Niğde, Turkey
[b] Department of Mathematics, Tuskegee University, Tuskegee, AL 36088, USA

ARTICLE INFO

Article history:
Received 15 December 2017
Accepted 10 April 2018
Available online 4 June 2018

2010 MSC:
26D15
54C30
26D10

Keywords:
Ostrowski
Trapezoid
Inequalities
Time scales

ABSTRACT

In this paper, some new Ostrowski and generalized Trapezoid type inequalities on time scales are established. The Ostrowski type inequality is presented via a parameter function $g : [0,1] \to [0,1]$. Our results generalize and extend the results of Dragomir, and Ujević. Furthermore, we apply our results to the discrete and continuous time scales to obtain some other interesting inequalities as special cases.

1. Introduction

The Ukraine born mathematician Alexander Ostrowski (1937) obtained the following interesting integral inequality.

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable in (a,b) and its derivative $f' : (a,b) \to \mathbb{R}$ is bounded in (a,b). Then for any $x \in [a,b]$, we have*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leqslant \left(\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right)(b-a)\|f'\|_\infty,$$

where $\|f'\|_\infty := \sup_{x\in(a,b)} |f'(x)| < \infty$. The inequality is sharp in the sense that the constant $\frac{1}{4}$ cannot be replaced by a smaller one.

Following thereafter, Cerone and Dragomir (2000) proved the following generalized trapezoid type inequality that is similar to the Theorem 1.

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable in (a,b) and its derivative $f' : (a,b) \to \mathbb{R}$ is bounded in (a,b). Then for any $x \in [a,b]$, we have*

$$\left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(t)dt \right| \leqslant \left(\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right)(b-a)\|f'\|_\infty. \tag{1}$$

In Dragomir (2003), Dragomir posed the following question: Suppose that the absolutely continuous function $g : [a,b] \to \mathbb{R}$ satisfies the standing condition:

$$-\infty < m \leqslant g'(t) \leqslant M < \infty \text{ for a.e } t \in [a,b].$$

Find inequalities that are analogous to the inequalities in Theorems 1 and 2 in terms of the difference $M - m$?

In this same paper, he answered his question by proving the following results:

* Corresponding author.
E-mail addresses: atuna@ohu.edu.tr (A. Tuna), enwaeze@tuskegee.edu (E.R. Nwaeze).

Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

https://doi.org/10.1016/j.jksus.2018.04.011
1018-3647/

Theorem 3. *Assume that the absolutely continuous function* $g:[a,b]\to\mathbb{R}$ *satisfies the condition* $-\infty<m\leqslant g'(t)\leqslant M<\infty$ *for a. e* $t\in[a,b]$. *Then we have*

$$\left|g(t)-\frac{1}{b-a}\int_a^b g(s)ds-\left(t-\frac{a+b}{2}\right)\left(\frac{m+M}{2}\right)\right| \leqslant\frac{M-m}{2}\left[\frac{1}{4}+\left(\frac{t-\frac{a+b}{2}}{b-a}\right)^2\right](b-a), \tag{2}$$

for all $t\in[a,b]$. *This inequality is sharp.*

Theorem 4. *Assume that the absolutely continuous function* $g:[a,b]\to\mathbb{R}$ *satisfies the condition* $-\infty<m\leqslant g'(t)\leqslant M<\infty$ *for a.e* $t\in[a,b]$. *Then we have*

$$\left|\frac{(b-t)g(b)+(t-a)g(a)}{b-a}-\frac{1}{b-a}\int_a^b g(s)ds+\left(\frac{m+M}{2}\right)\left(t-\frac{a+b}{2}\right)\right| \leqslant\frac{M-m}{2}\left[\frac{1}{4}+\left(\frac{t-\frac{a+b}{2}}{b-a}\right)^2\right](b-a), \tag{3}$$

for all $t\in[a,b]$. *This inequality is sharp.*

By introducing a parameter $\lambda\in[0,1]$, Ujević (2004) generalized Theorem 3 by proving the following result:

Theorem 5. *Let* $I\subset\mathbb{R}$ *be an open interval and* $a,b\in I, a<b$. *If* $f:I\to\mathbb{R}$ *is a differentiable function such that* $m\leqslant f'(t)\leqslant M$, *for all* $t\in[a,b]$, *for some constants* $m,M\in\mathbb{R}$, *then we have*

$$\left|\lambda\frac{f(a)+f(b)}{2}+(1-\lambda)f(x)-\frac{m+M}{2}(1-\lambda)\left(x-\frac{a+b}{2}\right) -\frac{1}{b-a}\int_a^b f(t)dt\right|\leqslant\frac{M-m}{2(b-a)}\left[\frac{(b-a)^2}{4}(\lambda^2+(1-\lambda)^2)+\left(x-\frac{a+b}{2}\right)^2\right],$$

where $a+\lambda((b-a)/2)\leqslant x\leqslant b-\lambda((b-a)/2)$.

In order to unify the difference and differential calculus, Stefan Hilger (1988) introduced the theory of time scales (see Section 2 for a brief overview). In the last few years, many classical integral inequalities have been extended to time scales, see for example the papers (Bohner and Matthews, 2008; Dinu, 2007; Karpuz and Ozkan, 2008; Kermausuor et al., 2017; Liu and Ngo, 2009; Liu and Tuna, 2012; Ngô and Liu, 2009; Nwaeze, 2017a,b, 2018a,b; Tuna and Daghan, 2010; Tuna et al., 2012) and references therein. Worthy of mention is the result due to Bohner and Matthews (2008) which extends Theorem 1 to time scales as follows:

Theorem 6. *Let* $a,b,s,t\in\mathbb{T}, a<b$ *and* $f:[a,b]\to\mathbb{R}$ *be differentiable on* (a,b). *Then*

$$\left|f(t)-\frac{1}{b-a}\int_a^b f(\sigma(s))\Delta s\right|\leqslant\frac{M}{b-a}(h_2(t,a)+h_2(t,b)), \tag{4}$$

where $M:=\sup_{a<t<b}|f^{\Delta}(t)|, \sigma(\cdot)$, *and* $h_2(\cdot,\cdot)$ *are given in Definitions 8 and 17, respectively. This inequality is sharp in the sense that the right-hand side of (4) cannot be replaced by a smaller one.*

Motivated by the above works, this present paper is set to achieve the following goals: firstly, we present a time scale generalization of Theorem 5 via a parameter function such that Theorem 5 is recaptured if the parameter function is the identity map and the time scale taken as the set of real numbers. Next, we extend Theorem 4 to time scales.

The paper is organized in the following manner: in Section 2, we present a brief background of the theory of time scales. Thereafter, we formulate and prove our results in Section 3. We wrap up this work by applying our results to the continuous and discrete cases.

2. Time scale essentials

For a general introduction to the time scales theory we refer the reader to Hilger's Ph.D. thesis (Hilger, 1988), the books (Bohner and Peterson, 2001, 2003; Lakshmikantham et al., 1996) and the survey (Agarwal et al., 2001).

Definition 7. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers.

We assume throughout that $\mathbb{T}$ has the topology that is inherited from the standard topology on $\mathbb{R}$. It is also assumed throughout that in $\mathbb{T}$ the interval $[a,b]$means the set $\{t\in\mathbb{T}:a\leqslant t\leqslant b\}$ for the points $a<b$ in $\mathbb{T}$. Since a time scale may not be connected, we need the following concept of jump operators.

Definition 8. For each $t\in\mathbb{T}$, the forward jump operator $\sigma:\mathbb{T}\to\mathbb{T}$ by $\sigma(t)=\inf\{s\in\mathbb{T}:s>t\}$ and the backward jump operator $\rho:\mathbb{T}\to\mathbb{T}$ is defined by $\rho(t)=\sup\{s\in\mathbb{T}:s<t\}$.

Definition 9. If $\sigma(t)>t$, then we say that t is right-scattered, while if $\rho(t)<t$, then we say that t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. If $\sigma(t)=t$, then t is called right-dense and if $\rho(t)=t$, then t is called left-dense. Points that are both right-dense and left-dense are called dense.

Definition 10. The mapping $\mu:\mathbb{T}\to[0,\infty)$ defined by $\mu(t)=\sigma(t)-t$ is called the graininess function. The set $\mathbb{T}^{\kappa}$ is defined as follows: if $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^{\kappa}=\mathbb{T}-\{m\}$; otherwise, $\mathbb{T}^{\kappa}=\mathbb{T}$.

If $\mathbb{T}=\mathbb{R}$, then $\mu(t)=0$, and when $\mathbb{T}=\mathbb{Z}$, we have $\mu(t)=1$.

Definition 11. Let $f:\mathbb{T}\to\mathbb{R}$ and $t\in\mathbb{T}^{\kappa}$. Then we define $f^{\Delta}(t)$ to be the number (provided it exists) with the property that for any given $\varepsilon>0$ there exists a neighborhood U of t such that

$$\left|f(\sigma(t))-f(s)-f^{\Delta}(t)[\sigma(t)-s]\right|\leqslant\varepsilon|\sigma(t)-s|,\quad\forall s\in U.$$

We call $f^{\Delta}(t)$ the delta derivative of f at t.

In the case $\mathbb{T}=\mathbb{R}$, $f^{\Delta}(t)=\frac{df(t)}{dt}$. In the case $\mathbb{T}=\mathbb{Z}$, $f^{\Delta}(t)=\Delta f(t)=f(t+1)-f(t)$, which is the usual forward difference operator.

Theorem 12. *If* $f,g:\mathbb{T}\to\mathbb{R}$ *are differentiable at* $t\in\mathbb{T}^{\kappa}$, *then the product* $fg:\mathbb{T}\to\mathbb{R}$ *is differentiable at* t *and*

$$(fg)^{\Delta}(t)=f^{\Delta}(t)g(t)+f(\sigma(t))g^{\Delta}(t).$$

Definition 13. The function $f:\mathbb{T}\to\mathbb{R}$ is said to be rd-continuous on $\mathbb{T}$ provided it is continuous at all right-dense points $t\in\mathbb{T}$ and its left-sided limits exist at all left-dense points $t\in\mathbb{T}$. The set of all rd-continuous function $f:\mathbb{T}\to\mathbb{R}$ is denoted by $C_{rd}(\mathbb{T},\mathbb{R})$. Also, the set of functions $f:\mathbb{T}\to\mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by $C^1_{rd}(\mathbb{T},\mathbb{R})$.

It is known that every rd-continuous function has an antiderivative.

Definition 14. Let $F:\mathbb{T}\to\mathbb{R}$ be a function. Then $F:\mathbb{T}\to\mathbb{R}$ is called the antiderivative of f on $\mathbb{T}$ if it satisfies $F^{\Delta}(t)=f(t)$ for any $t\in\mathbb{T}^{\kappa}$. In this case, the Cauchy integral is defined by

$$\int_a^b f(t)\Delta t = F(b)-F(a),\ a,b\in\mathbb{T}.$$

Theorem 15. *Let $f,g\in C_{\mathrm{rd}}(\mathbb{T},\mathbb{R}), a,b,c\in\mathbb{T}$ and $\alpha,\beta\in\mathbb{R}$. Then*

(1) $\int_a^b[\alpha f(t)+\beta g(t)]\Delta t=\alpha\int_a^b f(t)\Delta t+\beta\int_a^b g(t)\Delta t.$
(2) $\int_a^b f(t)\Delta t=-\int_b^a f(t)\Delta t.$
(3) $\int_a^b f(t)\Delta t=\int_a^c f(t)\Delta t+\int_c^b f(t)\Delta t.$
(4) $\int_a^b f(t)g^{\Delta}(t)\Delta t=(fg)(b)-(fg)(a)-\int_a^b f^{\Delta}(t)g(\sigma(t))\Delta t.$

Theorem 16. *If f is $\Delta-$integrable on $[a,b]$, then so is $|f|$, and*

$$\left|\int_a^b f(t)\Delta t\right|\leqslant\int_a^b |f(t)|\Delta t.$$

Definition 17. Let $h_k,g_k:\mathbb{T}^2\to\mathbb{R},\ k\in\mathbb{N}_0$ be defined by $h_0(t,s):=g_0(t,s):=1$, for all $s,t\in\mathbb{T}$ and then recursively by $g_{k+1}(t,s)=\int_s^t g_k(\sigma(\tau),s)\Delta\tau,\ h_{k+1}(t,s)=\int_s^t h_k(\tau,s)\Delta\tau$, for all $s,t\in\mathbb{T}$.

In view of the above definition, we make the following remarks that will come handy in the sequel.

– For $\mathbb{T}=\mathbb{R}, h_2(t,s)=g_2(t,s)=\frac{(t-s)^2}{2}$.
– For $\mathbb{T}=\mathbb{Z}, h_2(t,s)=\frac{(t-s)(t-s-1)}{2}$, and $g_2(t,s)=\frac{(t-s)(t-s+1)}{2}$.

3. Main results

Throughout this section, we assume that the continuous function $u:[a,b]\to\mathbb{R}$ satisfies the condition: for all $x\in[a,b]$,

$$-\infty<m\leqslant u^{\Delta}(x)\leqslant M<\infty. \tag{5}$$

For the proof of our theorems, we will need the following lemmas.

Lemma 18 Xu and Fang, 2016. *Suppose that $a,b,t,x\in\mathbb{T}, a<b, f:[a,b]\to\mathbb{R}$ is differentiable, and that g is a function of $[0,1]$ into $[0,1]$. We then have the inequality*

$$\begin{aligned}&\left|\frac{1+g(1-\lambda)-g(\lambda)}{2}f(x)+\frac{g(\lambda)f(a)+(1-g(1-\lambda))f(b)}{2}-\frac{1}{b-a}\int_a^b f(\sigma(t))\Delta t\right|\\ &\leqslant\frac{K}{b-a}\left[h_2\left(a,a+g(\lambda)\frac{b-a}{2}\right)+h_2\left(x,a+g(\lambda)\frac{b-a}{2}\right)\right.\\ &\left.+h_2\left(x,a+(1+g(1-\lambda))\frac{b-a}{2}\right)h_2\left(b,a+(1+g(1-\lambda))\frac{b-a}{2}\right)\right]\end{aligned}$$

for all $\lambda\in[0,1]$ such that $a+g(\lambda)\frac{b-a}{2}$ and $a+(1+g(1-\lambda))\frac{b-a}{2}$ are in $\mathbb{T}$, and $x\in\left[a+g(\lambda)\frac{b-a}{2},a+(1+g(1-\lambda))\frac{b-a}{2}\right]\cap\mathbb{T}$, where $K=\sup_{a<x<b}\left|f^{\Delta}(x)\right|<\infty$. This inequality is sharp provided

$$\frac{g^2(\lambda)-2s(\lambda)}{2}a-\frac{g^2(\lambda)}{2}b\geqslant\int_{a+g(\lambda)\frac{b-a}{2}}^{a} t\Delta t.$$

The next lemma is a generalized Trapezoid type inequality on time scales.

Lemma 19. *Let $a,b,t,x\in\mathbb{T}, a<b$. If $f\in C^1_{\mathrm{rd}}([a,b],\mathbb{R})$, then for all $x\in[a,b]$, we have*

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(\sigma(t))\Delta t\right|\leqslant\frac{K}{b-a}[h_2(a,x)+h(b,x)], \tag{6}$$

where $K=\sup_{a<x<b}\left|f^{\Delta}(x)\right|<\infty$.

Proof. Using the integration by parts formula, given in item (iv) of Theorem 15, we get

$$\int_a^b(t-x)f^{\Delta}(t)\Delta t=(b-x)f(b)-(a-x)f(a)-\int_a^b f(\sigma(t))\Delta t.$$

This implies that

$$\begin{aligned}&\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(\sigma(t))\Delta t\\ &=\frac{1}{b-a}\int_a^b(t-x)f^{\Delta}(t)\Delta t.\end{aligned} \tag{7}$$

Taking absolute values of both sides of (7) gives

$$\begin{aligned}&\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(\sigma(t))\Delta t\right|\\ &\leqslant\frac{1}{b-a}\int_a^b|x-t|\left|f^{\Delta}(t)\right|\Delta t\\ &\leqslant\frac{K}{b-a}\left[\int_a^x|x-t|\Delta t+\int_x^b|x-t|\Delta t\right]\\ &=\frac{K}{b-a}\left[\int_a^x(x-t)\Delta t+\int_x^b(t-x)\Delta t\right]\\ &=\frac{K}{b-a}[h_2(a,x)+h(b,x)].\end{aligned}$$

Therefore, the inequality (6) is proved. □

If we apply Lemma 19 to the continuous and discrete cases, we have the following results.

Remark 20. If we take $\mathbb{T}=\mathbb{R}$ in Lemma 19, we get Theorem 2.

Corollary 21. In the case of $\mathbb{T}=\mathbb{Z}$ in Lemma 19, we have

$$\begin{aligned}&\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\sum_{t=a}^{b-1}f(t+1)\right|\\ &\leqslant K\left[\left(\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right)(b-a)+\frac{1}{b-a}\left(x-\frac{a+b}{2}\right)\right],\end{aligned}$$

where $K=\sup_{a<x<b-1}|\Delta f(x)|<\infty$.

Example 22. By taking $a=0, b=n, f(k)=x_k$, and $t=i$ in Corollary 21, one gets that for $j\in\{0,1,\cdots,n-1\}$

$$\left|\frac{(n-j)x_n+jx_0}{n}-\frac{1}{n}\sum_{i=0}^{n-1}x_{i+1}\right|\leqslant K\left[\left(\frac{1}{4}+\left(\frac{j-\frac{n}{2}}{n}\right)^2\right)n+\frac{1}{n}\left(j-\frac{n}{2}\right)\right],$$

where $K=\sup_{0<j<n-1}\left|x_{j+1}-x_j\right|<\infty$.

In the following result, we obtain the Ostrowski type inequality on time scales.

Theorem 23. *Suppose that $a,b,t,x\in\mathbb{T}, a<b, u:[a,b]\to\mathbb{R}$ is differentiable, and that g is a function of $[0,1]$ into $[0,1]$. Then we have*

$$\left|\frac{1+g(1-\lambda)-g(\lambda)}{2}u(x)-\frac{1+g(1-\lambda)-g(\lambda)}{2}\left(x-\frac{a+b}{2}\right)\left(\frac{m+M}{2}\right)\right.$$
$$+\frac{g(\lambda)u(a)+(1-g(1-\lambda))u(b)}{2}-\left(\frac{b-a}{2}\right)\left(\frac{m+M}{2}\right)\frac{(1-g(1-\lambda)-g(\lambda))}{2}$$
$$\left.-\frac{1}{b-a}\int_a^b u(\sigma(t))\Delta t-\frac{m+M}{2(b-a)}\left(g_2\left(a,\frac{a+b}{2}\right)-g_2\left(b,\frac{a+b}{2}\right)\right)\right|$$
$$\leqslant\frac{M-m}{2(b-a)}\left[h_2\left(a,a+g(\lambda)\frac{b-a}{2}\right)+h_2\left(x,a+g(\lambda)\frac{b-a}{2}\right)\right.$$
$$\left.+h_2\left(x,a+(1+g(1-\lambda))\frac{b-a}{2}\right)+h_2\left(b,a+(1+g(1-\lambda))\frac{b-a}{2}\right)\right],$$

for all $\lambda\in[0,1]$ such that $a+g(\lambda)\frac{b-a}{2}$ and $a+(1+g(1-\lambda))\frac{b-a}{2}$ are in $\mathbb{T}$, and $x\in\left[a+g(\lambda)\frac{b-a}{2},a+(1+g(1-\lambda))\frac{b-a}{2}\right]\cap\mathbb{T}$.

Proof.

We consider the function $f:[a,b]\to\mathbb{R}$,

$$f(x)=u(x)-\left(x-\frac{a+b}{2}\right)\frac{m+M}{2}.$$

The function f is continuous and $f^{\Delta}(x)=u^{\Delta}(x)-\frac{m+M}{2}$. From (5), we have that for all $x\in[a,b]$,

$$\left|f^{\Delta}(x)\right|\leqslant\frac{M-m}{2}.$$

Hence, $f^{\Delta}\in L_{\infty}[a,b]_{\mathbb{T}}$ and $\left\|f^{\Delta}\right\|_{\infty}\leqslant\frac{M-m}{2}$.

Applying the inequality in Lemma 18 to the function f, we obtain

$$\left|\frac{1+g(1-\lambda)-g(\lambda)}{2}\left(u(x)-\left(x-\frac{a+b}{2}\right)\frac{m+M}{2}\right)\right.$$
$$-\frac{1}{b-a}\int_a^b\left[u(\sigma(t))-\left(\sigma(t)-\frac{a+b}{2}\right)\left(\frac{m+M}{2}\right)\right]\Delta t$$
$$\left.+\frac{g(\lambda)\left(u(a)-\left(a-\frac{a+b}{2}\right)\frac{m+M}{2}\right)+(1-g(1-\lambda))\left(u(b)-\left(b-\frac{a+b}{2}\right)\frac{m+M}{2}\right)}{2}\right|$$
$$\leqslant\frac{M-m}{2(b-a)}\left[h_2\left(a,a+g(\lambda)\frac{b-a}{2}\right)+h_2\left(x,a+g(\lambda)\frac{b-a}{2}\right)\right.$$
$$\left.+h_2\left(x,a+(1+g(1-\lambda))\frac{b-a}{2}\right)+h_2\left(b,a+(1+g(1-\lambda))\frac{b-a}{2}\right)\right],$$

from which we get the intended result. □

If we apply Theorem 23 to the continuous and discrete cases, we have the following results. For this, we use the definitions of $h_2(\cdot,\cdot)$ and $g_2(\cdot,\cdot)$ as outlined in Definition 17.

Corollary 24. *In the case when $\mathbb{T}=\mathbb{R}$ in Theorem 23, we get*

$$\left|\frac{1+g(1-\lambda)-g(\lambda)}{2}u(x)-\frac{1+g(1-\lambda)-g(\lambda)}{2}\left(x-\frac{a+b}{2}\right)\left(\frac{m+M}{2}\right)-\frac{1}{b-a}\int_a^b u(t)dt\right.$$
$$\left.+\frac{g(\lambda)u(a)+(1-g(1-\lambda))u(b)}{2}-\left(\frac{b-a}{2}\right)\left(\frac{m+M}{2}\right)\frac{(1-g(1-\lambda)-g(\lambda))}{2}\right|$$
$$\leqslant\frac{M-m}{2(b-a)}\left[\frac{g^2(\lambda)(b-a)^2}{8}+\frac{1}{2}\left(\frac{(b-a)g(\lambda)-2(x-a)}{2}\right)^2\right.$$
$$\left.+\frac{((b-a)g(1-\lambda)+a+b-2x)^2}{8}+\frac{(b-a)^2(g(1-\lambda)-1)^2}{8}\right].$$

Corollary 25. *In the case of $\mathbb{T}=\mathbb{Z}$ in Theorem 23, we have*

$$\left|\frac{1+g(1-\lambda)-g(\lambda)}{2}u(x)-\frac{1+g(1-\lambda)-g(\lambda)}{2}\left(x-\frac{a+b}{2}\right)\left(\frac{m+M}{2}\right)\right.$$
$$+\frac{g(\lambda)u(a)+(1-g(1-\lambda))u(b)}{2}-\left(\frac{b-a}{2}\right)\left(\frac{m+M}{2}\right)\frac{(1-g(1-\lambda)-g(\lambda))}{2}$$
$$\left.-\frac{1}{b-a}\sum_{t=a}^{b-1}u(t+1)+\frac{m+M}{4}\right|$$
$$\leqslant\frac{M-m}{2(b-a)}\left[\frac{b-a}{8}g(\lambda)((b-a)g(\lambda)+2)+\frac{1}{2}\left(\frac{(b-a)g(\lambda)-2(x-a)}{2}\right)^2\right.$$
$$+\frac{(b-a)g(\lambda)-2(x-a)}{4}+\frac{((b-a)g(1-\lambda)+a+b-2x)^2}{8}$$
$$\left.+\frac{((b-a)g(1-\lambda)+a+b-2x)}{4}+\frac{(b-a)(g(1-\lambda)-1)}{8}((b-a)(g(1-\lambda)-1)+2)\right],$$

for all $\lambda\in[0,1]$ such that $a+g(\lambda)\frac{b-a}{2}$ and $a+(1+g(1-\lambda))\frac{b-a}{2}$ are in $\mathbb{Z}$, and $x\in\left[a+g(\lambda)\frac{b-a}{2},a+(1+g(1-\lambda))\frac{b-a}{2}\right]\cap\mathbb{Z}$.

Corollary 26. *In the case of $g(\lambda)=\lambda$in Theorem 23, we get*

$$\left|(1-\lambda)u(x)-(1-\lambda)\left(x-\frac{a+b}{2}\right)\left(\frac{m+M}{2}\right)+\lambda\frac{u(a)+u(b)}{2}\right.$$
$$\left.-\frac{1}{b-a}\int_a^b u(\sigma(t))\Delta t-\frac{m+M}{2(b-a)}\left(g_2\left(a,\frac{a+b}{2}\right)-g_2\left(b,\frac{a+b}{2}\right)\right)\right|$$
$$\leqslant\frac{M-m}{2(b-a)}\left[h_2\left(a,a+\lambda\frac{b-a}{2}\right)+h_2\left(x,a+\lambda\frac{b-a}{2}\right)\right.$$
$$\left.+h_2\left(x,a+(2-\lambda)\frac{b-a}{2}\right)h_2\left(b,a+(2-\lambda)\frac{b-a}{2}\right)\right],$$

for all $\lambda\in[0,1]$such that $a+\lambda\frac{b-a}{2}$ and $a+(2-\lambda)\frac{b-a}{2}$ are in $\mathbb{T}$, and $x\in\left[a+\lambda\frac{b-a}{2},a+(2-\lambda)\frac{b-a}{2}\right]\cap\mathbb{T}$.

Proposition 27. *In the case of $\lambda=0$ in Corollary 26, we get*

$$\left|u(x)-\left(x-\frac{a+b}{2}\right)\left(\frac{m+M}{2}\right)-\frac{m+M}{2(b-a)}\left(g_2\left(a,\frac{a+b}{2}\right)-g_2\left(b,\frac{a+b}{2}\right)\right)\right.$$
$$\left.-\frac{1}{b-a}\int_a^b u(\sigma(t))\Delta t\right|\leqslant\frac{M-m}{2(b-a)}[h_2(x,a)+h_2(x,b)],$$

for all $a,b,t,x\in\mathbb{T}$.

Remark 28. If we take $\mathbb{T}=\mathbb{R}$ in Corollary 26, then we recapture Theorem 5. Also, Proposition 27 boils down to Theorem 3.

Proposition 29. *In the case of $\lambda=\frac{1}{2}$ in Corollary 26, we get*

$$\left|\frac{1}{2}u(x)-\frac{1}{2}\left(x-\frac{a+b}{2}\right)\left(\frac{m+M}{2}\right)+\frac{u(a)+u(b)}{4}\right.$$
$$\left.-\frac{1}{b-a}\int_a^b u(\sigma(t))\Delta t-\frac{m+M}{2(b-a)}\left(g_2\left(a,\frac{a+b}{2}\right)-g_2\left(b,\frac{a+b}{2}\right)\right)\right|$$
$$\leqslant\frac{M-m}{2(b-a)}\left[h_2\left(a,\frac{3a+b}{4}\right)+h_2\left(x,\frac{3a+b}{4}\right)+h_2\left(x,\frac{a+3b}{4}\right)\right.$$
$$\left.+h_2\left(b,\frac{a+3b}{4}\right)\right],$$

where $\frac{3a+b}{4}$ and $\frac{a+3b}{4}$ are in $\mathbb{T}$, and $x\in\left[\frac{3a+b}{4},\frac{a+3b}{4}\right]\cap\mathbb{T}$.

Proposition 30. *In the case of $\lambda=1$ in Corollary 26, we get*

$$\left|\frac{u(a)+u(b)}{2}-\frac{1}{b-a}\int_a^b u(\sigma(t))\Delta t-\frac{m+M}{2(b-a)}\left(g_2\left(a,\frac{a+b}{2}\right)-g_2\left(b,\frac{a+b}{2}\right)\right)\right|$$
$$\leqslant\frac{M-m}{2(b-a)}\left[h_2\left(a,\frac{a+b}{2}\right)+2h_2\left(x,\frac{a+b}{2}\right)+h_2\left(b,\frac{a+b}{2}\right)\right]$$

where $\frac{a+b}{2} \in \mathbb{T}$.

Our next result is the generalized Trapezoid type inequality on time scales.

Theorem 31. *Let* $a, b, t, x \in \mathbb{T}, a < b$. *If* $u \in C^1_{rd}([a,b], \mathbb{R})$, *then for all* $x \in [a,b]$, *we have*

$$\left|\frac{(b-x)u(b)+(x-a)u(a)}{b-a} + \frac{m+M}{2}\left(x - \frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b u(\sigma(t))\Delta t - \frac{m+M}{2(b-a)}\left(g_2\left(a,\frac{a+b}{2}\right) - g_2\left(b,\frac{a+b}{2}\right)\right)\right| \leqslant \frac{M-m}{2(b-a)}[h_2(a,x)+h(b,x)].$$

Proof.

Let the function $f : [a,b] \to \mathbb{R}$, be defined by

$$f(x) = u(x) - \left(x - \frac{a+b}{2}\right)\frac{m+M}{2}.$$

Then, the function f is continuous and $f^\Delta(x) = u^\Delta(x) - \frac{m+M}{2}$. From (5), we get that for all $x \in [a,b]$,

$$\left|f^\Delta(x)\right| \leqslant \frac{M-m}{2},$$

which shows that $f^\Delta \in L_\infty[a,b]_\mathbb{T}$ and $\left\|f^\Delta\right\|_\infty \leqslant \frac{M-m}{2}$.

Applying Lemma 19 to the mapping f, we obtain

$$\left|\frac{(b-x)\left(u(b) - \left(b - \frac{a+b}{2}\right)\frac{m+M}{2}\right) + (x-a)\left(u(a) - \left(a - \frac{a+b}{2}\right)\frac{m+M}{2}\right)}{b-a} - \frac{1}{b-a}\int_a^b \left[u(\sigma(t)) - \left(\sigma(t) - \frac{a+b}{2}\right)\left(\frac{m+M}{2}\right)\right]\Delta t\right|$$

$$\leqslant \frac{M-m}{2(b-a)}[h_2(a,x)+h(b,x)],$$

from which we get the desired inequality. □

If we apply Theorem 31 to the discrete and continuous cases, we arrive at the following results.

Remark 32. We observe that Theorem 31 amounts to Theorem 4 if the time scale is the set of real numbers.

Corollary 33. *Setting* $\mathbb{T} = \mathbb{Z}$ *in Theorem 31, we obtain*

$$\left|\frac{(b-x)u(b)+(x-a)u(a)}{b-a} + \left(\frac{m+M}{2}\right)\left(x - \frac{a+b}{2}\right) - \frac{1}{b-a}\sum_{t=a}^{b-1}u(t+1) + \frac{m+M}{4}\right|$$

$$\leqslant \frac{M-m}{2}\left[\left(\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right)(b-a) + \frac{1}{b-a}\left(x - \frac{a+b}{2}\right)\right].$$

Proposition 34. *In the case of* $x = \frac{a+b}{2} \in \mathbb{T}$ *in the Theorem 31, we get*

$$\left|\frac{u(b)+u(a)}{2} - \frac{1}{b-a}\int_a^b u(\sigma(t))\Delta t - \frac{m+M}{2(b-a)}\left(g_2\left(a,\frac{a+b}{2}\right) - g_2\left(b,\frac{a+b}{2}\right)\right)\right|$$

$$\leqslant \frac{M-m}{2(b-a)}\left[h_2\left(a,\frac{a+b}{2}\right) + h\left(b,\frac{a+b}{2}\right)\right].$$

4. Conclusion

Two main theorems have been established. Our results extend and generalize results of Ujević (2004), and Dragomir (2003). By taking the identity map as the parameter function and then choosing different values of the parameter $\lambda \in [0,1]$, and/or a different time scale (other than the ones presented, for example, the quantum time scale), one can obtain many more interesting inequalities in this direction.

References

Agarwal, R., Bohner, M., Peterson, A., 2001. Inequalities on time scales: a survey. Math. Inequal. Appl. 4, 535–557.

Bohner, M., Matthews, T., 2008. Ostrowski inequalities on time scales. J. Inequal. Pure Appl. Math. 9. Art. 6.

Bohner, M., Peterson, A., 2001. Dynamic Equations on Time Scales: An Introduction with Applications. Birkhäuser Boston Inc., Boston, MA.

Bohner, M., Peterson, A., 2003. Advances in Dynamic Equations on Time Scales. Birkhäuser Boston, Boston, MA.

Cerone, P., Dragomir, S.S., 2000. Trapezoidal type rules from an inequalities point of view. In: Anastassiou, G.A. (Ed.), Analytic-Computational Methods in Applied Mathematics. CRC Press, New York, pp. 65–134.

Dinu, C., 2007. Ostrowski type inequalities on time scales. An. Univ. Craiova Ser. Mater. Inform. 34, 43–58.

Dragomir, S.S., 2003. Improvements of Ostrowski and generalized trapezoid inequality in terms of the upper and lower bounds of the first derivative. Tamkang J. Math. https://doi.org/10.5556/j.tkjm.34.2003.312.

Hilger, S., 1988. Ein Maβkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten (Ph.D. thesis). Universität Würzburg, W ürzburg, Germany.

Karpuz, B., Ozkan, U.M., 2008. Generalized Ostrowski's inequality on time scales. J. Inequal. Pure Appl. Math. 9. Art. 112.

Kermausuor, S., Nwaeze, E.R., Torres, D.F.M., 2017. Generalized weighted Ostrowski and Ostrowski-Grüss type inequalities on time scale via a parameter function. J. Math. Inequal. 11, 1185–1199.

Lakshmikantham, V., Sivasundaram, S., Kaymakcalan, B., 1996. Dynamic Systems on Measure Chains. Mathematics and Its Applications. 370, Kluwer Academic Publishers Group, Dordrecht.

Liu, W.J., Ngô, Q.-A., 2009. An Ostrowski-Grüss type inequality on time scales. Comput. Math. Appl. 58, 1207–1210.

Liu, W.J., Tuna, A., 2012. Weighted Ostrowski, trapezoid and Grüss type inequalities on time scales. J. Math. Inequal. 6, 381–399.

Ngô, Q.-A., Liu, W.J., 2009. A sharp Grüss type inequality on time scales and application to the sharp Ostrowski-Grüss inequality. Commun. Math. Anal. 6, 33–41.

Nwaeze, E.R., 2017a. A new weighted Ostrowski type inequality on arbitrary time scale. J. King Saud Univ. Sci. 29 (2), 230–234.

Nwaeze, E.R., 2017b. Generalized weighted trapezoid and Grüss type inequalities on time scales. Aust. J. Math. Anal. Appl. 11 (1). Art. 4.

Nwaeze, E.R., 2018a. Time scale versions of the Ostrowski-Grüss type inequality with a parameter function. J. Math. Inequal 12 (2), 531–543.

Nwaeze, E.R., 2018b. New integral inequalities on time scales with applications to the continuous and discrete calculus. Commun. Appl. Anal. 22 (1), 1–17.

Ostrowski, A., 1937. Uber die Absolutabweichung einer differentiierbaren Funktion von ihrem Integralmittelwert. Comment. Math. Helv. 10, 226–227.

Tuna, A., Daghan, D., 2010. Generalization of Ostrowski and Ostrowski-Grüss type inequalities on time scales. Comput. Math. Appl. 60, 803–811.

Tuna, A., Jiang, Y., Liu, W.J., 2012. Weighted Ostrowski, Ostrowski-Grüss and Ostrowski-Čebyšev type inequalities on time scales. Publ. Math. Debrecen 81, 81–102.

Ujević, N., 2004. A generalization of Ostrowski's inequality and applications in numerical integration. Appl. Math. Lett. 17, 133–137.

Xu, G., Fang, B.Z., 2016. A new Ostrowski type inequality on time scales. J. Math. Inequal. 10 (3), 751–760.