



Contents lists available at ScienceDirect

Journal of King Saud University – Science

journal homepage: www.sciencedirect.com

Original article

Computation of vertex and edge resolvability of benzenoid tripod structure

Ali N.A. Koam^a, Ali Ahmad^{b,*}, M.A. Asim^b, Muhammad Azeem^c^a Department of Mathematics, College of Science, Jazan University, New Campus, Jazan 2097, Saudi Arabia^b College of Computer Science & Information Technology, Jazan University, Jazan, Saudi Arabia^c Department of Mathematics, Riphah Institute of Computing and Applied Sciences, Riphah International University, Lahore, Pakistan

ARTICLE INFO

Article history:

Received 14 June 2021

Revised 27 October 2021

Accepted 23 June 2022

Available online 28 June 2022

Subject classifications:

05C12

05C90

05C05

Keywords:

Node-resolvability

Edge-resolvability

Benzenoid structure

Benzenoid tripod

ABSTRACT

In the chemistry, the chemical compound's structures are commonly shown by the graphs. In the scenario, atoms are replaced with vertices or nodes and bond types with simple lines known as edges. When the each node and edge of a graph have distinct representation or location with respect to the chosen vertices in a subset from the graph is known as the resolving set of mixed metric dimension say F_m . This concept is known as vertex and edge resolvability parameter. This concept is usually helps to acquired a unique location or position of a structure or a graph. It is used to know the patterns of different drugs in the pharmaceutical research. There are many other applications of this variant. The exact metric dimension or resolving set of mixed metric dimension of benzenoid tripod structure is found in this research work. We proved that, this parameter is constant for the particular structure of benzenoid tripod graph.

© 2022 The Author(s). Published by Elsevier B.V. on behalf of King Saud University. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>).

1. Introduction

Understanding the chemical structures that support current chemical ideas, establishing and exploring unique mathematical models of chemical processes, and applying mathematical concepts and techniques to chemistry are just a few of the challenges that mathematical chemistry has lately addressed. Throughout the history of science, some countable researchers have been persuaded to utilize links between both major fields in which mathematics and the second most important is chemistry, as well as the prospect of using basic operations of arithmetic to deduce established and anticipate new chemical properties. Mathematical techniques are frequently employed in a variety of fields of physical chemistry, most notably thermodynamics and chemical energy.

After physicists demonstrated in the early 20th century that the main properties of chemical compounds could be foresee using the techniques of quantum theory, there was a considerable demand for math in chemistry. The realization that chemistry cannot be understood without understanding of quantum physics, including its sophisticated mathematical tools, was the fundamental driving factor that brought the math and its concepts into the laboratories of chemistry. We recommend various material for distinct studies of related to the topic of mathematical chemistry in the form of graph theory (Yang et al., 2019; Manzoor et al., 2020; Siddiqui et al., 2016).

To characterize the structural properties of clusters, polymers, crystals, processes, molecules and many other materials, chemical graph theory is used which is very deep and novel field of mathematical science. The nodes or vertices of this field can be orbitals, intermediates, group of atoms, molecule, electron and many other items. Some forces such as Debye, Keesom, van der Waals forces may all be used to show the relationships between a structure's vertices, moreover, also some non-bonded and bonded connection, intermolecular bonding and basic reactions.

For the literature review and base case parameter of this case study, we will start from Nadeem et al. (2020), in which authors

* Corresponding author.

E-mail address: akoum@jazanu.edu.sa (A.N.A. Koam).

Peer review under responsibility of King Saud University.



Production and hosting by Elsevier

made a three-dimensional hexagonal structure and proved that the resolving set of this is constant, moreover this concept is named as locating number and discussed some applications. Particular type of chemical structure are also studied by this parameter, for example α -boron nanotubes are discussed in Hussain et al. (2018), silicate star structure and its metric are determined in Simonraj and George (2015), on the cellulose network (Siddiqui and Imran, 2015). And another structure of VC_5C_7 nanotubes are reshaped with this parameter are available in Imran et al. (2019). Mixed metric generators for generalized Peterson graphs and some generalize classes of rotational symmetric graphs are studied in Raza and Ji (2020) and Raza et al. (2020). The graph generated from different path graphs and discussed its edge and vertex resolvability in Raza et al. (2020). The vertex and edge resolvability of polycyclic hydrocarbons from aromatic family are found in Azeem and Nadeem (2021).

The seminal paper of the notion of resolving sets, are available at Slater (1975), the same idea presented by Harary and Melter (1976) with the name of locating parameter. In the pure graph theoretical parameter (Chartrand et al., 2000; Chartrand et al., 2000) named it as metric generator. This parameter have distinct applications and implementations like, for determining the patterns of variant of drugs in the field of pharmaceutical research, the statement enclosed by Johnson (1993). Metric resolving parameters have different variant of applications, such as image processing, facility location problems, sonar and coast-guard loran (Slater, 1975), combinatorial optimization (Sebö and Tannier, 2004), computer networks (Manuel et al., 2008), weighing problems (Söderberg and Shapiro, 1963), robot navigation (Khuller et al., 1996), for detailed and further study on this parameter see (Perc et al., 2013; Perc and Szolnoki, 2010). The notion of metric dimension is commonly employed to address many complex issues due to its vast range of applications. We refer to for resolvability parameters of various chemical structures (Hussain et al., 2018; Krishnan and Rajan, 2016; Siddiqui and Imran, 2014).

Two edges are separated in the similar manner of vertices. For example, instead of nodes of a graph two edges may have unique representation or locations in terms of with respect to the chosen vertices in a subset of a graph. Taking this reason behind (Kelenc et al., 2018) put forward another parameter known as the edge metric resolving set or the edge metric dimension. In which the used the metric of a graph in terms of edges to locate every pair of edges and it is purely depend on the selection of vertices. Later on Kelenc et al. (Dec 2017), developed a mixed metric dimension in which both notions (recognising vertices and edges) are combined, with the additional restriction that no two elements (vertices and edges) have the same distances with regard to the selected subset. For the computational costs of all above topics we suggested to see (Hauptmann et al., 2012; Lewis et al., 1983; Johnson, 1993; Johnson, 1998).

Next to this, some basic definitions are provided for the basic use of them in our main task. In the Section 2, construction of our main structure or chemical compound is drawn for the better understanding to readers. Also the same section included the results of main task in terms of lemmas and remarks. In the Section 3, conclusion is drawn and closing remarks are added. At the end appropriate references are given related to this research work.

Given below are basic concepts and background results related to study of this work.

Definition 1.1 Nadeem et al. (2020). “Assume C be an associated graph of chemical structure/network, whose vertex/node set we will denote with symbol $N(C)$ or simply N , while $B(C)$ or B is the edge/bond set, the shortest distance between two bonds

$b_1, b_2 \in N(C)$ denoted by S_{b_1, b_2} , and calculated by counting the number of bonds while moving through the $b_1 - b_2$ path. The distance between an edge $e = b_1 b_2 \in B(C)$, and a node $b \in N(C)$ is counted by the relation $S_{e, b} = \min\{S_{b_1, b}, S_{b_2, b}\}$.”

Definition 1.2 Deng et al. (mar 2021). “A vertex b of a connected associated graph C of a chemical structure, differentiates nodes (b_1) and edges (e_1), if $S_{b_1, b} \neq S_{b, e_1}$. A subset F_m is a mixed resolving set if any different pair of components of C are separated by a node of F_m . The minimum number of nodes in mixed resolving set for C is named the mixed metric dimension and is denoted by $dim_m(C)$. It is also known as blended version of both metric (Slater, 1975) and edge metric dimension (Chartrand et al., 2000).”

Theorem 1.3 Kelenc et al. (Dec 2017). “If $dim_m(C)$ is the mixed metric dimension, then $dim_m(C) = 2$, iff C is a path P_n .”

2. Construction of tripod structure

Benzenoid hydrocarbons are considered as very important and natural representations of graph of benzenoid systems, because of their prominence in theoretical chemistry (Chou et al., 2012). According to Ali et al. (2018), it is an established theory that benzenoids based hydrocarbons are vital and advantageous in the environmental, food and chemical sectors. Various catacondensed and pericondensed benzenoid structures were considered in respect to polynomial types. This is a tripod made of pericondensed benzenoid. It has $4(\theta_1 + \theta_2 + \theta_3) - 8$ nodes and $5(\theta_1 + \theta_2 + \theta_3) - 11$ bonds, with all the running parameters $\theta_1, \theta_2, \theta_3 \geq 2$. In addition, Jamil et al. (2020) is available for comprehensive topological studies of benzenoid structures and in Koam et al. (2021), some metric based study is recently came up in the literature. The bond and node or we can say simply edge set and vertex set for this benzenoid structure $T(\theta_1, \theta_2, \theta_3)$ is shown below. We use the labelling of edges and nodes provided in Fig. 1 in our primary results (Ali et al., 2021).

$$\begin{aligned} N(T(\theta_1, \theta_2, \theta_3)) &= \{a_\delta : 1 \leq \delta \leq 2\theta_3\} \cup \{b_\delta : 1 \leq \delta \leq 2\theta_1\} \cup \{c_\delta, c'_\delta : 1 \leq \delta \leq 2\theta_2 - 1\} \\ &\cup \{a'_\delta : 1 \leq \delta \leq 2\theta_3 - 3\} \cup \{b'_\delta : 1 \leq \delta \leq 2\theta_1 - 3\}, \\ B(T(\theta_1, \theta_2, \theta_3)) &= \{a_\delta a_{\delta+1} : 1 \leq \delta \leq 2\theta_3 - 1\} \cup \{b_\delta b_{\delta+1} : 1 \leq \delta \leq 2\theta_1 - 1\} \\ &\cup \{c_\delta c_{\delta+1}, c'_\delta c'_{\delta+1} : 1 \leq \delta \leq 2\theta_2 - 2\} \cup \{a'_\delta a'_{\delta+1} : 1 \leq \delta \leq 2\theta_3 - 4\} \\ &\cup \{b'_\delta b'_{\delta+1} : 1 \leq \delta \leq 2\theta_1 - 4\} \cup \{a_\delta a'_\delta : 1 \leq \delta(\text{odd}) \leq 2\theta_3 - 3\} \\ &\cup \{b_\delta b'_\delta : 1 \leq \delta(\text{odd}) \leq 2\theta_1 - 3\} \cup \{c_\delta c'_\delta : 1 \leq \delta(\text{odd}) \leq 2\theta_2 - 1\} \\ &\cup \{a_{2\theta_3} b_1, a_{2\theta_3-1} c'_1, b_2 c_1, a_{2\theta_3-3} c'_2, b'_1 c_2, \}. \end{aligned}$$

Lemma 2.1. If $T(\theta_1, \theta_2, \theta_3)$ is a benzenoid tripod graph with any of $\theta_1, \theta_2, \theta_3 = 2$, then the mixed metric resolving set has minimum three members.

Proof. The order or the total amount of vertices in the benzenoid tripod's associated graph with any of $\theta_1, \theta_2, \theta_3 = 2$, are $4(\theta_1 + \theta_2 + \theta_3) - 8$, and to look over the likelihood of mixed metric resolving set having three members in it, are $C(4(\theta_1 + \theta_2 + \theta_3) - 8, 3) = \frac{(4(\theta_1 + \theta_2 + \theta_3) - 8)!}{2 \times (4(\theta_1 + \theta_2 + \theta_3) - 11)!}$. Over here we are focusing on three members in the set, from the result of Theorem 1.3 in the literature, the only two member of mixed metric resolving set is just for the basic graph f path. Now because of its computational cost of selecting mixed metric resolving set, for a graph, we can't discover the precise number of mixed resolving sets, but from $\frac{(4(\theta_1 + \theta_2 + \theta_3) - 8)!}{2 \times (4(\theta_1 + \theta_2 + \theta_3) - 11)!}$ possibilities we selected a subset F_m and described

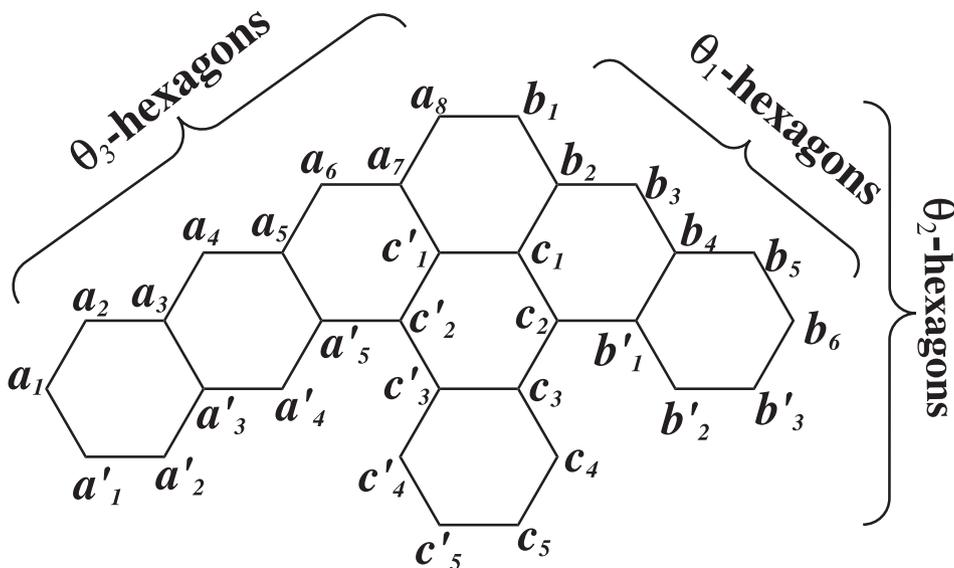


Fig. 1. Benzenoid Tripod with $\{\theta_1, \theta_2, \theta_3\} = \{3, 3, 4\}$.

as: $F_m = \{a_1, a_{2\theta_3}, b_{2\theta_1}\}$. Proving of this claim that F_m is also a subset having ability to be the candidate of mixed metric resolvability set of benzenoid tripod graph or $T(\theta_1, \theta_2, \theta_3)$, we will stick to 1.2's definition. We will also look over the distinct placements or locations of each node and edge to meet the definition's requirements, and the approach is explained above in Definition 1.2.

Positions $p(a_\delta|F_m)$ with respect to F_m , for the nodes a_δ with $\delta = 1, 2, \dots, 2\theta_3$, are described as:

$$p(a_\delta|F_m) = (\delta - 1, 2\theta_3 - \delta, 2(\theta_3 + \theta_1) - \delta).$$

Positions $p(b_\delta|F_m)$ with respect to F_m , for the nodes b_δ with $\delta = 1, 2, \dots, 2\theta_1$, are described as:

$$p(b_\delta|F_m) = (2\theta_3 + \delta - 1, \delta, 2\theta_1 - \delta).$$

Positions $p(c_\delta|F_m)$ with respect to F_m , for the nodes c_δ with $\delta = 1, 2, \dots, 2\theta_2 - 1$, are described as:

$$p(c_\delta|F_m) = \begin{cases} (2\theta_3 + \delta - 1, \delta + 2, 2\theta_1 - 1), & \text{if } \delta = 1; \\ (2\theta_3 + \delta - 1, \delta + 2, 2(\theta_1 - 2) + \delta), & \text{if } \delta = 2; \\ (2\theta_3 + \delta - 3, \delta + 2, 2(\theta_1 - 2) + \delta), & \text{if } \delta = 3, 4, \dots, 2\theta_2 - 1. \end{cases}$$

Positions $p(a'_\delta|F_m)$ with respect to F_m , for the nodes a'_δ with $\delta = 1, 2, \dots, 2\theta_3 - 3$, are described as:

$$p(a'_\delta|F_m) = (\delta, 2\theta_3 + 1 - \delta, 2(\theta_3 + \theta_1) - 1 - \delta).$$

Positions $p(b'_\delta|F_m)$ with respect to F_m , for the nodes b'_δ with $\delta = 1, 2, \dots, 2\theta_1 - 3$, are described as:

$$p(b'_\delta|F_m) = (2\theta_3 + \delta + 1, \delta + 4, 2(\theta_1 - 1) - \delta).$$

Positions $p(c'_\delta|F_m)$ with respect to F_m , for the nodes c'_δ with $\delta = 1, 2, \dots, 2\theta_2 - 1$, are described as:

$$p(c'_\delta|F_m) = \begin{cases} (2\theta_3 - 1, \delta + 1, 2\theta_1 - 1 + \delta), & \text{if } \delta = 1; \\ (2(\theta_3 - 2) + \delta, \delta + 1, 2\theta_1 - 1 + \delta), & \text{if } \delta = 2; \\ (2(\theta_3 - 2) + \delta, \delta + 1, 2\theta_1 - 3 + \delta), & \text{if } \delta = 3, 4, \dots, 2\theta_2 - 1. \end{cases}$$

Given above are the positions of nodes only, with respect to the chosen vertices in F_m , for the fulfillment of definition of mixed resolving set we still need to investigate the edges or bonds of graph. Following are positions of bonds that will complete the definition.

Positions $p(a_\delta a_{\delta+1}|F_m)$ with regard to F_m , for the bonds $a_\delta a_{\delta+1}$ with $\delta = 1, 2, \dots, 2\theta_3 - 1$, are described as:

$$p(a_\delta a_{\delta+1}|F_m) = \begin{cases} (\delta - 1, 2\theta_3 - 1 - \delta, 2(\theta_3 + \theta_1) - 1 - \delta), & \text{if } \delta = 1, 2, \dots, 2\theta_3 - 2; \\ (\delta - 1, 2\theta_3 - 1 - \delta, 2\theta_3), & \text{if } \delta = 2\theta_3 - 1. \end{cases}$$

Positions $p(b_\delta b_{\delta+1}|F_m)$ with regard to F_m , for the bonds $b_\delta b_{\delta+1}$ with $\delta = 1, 2, \dots, 2\theta_1 - 1$, are described as:

$$p(b_\delta b_{\delta+1}|F_m) = (2\theta_3 + \delta - 1, \delta, 2\theta_1 - 1 - \delta).$$

Positions $p(c_\delta c_{\delta+1}|F_m)$ with regard to F_m , for the bonds $c_\delta c_{\delta+1}$ with $\delta = 1, 2, \dots, 2\theta_2 - 2$, are described as:

$$p(c_\delta c_{\delta+1}|F_m) = \begin{cases} (2\theta_3, \delta + 2, 2\theta_1 - 2), & \text{if } \delta = 1; \\ (2\theta_3, \delta + 2, 2(\theta_1 - 2) + \delta), & \text{if } \delta = 2; \\ (2\theta_3 + \delta - 3, \delta + 2, 2(\theta_1 - 2) + \delta), & \text{if } \delta = 3, 4, \dots, 2\theta_2 - 1. \end{cases}$$

Positions $p(a'_\delta a'_{\delta+1}|F_m)$ with regard to F_m , for the bonds $a'_\delta a'_{\delta+1}$ with $\delta = 1, 2, \dots, 2\theta_3 - 4$, are described as:

$$p(a'_\delta a'_{\delta+1}|F_m) = (\delta, 2\theta_3 - \delta, 2(\theta_3 + \theta_1) - 1 - \delta).$$

Positions $p(b'_\delta b'_{\delta+1}|F_m)$ with regard to F_m , for the bonds $b'_\delta b'_{\delta+1}$ with $\delta = 1, 2, \dots, 2\theta_1 - 4$, are described as:

$$p(b'_\delta b'_{\delta+1}|F_m) = (2\theta_3 + \delta + 1, \delta + 4, 2\theta_1 - 3 - \delta).$$

Positions $p(c'_\delta c'_{\delta+1}|F_m)$ with regard to F_m , for the bonds $c'_\delta c'_{\delta+1}$ with $\delta = 1, 2, \dots, 2\theta_2 - 2$, are described as:

$$p(c'_\delta c'_{\delta+1}|F_m) = \begin{cases} (2(\theta_3 - 1), \delta + 1, 2\theta_1), & \text{if } \delta = 1; \\ (2(\theta_3 - 1), \delta + 1, 2\theta_1), & \text{if } \delta = 2; \\ (2(\theta_3 - 2) + \delta, \delta + 1, 2\theta_1 - 3 + \delta), & \text{if } \delta = 3, 4, \dots, 2\theta_2 - 1. \end{cases}$$

Positions $p(a_\delta a'_\delta|F_m)$ with regard to F_m , for the bonds $a_\delta a'_\delta$ with $\delta = 1, 3, \dots, 2\theta_3 - 3$, are described as:

$$p(a_\delta a'_\delta|F_m) = (\delta - 1, 2\theta_3 - \delta, 2(\theta_3 + \theta_1) - 1 - \delta).$$

Positions $p(b_{\delta+3} b'_\delta|F_m)$ with regard to F_m , for the bonds $b_{\delta+3} b'_\delta$ with $\delta = 1, 3, \dots, 2\theta_1 - 3$, are described as:

$$p(b_{\delta+3} b'_\delta|F_m) = (2\theta_3 + \delta + 1, \delta + 3, 2\theta_1 - 3 - \delta).$$

Positions $p(c_\delta c'_\delta|F_m)$ with regard to F_m , for the bonds $c_\delta c'_\delta$ with $\delta = 1, 3, \dots, 2\theta_2 - 3$, are described as:

$$p(c_\delta c'_\delta | F_m) = \begin{cases} (2\theta_3 - 1, \delta + 1, 2\theta_1 - 1), & \text{if } \delta = 1; \\ (2(\theta_3 - 2) + \delta, \delta + 1, 2(\theta_1 - 2) + \delta), & \text{if } \delta = 2, 3, \dots, 2\theta_2 - 1. \end{cases}$$

Positions of the joint edges are described as:

$$\begin{aligned} p(a_{2\theta_3} b_1 | F_m) &= (2\theta_3 - 1, 0, 2\theta_1 - 1), \\ p(a_{2\theta_3} c'_1 | F_m) &= (2\theta_3 - 2, 1, 2\theta_1), \\ p(b_2 c_1 | F_m) &= (2\theta_3, 2, 2\theta_1 - 2), \\ p(a'_{2\theta_3-3} c'_2 | F_m) &= (2\theta_3 - 3, 3, 2\theta_1 + 1), \\ p(b'_1 c_2 | F_m) &= (2\theta_3 + 1, 4, 2\theta_1 - 3). \end{aligned}$$

By the given positions $p(\cdot | F_m)$ of all $4(\theta_1 + \theta_2 + \theta_3) - 8$ -nodes and $5(\theta_1 + \theta_2 + \theta_3) - 11$ -edges of $T(\theta_1, \theta_2, \theta_3)$ graph of benzenoid tripod with any of $\theta_1, \theta_2, \theta_3 = 2$, with respect to F_m , are unique and no two nodes, no two edges and not a single edge with a vertex have same position p . As a result, we may infer that the nodes and edges of the $T(\theta_1, \theta_2, \theta_3)$ graph have been resolved with only three nodes. So it is concluded that the members with minimality conditions of mixed metric resolving set of $T(\theta_1, \theta_2, \theta_3)$ are three. \square

Remark 1. If $T(\theta_1, \theta_2, \theta_3)$ is a of benzenoid tripod graph with any of $\theta_1, \theta_2, \theta_3 = 2$, then

$$\dim_m(T(\theta_1, \theta_2, \theta_3)) = 3.$$

Proof. As based on the definition of mixed metric dimension, the concept is solemnly based on the selected subset (F_m) chooses in a manner that each edge and vertex have distinct location in regards to the chosen vertices in subset. The Lemma 2.1 shows and proved the chosen subset for the mixed metric resolving set and their hardness to be chosen with the condition of minimality. We proved in that lemma the subset $F_m = \{a_1, a_{2\theta_3}, b_{2\theta_1}\}$ is one of the candidate for the mixed metric resolving set for a particular benzenoid tripod or $T(\theta_1, \theta_2, \theta_3)$ graph for all the possible combinatorial values of with any of $\theta_1, \theta_2, \theta_3 = 2$. Moreover, we discussed in the same lemma that $|F_m| = 3$ is fulfilling the minimality conditions on their members. It is impossible to get a subset with two members in a selected subset. It is sufficient to verify what we assert in the assertion that the mixed metric dimension of the benzenoid tripod is three, which concludes the proof. \square

Lemma 2.2. If $T(\theta_1, \theta_2, \theta_3)$ is a benzenoid tripod graph with $\theta_1, \theta_2, \theta_3 \geq 3$, then the mixed metric resolving set has minimum four members.

Proof. The order or the total amount of vertices in the benzenoid tripod's associated graph with any of $\theta_1, \theta_2, \theta_3 \geq 3$, are $4(\theta_1 + \theta_2 + \theta_3) - 8$, and to look over the likelihood of mixed metric resolvability set having four members in it, are $C(4(\theta_1 + \theta_2 + \theta_3) - 8, 4) = \frac{(4(\theta_1 + \theta_2 + \theta_3) - 8)!}{2 \times (4(\theta_1 + \theta_2 + \theta_3) - 12)!}$. Over here we are focusing on four members in the set, for the three cardinality of mixed resolving set we will discuss in next part of this proof. Now because of its computational cost of selecting mixed metric resolving set, for a graph, we can't discover the precise number of mixed resolving sets, but from $\frac{(4(\theta_1 + \theta_2 + \theta_3) - 8)!}{2 \times (4(\theta_1 + \theta_2 + \theta_3) - 12)!}$ possibilities we selected the subset F_m and described as: $F_m = \{a_1, a_{2\theta_3}, b_{2\theta_1}, c_{2\theta_2-1}\}$. Proving of this claim that F_m is also a subset having ability to be the candidate of mixed metric resolving set of benzenoid tripod graph or $T(\theta_1, \theta_2, \theta_3)$, we will stick to 1.2's definition. We will also look over the distinct placements or locations of each node and edge to meet the definition's requirements, and the approach is explained above in Definition 1.2.

Positions $p(a_\delta | F_m)$ with respect to F_m , for the nodes a_δ with $\delta = 1, 2, \dots, 2\theta_3$, are described as:

$$p(a_\delta | F_m) = \begin{cases} (\delta - 1, 2\theta_3 - \delta, 2(\theta_3 + \theta_1) - \delta, 2(\theta_3 + \theta_2) - 3 - \delta), & \text{if } \delta = 1, 2, \dots, 2\theta_3 - 3; \\ (\delta - 1, 2\theta_3 - \delta, 2(\theta_3 + \theta_1) - \delta, 2\theta_2 + 1), & \text{if } \delta = 2\theta_3 - 2; \\ (\delta - 1, 2\theta_3 - \delta, 2(\theta_3 + \theta_1) - \delta, 2\theta_2), & \text{if } \delta = 2\theta_3 - 1; \\ (\delta - 1, 2\theta_3 - \delta, 2(\theta_3 + \theta_1) - \delta, 2\theta_2 + 1), & \text{if } \delta = 2\theta_3. \end{cases}$$

Positions $p(b_\delta | F_m)$ with respect to F_m , for the nodes b_δ with $\delta = 1, 2, \dots, 2\theta_1$, are described as:

$$p(b_\delta | F_m) = \begin{cases} (2\theta_3 + \delta - 1, \delta, 2\theta_1 - \delta, 2\theta_2 + 1 - \delta), & \text{if } \delta = 1, 2; \\ (2\theta_3 + \delta - 1, \delta, 2\theta_1 - \delta, 2\theta_2), & \text{if } \delta = 3; \\ (2\theta_3 + \delta - 1, \delta, 2\theta_1 - \delta, 2\theta_2 - 5 + \delta), & \text{if } \delta = 4, 5, \dots, 2\theta_1. \end{cases}$$

Positions $p(c_\delta | F_m)$ with respect to F_m , for the nodes c_δ with $\delta = 1, 2, \dots, 2\theta_2 - 1$, are described as:

$$p(c_\delta | F_m) = \begin{cases} (2\theta_3 + \delta - 1, \delta + 2, 2\theta_1 - 1, 2\theta_2 - \delta - 1), & \text{if } \delta = 1; \\ (2\theta_3 + \delta - 1, \delta + 2, 2(\theta_1 - 2) + \delta, 2\theta_2 - \delta - 1), & \text{if } \delta = 2; \\ (2\theta_3 + \delta - 3, \delta + 2, 2(\theta_1 - 2) + \delta, 2\theta_2 - \delta - 1), & \text{if } \delta = 3, 4, \dots, 2\theta_2 - 1. \end{cases}$$

Positions $p(a'_\delta | F_m)$ with respect to F_m , for the nodes a'_δ with $\delta = 1, 2, \dots, 2\theta_3 - 3$, are described as:

$$p(a'_\delta | F_m) = (\delta, 2\theta_3 + 1 - \delta, 2(\theta_3 + \theta_1) - 1 - \delta, 2(\theta_3 + \theta_1 - 2) - \delta).$$

Positions $p(b'_\delta | F_m)$ with respect to F_m , for the nodes b'_δ with $\delta = 1, 2, \dots, 2\theta_1 - 3$, are described as:

$$p(b'_\delta | F_m) = (2\theta_3 + \delta + 1, \delta + 4, 2(\theta_1 - 1) - \delta, 2\theta_2 - 3 + \delta).$$

Positions $p(c'_\delta | F_m)$ with respect to F_m , for the nodes c'_δ with $\delta = 1, 2, \dots, 2\theta_2 - 1$, are described as:

$$p(c'_\delta | F_m) = \begin{cases} (2\theta_3 - 1, \delta + 1, 2\theta_1 - 1 + \delta, 2\theta_2 - \delta), & \text{if } \delta = 1; \\ (2(\theta_3 - 2) + \delta, \delta + 1, 2\theta_1 - 1 + \delta, 2\theta_2 - \delta), & \text{if } \delta = 2; \\ (2(\theta_3 - 2) + \delta, \delta + 1, 2\theta_1 - 3 + \delta, 2\theta_2 - \delta), & \text{if } \delta = 3, 4, \dots, 2\theta_2 - 1. \end{cases}$$

Given above are the positions of nodes only, with respect to the chosen vertices in F_m , for the fulfillment of definition of mixed resolving set we still need to investigate the edges or bonds of graph. Following are positions of bonds that will complete the definition.

Positions $p(a_\delta a_{\delta+1} | F_m)$ with respect to F_m , for the bonds $a_\delta a_{\delta+1}$ with $\delta = 1, 2, \dots, 2\theta_3 - 1$, are described as:

$$p(a_\delta a_{\delta+1} | F_m) = \begin{cases} (\delta - 1, 2\theta_3 - 1 - \delta, 2(\theta_3 + \theta_1) - 1 - \delta, 2(\theta_3 + \theta_2 - 2) - \delta), & \\ \text{if } \delta = 1, 2, \dots, 2\theta_3 - 4; \\ (\delta - 1, 2\theta_3 - 1 - \delta, 2(\theta_3 + \theta_1) - 1 - \delta, 2\theta_2), & \\ \text{if } \delta = 2\theta_3 - 3, 2\theta_3 - 2; \\ (\delta - 1, 2\theta_3 - 1 - \delta, 2\theta_3, 2\theta_2), & \\ \text{if } \delta = 2\theta_3 - 1. \end{cases}$$

Positions $p(b_\delta b_{\delta+1} | F_m)$ in regards to F_m , for the bonds $b_\delta b_{\delta+1}$ with $\delta = 1, 2, \dots, 2\theta_1 - 1$, are described as:

$$p(b_\delta b_{\delta+1} | F_m) = \begin{cases} (2\theta_3 + \delta - 1, \delta, 2\theta_1 - 1 - \delta, 2\theta_2 - 1), & \text{if } \delta = 1, 2, 3; \\ (2\theta_3 + \delta - 1, \delta, 2\theta_1 - 1 - \delta, 2\theta_2 - 5 + \delta), & \text{if } \delta = 4, 5, \dots, 2\theta_1 - 1. \end{cases}$$

Positions $p(c_\delta c_{\delta+1} | F_m)$ in regards to F_m , for the bonds $c_\delta c_{\delta+1}$ with $\delta = 1, 2, \dots, 2\theta_2 - 2$, are described as:

$$p(c_\delta c_{\delta+1} | F_m) = \begin{cases} (2\theta_3, \delta + 2, 2\theta_1 - 2, 2\theta_2 - 2 - \delta), & \text{if } \delta = 1; \\ (2\theta_3, \delta + 2, 2(\theta_1 - 2) + \delta, 2\theta_2 - 2 - \delta), & \text{if } \delta = 2; \\ (2\theta_3 + \delta - 3, \delta + 2, 2(\theta_1 - 2) + \delta, 2\theta_2 - 2 - \delta), & \text{if } \delta = 3, 4, \dots, 2\theta_2 - 1. \end{cases}$$

Positions $p(a'_\delta a'_{\delta+1} | F_m)$ in regards to F_m , for the bonds $a'_\delta a'_{\delta+1}$ with $\delta = 1, 2, \dots, 2\theta_3 - 4$, are described as:

$$p(a'_\delta a'_{\delta+1} | F_m) = (\delta, 2\theta_3 - \delta, 2(\theta_3 + \theta_1 - 1) - \delta, 2(\theta_3 + \theta_1) - 5 - \delta).$$

Positions $p(b'_\delta b'_{\delta+1} | F_m)$ in regards to F_m , for the bonds $b'_\delta b'_{\delta+1}$ with $\delta = 1, 2, \dots, 2\theta_1 - 4$, are described as:

$$p(b'_\delta b'_{\delta+1}|F_m) = (2\theta_3 + \delta + 1, \delta + 4, 2\theta_1 - 3 - \delta, 2\theta_2 - 3 + \delta).$$

Positions $p(c'_\delta c'_{\delta+1}|F_m)$ in regards to F_m , for the bonds $c'_\delta c'_{\delta+1}$ with $\delta = 1, 2, \dots, 2\theta_2 - 2$, are described as:

$$p(c'_\delta c'_{\delta+1}|F_m) = \begin{cases} (2(\theta_3 - 1), \delta + 1, 2\theta_1, 2\theta_2 - 3 + \delta), & \text{if } \delta = 1; \\ (2(\theta_3 - 1), \delta + 1, 2\theta_1, 2\theta_2 - 3 + \delta), & \text{if } \delta = 2; \\ (2(\theta_3 - 2) + \delta, \delta + 1, 2\theta_1 - 3 + \delta, 2\theta_2 - 3 + \delta), & \text{if } \delta = 3, 4, \dots, 2\theta_2 - 1. \end{cases}$$

Positions $p(a'_\delta a'_\delta|F_m)$ in regards to F_m , for the bonds $a'_\delta a'_\delta$ with $\delta = 1, 3, \dots, 2\theta_3 - 3$, are described as:

$$p(a'_\delta a'_\delta|F_m) = (\delta - 1, 2\theta_3 - \delta, 2(\theta_3 + \theta_1) - 1 - \delta, 2(\theta_3 + \theta_2 - 2) - \delta).$$

Positions $p(b_{\delta+3} b'_\delta|F_m)$ in regards to F_m , for the bonds $b_{\delta+3} b'_\delta$ with $\delta = 1, 3, \dots, 2\theta_1 - 3$, are described as:

$$p(b_{\delta+3} b'_\delta|F_m) = (2\theta_3 + \delta + 1, \delta + 3, 2\theta_1 - 3 - \delta, 2\theta_2 - 3 + \delta).$$

Positions $p(c_\delta a'_\delta|F_m)$ in regards to F_m , for the bonds $c_\delta a'_\delta$ with $\delta = 1, 3, \dots, 2\theta_2 - 3$, are described as:

$$p(c_\delta a'_\delta|F_m) = \begin{cases} (2\theta_3 - 1, \delta + 1, 2\theta_1 - 1, 2\theta_2 - 1 - \delta), & \text{if } \delta = 1; \\ (2(\theta_3 - 2) + \delta, \delta + 1, 2(\theta_1 - 2) + \delta, 2\theta_2 - 1 - \delta), & \text{if } \delta = 2, 3, \dots, 2\theta_2 - 1. \end{cases}$$

Positions of the joint bonds are described as:

$$\begin{aligned} p(a_{2\theta_3} b_1|F_m) &= (2\theta_3 - 1, 0, 2\theta_1 - 1, 2\theta_2), \\ p(a_{2\theta_3} c'_1|F_m) &= (2\theta_3 - 2, 1, 2\theta_1, 2\theta_2 - 1), \\ p(b_2 c_1|F_m) &= (2\theta_3, 2, 2\theta_1 - 2, 2\theta_2 - 2), \\ p(a'_{2\theta_3-3} c'_2|F_m) &= (2\theta_3 - 3, 3, 2\theta_1 + 1, 2\theta_2 - 2), \\ p(b'_1 c_2|F_m) &= (2\theta_3 + 1, 4, 2\theta_1 - 3, 2\theta_2 - 3). \end{aligned}$$

In support of a particular fact in order to meet the definition's requirements of mixed metric resolving set, we can say that F_m having four members in it is achievable, however when we discuss the optimized value of $|F_m|$, we still have to look at the least number for $|F_m|$. The given bellow are some scenarios to consider while determining whether $|F_m| = 3$ is achievable or not. Though we have found the set of mixed metric resolvability by doing with the help of algorithm and satisfied that $|F_m| \neq 3$, but for the proving purpose we build some general cases and try to conclude that only $|F_m| > 3$ is possible.

Case1: Assume that $F'_m \subset \{a_\delta : \delta = 1, 2, \dots, 2\theta_3\}$, considering the constraint that, according to the demand of our theorem $|F'_m| = 3$. In the same vertex's position and/or edge's position, the outcome is inferred and our supposition is contradicted by the fact that $p(a_r a_{r+1}|F'_m) = p(a'_s a'_s|F'_m)$, where $1 \leq r \leq 2\theta_3 - 1$ and $1 \leq s(\text{odd}) \leq 2\theta_3 - 3$. In the vertex's position the similarities are in $p(a_u|F'_m) = p(c'_v|F'_m)$, where $1 \leq u \leq 2\theta_3$ and $1 \leq v \leq 2\theta_2 - 1$.

Case2: Assume that $F'_m \subset \{b_\delta : \delta = 1, 2, \dots, 2\theta_1\}$, considering the constraint that, according to the demand of our theorem $|F'_m| = 3$. In the same vertex's position and/or edge's position, the outcome is inferred and our supposition is contradicted by the fact that $p(a_r a_{r+1}|F'_m) = p(a'_s a'_{s+1}|F'_m)$, where $1 \leq r \leq 2\theta_3 - 1$ and $1 \leq s \leq 2\theta_3 - 4$. In the vertex's position the similarities are in $p(a_u|F'_m) = p(a'_v|F'_m)$, where $1 \leq u \leq 2\theta_3$ and $1 \leq v \leq 2\theta_3 - 3$.

Case3: Assume that $F'_m \subset \{c_\delta : \delta = 1, 2, \dots, 2\theta_2 - 1\}$, considering the constraint that, according to the demand of our theorem $|F'_m| = 3$. In the same vertex's position and/or edge's position, the outcome is inferred and our supposition is contradicted by the fact that $p(b_{r+3} b'_r|F'_m) = p(a'_s a'_{s+1}|F'_m)$, where $1 \leq r(\text{odd}) \leq 2\theta_1 - 3$ and $1 \leq s \leq 2\theta_3 - 1$. In the vertex's position the similarities are in $p(a_u|F'_m) = p(b'_v|F'_m)$, where $1 \leq u \leq 2\theta_3$ and $1 \leq v \leq 2\theta_1$.

Case4: Assume that $F'_m \subset \{a_{\delta'} : \delta = 1, 2, \dots, 2\theta_3 - 3\}$, considering the constraint that, according to the demand of our theorem $|F'_m| = 3$. In the same vertex's position and/or edge's position, the outcome is inferred and our supposition is contradicted by the fact

that $p(a_r a_{r+1}|F'_m) = p(c'_s c'_s|F'_m)$, where $1 \leq r \leq 2\theta_3 - 1$ and $1 \leq s(\text{odd}) \leq 2\theta_2 - 1$. In the vertex's position the similarities are in $p(a_u|F'_m) = p(c'_v|F'_m)$, where $1 \leq u \leq 2\theta_3$ and $1 \leq v \leq 2\theta_2 - 1$.

Case5: Assume that $F'_m \subset \{b_{\delta'} : \delta = 1, 2, \dots, 2\theta_1 - 3\}$, considering the constraint that, according to the demand of our theorem $|F'_m| = 3$. In the same vertex's position and/or edge's position, the outcome is inferred and our supposition is contradicted by the fact that $p(a_r a_{r+1}|F'_m) = p(a'_s a'_{s+1}|F'_m)$, where $1 \leq r \leq 2\theta_3 - 1$ and $1 \leq s \leq 2\theta_3 - 4$. In the vertex's position the similarities are in $p(a_u|F'_m) = p(c'_v|F'_m)$, where $1 \leq u \leq 2\theta_3$ and $1 \leq v \leq 2\theta_2 - 1$.

Case6: Assume that $F'_m \subset \{c_{\delta'} : \delta = 1, 2, \dots, 2\theta_2 - 1\}$, considering the constraint that, according to the demand of our theorem $|F'_m| = 3$. In the same vertex's position and/or edge's position, the outcome is inferred and our supposition is contradicted by the fact that $p(a_r a_{r+1}|F'_m) = p(a'_s a'_{s+1}|F'_m)$, where $1 \leq r, s \leq 2\theta_3 - 1$. In the vertex's position the similarities are in $p(a_u|F'_m) = p(b'_v|F'_m)$, where $1 \leq u \leq 2\theta_3$ and $1 \leq v \leq 2\theta_1$.

Case7: Assume that $F'_m \subset \{a_\delta, b_j : \delta = 1, 2, \dots, 2\theta_3, j = 1, 2, \dots, 2\theta_1\}$, considering the constraint that, according to the demand of our theorem $|F'_m| = 3$. In the same vertex's position and/or edge's position, the outcome is inferred and our supposition is contradicted by the fact that $p(b_r b_{r+1}|F'_m) = p(b'_s b'_{s+1}|F'_m)$, where $1 \leq r \leq 2\theta_1 - 1$ and $1 \leq s \leq 2\theta_1 - 4$. In the vertex's position the similarities are in $p(a_u|F'_m) = p(c'_v|F'_m)$, where $1 \leq u \leq 2\theta_3$ and $1 \leq v \leq 2\theta_2 - 1$.

Case8: Assume that $F'_m \subset \{a_\delta, c_j : \delta = 1, 2, \dots, 2\theta_3, j = 1, 2, \dots, 2\theta_2 - 1\}$, considering the constraint that, according to the demand of our theorem $|F'_m| = 3$. In the same vertex's position and/or edge's position, the outcome is inferred and our supposition is contradicted by the fact that $p(b_{r+3} b'_r|F'_m) = p(b'_s b'_{s+1}|F'_m)$, where $1 \leq r(\text{odd}) \leq 2\theta_1 - 3$ and $1 \leq s \leq 2\theta_1 - 4$. In the vertex's position the similarities are in $p(a_u|F'_m) = p(c'_v|F'_m)$, where $1 \leq u \leq 2\theta_3$ and $1 \leq v \leq 2\theta_2 - 1$.

Case9: Assume that $F'_m \subset \{a_\delta, a'_j : \delta = 1, 2, \dots, 2\theta_3, j = 1, 2, \dots, 2\theta_3 - 3\}$, considering the constraint that, according to the demand of our theorem $|F'_m| = 3$. In the same vertex's position and/or edge's position, the outcome is inferred and our supposition is contradicted by the fact that $p(b_r b_{r+1}|F'_m) = p(b_{s+3} b'_r|F'_m)$, where $1 \leq r \leq 2\theta_1 - 1$ and $1 \leq s(\text{odd}) \leq 2\theta_1 - 3$. In the vertex's position the similarities are in $p(b_u|F'_m) = p(b'_v|F'_m)$, where $1 \leq u \leq 2\theta_1 - 1$ and $1 \leq v \leq 2\theta_1 - 3$.

Case10: Assume that $F'_m \subset \{a_\delta, b'_j : \delta = 1, 2, \dots, 2\theta_3, j = 1, 2, \dots, 2\theta_1 - 3\}$, considering the constraint that, according to the demand of our theorem $|F'_m| = 3$. In the same vertex's position and/or edge's position, the outcome is inferred and our supposition is contradicted by the fact that $p(a_r a_{r+1}|F'_m) = p(a'_s a'_s|F'_m)$, where $1 \leq r \leq 2\theta_3 - 1$ and $1 \leq s(\text{odd}) \leq 2\theta_3 - 3$. In the vertex's position the similarities are in $p(a_u|F'_m) = p(a'_v|F'_m)$, where $1 \leq u \leq 2\theta_3$ and $1 \leq v \leq 2\theta_3 - 3$.

Case11: Assume that $F'_m \subset \{a_\delta, c'_j : \delta = 1, 2, \dots, 2\theta_3, j = 1, 2, \dots, 2\theta_2 - 1\}$, considering the constraint that, according to the demand of our theorem $|F'_m| = 3$. In the same vertex's position and/or edge's position, the outcome is inferred and our supposition is contradicted by the fact that $p(b_r b_{r+1}|F'_m) = p(b'_s b'_{s+1}|F'_m)$, where $1 \leq r \leq 2\theta_1 - 1$ and $1 \leq s \leq 2\theta_1 - 4$. In the vertex's position the similarities are in $p(b_u|F'_m) = p(b'_v|F'_m)$, where $1 \leq u \leq 2\theta_1$ and $1 \leq v \leq 2\theta_1 - 3$.

Case12: Assume that $F'_m \subset \{b_\delta, c_j : \delta = 1, 2, \dots, 2\theta_1, j = 1, 2, \dots, 2\theta_2 - 1\}$, considering the constraint that, according to the demand of our theorem $|F'_m| = 3$. In the same vertex's position and/or edge's position, the outcome is inferred and our supposition is

contradicted by the fact that $p(a_r a_{r+1} | F'_m) = p(a_s a'_s | F'_m)$, where $1 \leq r \leq 2\theta_3 - 1$ and $1 \leq s(\text{odd}) \leq 2\theta_3 - 3$. In the vertex's position the similarities are in $p(a'_u | F'_m) = p(b'_v | F'_m)$, where $1 \leq u \leq 2\theta_3 - 3$ and $1 \leq v \leq 2\theta_1 - 3$.

Case13: Assume that $F'_m \subset \{b_\delta, a'_j : \delta = 1, 2, \dots, 2\theta_1, j = 1, 2, \dots, 2\theta_3 - 3\}$, considering the constraint that, according to the demand of our theorem $|F'_m| = 3$. In the same vertex's position and/or edge's position, the outcome is inferred and our supposition is contradicted by the fact that $p(a_r a_{r+1} | F'_m) = p(a_s a'_s | F'_m)$, where $1 \leq r \leq 2\theta_3 - 1$ and $1 \leq s(\text{odd}) \leq 2\theta_3 - 3$. In the vertex's position the similarities are in $p(a'_u | F'_m) = p(c'_v | F'_m)$, where $1 \leq u \leq 2\theta_3 - 3$ and $1 \leq v \leq 2\theta_2 - 1$.

Case14: Assume that $F'_m \subset \{b_\delta, b'_j : \delta = 1, 2, \dots, 2\theta_1, j = 1, 2, \dots, 2\theta_3 - 3\}$, considering the constraint that, according to the demand of our theorem $|F'_m| = 3$. In the same vertex's position and/or edge's position, the outcome is inferred and our supposition is contradicted by the fact that $p(a_r a_{r+1} | F'_m) = p(a_s a'_s | F'_m)$, where $1 \leq r \leq 2\theta_3 - 1$ and $1 \leq s(\text{odd}) \leq 2\theta_3 - 3$. In the vertex's position the similarities are in $p(b_u | F'_m) = p(b'_v | F'_m)$, where $1 \leq u \leq 2\theta_1$ and $1 \leq v \leq 2\theta_1 - 3$.

Case15: Assume that $F'_m \subset \{b_\delta, c'_j : \delta = 1, 2, \dots, 2\theta_1, j = 1, 2, \dots, 2\theta_2 - 1\}$, considering the constraint that, according to the demand of our theorem $|F'_m| = 3$. In the same vertex's position and/or edge's position, the outcome is inferred and our supposition is contradicted by the fact that $p(a_r a_{r+1} | F'_m) = p(a'_s a'_{s+1} | F'_m)$, where $1 \leq r \leq 2\theta_3 - 1$ and $1 \leq s \leq 2\theta_3 - 4$. In the vertex's position the similarities are in $p(c_u | F'_m) = p(c'_v | F'_m)$, where $1 \leq u, v \leq 2\theta_2 - 1$.

Case16: Assume that $F'_m \subset \{c_\delta, a'_j : \delta = 1, 2, \dots, 2\theta_2 - 1, j = 1, 2, \dots, 2\theta_3 - 3\}$, considering the constraint that, according to the demand of our theorem $|F'_m| = 3$. In the same vertex's position and/or edge's position, the outcome is inferred and our supposition is contradicted by the fact that $p(b_r b_{r+1} | F'_m) = p(b'_s b'_{s+1} | F'_m)$, where $1 \leq r \leq 2\theta_1 - 1$ and $1 \leq s \leq 2\theta_1 - 4$. In the vertex's position the similarities are in $p(b_u | F'_m) = p(b_v | F'_m)$, where $1 \leq u, v \leq 2\theta_1$.

Case17: Assume that $F'_m \subset \{c_\delta, b'_j : \delta = 1, 2, \dots, 2\theta_2 - 1, j = 1, 2, \dots, 2\theta_1 - 3\}$, considering the constraint that, according to the demand of our theorem $|F'_m| = 3$. In the same vertex's position and/or edge's position, the outcome is inferred and our supposition is contradicted by the fact that $p(a_r a_{r+1} | F'_m) = p(a'_s a'_{s+1} | F'_m)$, where $1 \leq r \leq 2\theta_3 - 1$ and $1 \leq s \leq 2\theta_3 - 4$. In the vertex's position the similarities are in $p(a_u | F'_m) = p(a'_v | F'_m)$, where $1 \leq u \leq 2\theta_3$ and $1 \leq v \leq 2\theta_3 - 3$.

Case18: Assume that $F'_m \subset \{c_\delta, c'_j : \delta, j = 1, 2, \dots, 2\theta_2 - 1\}$, considering the constraint that, according to the demand of our theorem $|F'_m| = 3$. In the same vertex's position and/or edge's position, the outcome is inferred and our supposition is contradicted by the fact that $p(b_r b_{r+1} | F'_m) = p(b_s b_{s+1} | F'_m)$, where $1 \leq r, s \leq 2\theta_1 - 1$. In the vertex's position the similarities are in $p(b_u | F'_m) = p(b'_v | F'_m)$, where $1 \leq u \leq 2\theta_1$ and $1 \leq v \leq 2\theta_1 - 3$.

Case19: Assume that $F'_m \subset \{a'_\delta, b'_j : \delta = 1, 2, \dots, 2\theta_3 - 3, j = 1, 2, \dots, 2\theta_1 - 3\}$, considering the constraint that, according to the demand of our theorem $|F'_m| = 3$. In the same vertex's position and/or edge's position, the outcome is inferred and our supposition is contradicted by the fact that $p(b_r b_{r+1} | F'_m) = p(b'_s b'_{s+1} | F'_m)$, where $1 \leq r \leq 2\theta_1 - 1, 1 \leq s \leq 2\theta_1 - 4$. In the vertex's position the similarities are in $p(a_u | F'_m) = p(a'_v | F'_m)$, where $1 \leq u \leq 2\theta_3$ and $1 \leq v \leq 2\theta_3 - 3$.

Case20: Assume that $F'_m \subset \{a'_\delta, c'_j : \delta = 1, 2, \dots, 2\theta_3 - 3, j = 1, 2, \dots, 2\theta_2 - 1\}$, considering the constraint that, according to the demand of our theorem $|F'_m| = 3$. In the same vertex's position and/or

edge's position, the outcome is inferred and our supposition is contradicted by the fact that $p(b_r b_{r+1} | F'_m) = p(b'_s b'_{s+1} | F'_m)$, where $1 \leq r \leq 2\theta_1 - 1, 1 \leq s \leq 2\theta_1 - 4$. In the vertex's position the similarities are in $p(b_u | F'_m) = p(b_v | F'_m)$, where $1 \leq u, v \leq 2\theta_1$.

Case21: Assume that $F'_m \subset \{b'_\delta, c'_j : \delta = 1, 2, \dots, 2\theta_1 - 3, j = 1, 2, \dots, 2\theta_2 - 1\}$, considering the constraint that, according to the demand of our theorem $|F'_m| = 3$. In the same vertex's position and/or edge's position, the outcome is inferred and our supposition is contradicted by the fact that $p(a_r a_{r+1} | F'_m) = p(a'_s a'_{s+1} | F'_m)$, where $1 \leq r, s \leq 2\theta_1$. In the vertex's position the similarities are in $p(c_u | F'_m) = p(c'_v | F'_m)$, where $1 \leq u, v \leq 2\theta_2 - 1$.

By the given positions $p(\cdot | F_m)$ of all $4(\theta_1 + \theta_2 + \theta_3) - 8$ -nodes and $5(\theta_1 + \theta_2 + \theta_3) - 11$ -edges of $T(\theta_1, \theta_2, \theta_3)$ graph of benzenoid tripod with $\theta_1, \theta_2, \theta_3 \geq 3$, in regards to F_m , are unique and no two nodes, no two edges and not a single edge with a vertex have same position p . As a result, we may infer that the nodes and edges of $T(\theta_1, \theta_2, \theta_3)$ have been resolved with four vertices. We also checked that the mixed resolving set F_m with $|F_m| = 3$ are resulted in with either two edges, two vertices or a single edge with a vertex have same position p . As a result, we may infer that the nodes and edges of $T(\theta_1, \theta_2, \theta_3)$ with four vertices. It is finalized that the subset of mixed metric resolvability has minimal members and the graph $T(\theta_1, \theta_2, \theta_3)$ has cardinal mixed metric dimension. \square

Remark 2. If $T(\theta_1, \theta_2, \theta_3)$ is a benzenoid tripod graph with $\theta_1, \theta_2, \theta_3 \geq 3$, then

$$\dim_m(T(\theta_1, \theta_2, \theta_3)) = 4.$$

Proof. As based on the definition of mixed metric dimension, the concept is solemnly based on the selected subset (F_m) chooses in a manner that each edge and vertex have distinct location in regards to the chosen vertices in subset. The Lemma 2.2 shows and proved the chosen subset for the mixed metric resolving set and their hardness to be chosen with the condition of minimality. We proved in that lemma the subset $F_m = \{a_1, a_{2\theta_3}, b_{2\theta_1}, c_{2\theta_2-1}\}$ is one of the candidate for the mixed metric resolving set for a particular benzenoid tripod or $T(\theta_1, \theta_2, \theta_3)$ graph for all the possible combinatorial values of with any of $\theta_1, \theta_2, \theta_3 \geq 3$. Moreover, we discussed in the same lemma that $|F_m| = 4$ is fulfilling the minimality conditions on their members. It is impossible to get a subset with three members in a selected subset. It is sufficient to verify what we assert in the assertion that the mixed metric dimension of the benzenoid tripod is three, which concludes the proof. \square

3. Conclusion

The study of intricate chemical networks and structures in their most fundamental forms has become easier thanks to only the field of mathematical chemistry, particularly graph theoretical chemistry. Resolvability is a parameter that controls how the whole node and edge set reconfigures itself into distinct ensembles to called or accessed them. A parameter having this feature is called as the mixed metric dimension, which transforms each node and edge of a structure into a unique shape. In this paper, we look at the benzenoid tripod structure in order to find the smallest edge and node combined resolving set. For this structure, we determined that the node and edge combined resolving set has a constant and precise number of members. In the scope of future work, this work can be done for any regular polygons instead of hexagons. This direction will be quite interesting and this help for readers.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

- Ahmad, Ali, Husain, Sadia, Azeem, Muhammad, Elahi, Kashif, Siddiqui, M.K., 2021. Computation of edge resolvability of benzenoid tripod structure 2021, 1–8.
- Ali, A., Nazeer, W., Munir, M., Kang, S.M., 2018. M-polynomials and topological indices of zigzag and rhombic benzenoid systems. *Open Chem.* 16 (73–78), 122–135.
- Azeem, M., Nadeem, M.F., 2021. Metric-based resolvability of polycyclic aromatic hydrocarbons. *Eur. Phys. J. Plus* 136 (395).
- Chartrand, G., Eroh, L., Johnson, M.A.O., Ortrud, R., 2000. Resolvability in graphs and the metric dimension of a graph. *Discrete Appl. Math.* 105, 99–113.
- Chartrand, G., Salehi, E., Zhang, P., 2000. The partition dimension of graph. *Aequationes Math.* 59, 45–54.
- Chou, C.P., Li, Y., Witek, H.A., 2012. Zhang–zhang polynomials of various classes of benzenoid systems. *MATCH Commun. Math. Comput. Chem.* 68, 31–64.
- Deng, Bo, Nadeem, Muhammad Faisal, Azeem, Muhammad, mar 2021. On the edge metric dimension of different families of möbius networks. *Math. Probl. Eng.* 2021, 1–9.
- Harary, F., Melter, R.A., 1976. On the metric dimension of a graph. *Ars Comb.* 2, 191–195.
- Hauptmann, M., Schmied, R., Viehmann, C., 2012. Approximation complexity of metric dimension problem. *J. Discrete Algorithms* 14, 214–222.
- Hussain, Z., Munir, M., Choudhary, M., Kang, S.M., 2018. Computing metric dimension and metric basis of 2d lattice of alpha-boron nanotubes. *Symmetry* 10.
- Imran, S., Siddiqui, M.K., Hussain, M., 2019. Computing the upper bounds for the metric dimension of cellulose network. *Appl. Math. E-notes* 19, 585–605.
- Jamil, M.K., Imran, M., Sattar, K.A., 2020. Novel face index for benzenoid hydrocarbons. *Mathematics* 8 (312).
- Johnson, M.A., 1993. Structure-activity maps for visualizing the graph variables arising in drug design. *J. Biopharm. Stat.* 3, 203–236.
- Johnson, M.A., 1998. Browseable structure-activity datasets, advances in molecular similarity. JAI Press Connecticut, pp. 153–170.
- Kelenc, A., Tratnik, N., Yero, I.G., 2018. Uniquely identifying the edges of a graph: the edge metric dimension. *Discrete Appl. Math.* 251, 204–220.
- Kelenc, Aleksander, Kuziak, Dorota, Taranenko, Andrej, Yero, Ismael G., Dec 2017. Mixed metric dimension of graphs. *Appl. Math. Comput.* 314, 429–438.
- Khuller, S., Raghavachari, B., Rosenfeld, A., 1996. Landmarks in graphs. *Discrete Appl. Math.* 70 (3), 217–229.
- Koam, A.N.A., Ahmad, A., Abdelhag, M.E., Azeem, M., 2021. Metric and fault-tolerant metric dimension of hollow coronoid. *IEEE Access* 9, 81527–81534.
- Krishnan, S., Rajan, B., 2016. Fault-tolerant resolvability of certain crystal structures. *Appl. Math.* 7, 599–604.
- Lewis, H.R., Garey, M.R., Johnson, D.S., 1983. Computers and intractability: a guide to the theory of np-completeness. *J. Symb. Log.* 48 (2), 498–500.
- Manuel, P., Bharati, R., Rajasingh, I., Monica, M.C., 2008. On minimum metric dimension of honeycomb networks. *J. Discrete Algorithms* 6 (1), 20–27.
- Manzoor, S., Siddiqui, M.K., Ahmad, S., 2020. On entropy measures of polycyclic hydroxychloroquine used for novel coronavirus (covid-19) treatment. *Polycyclic Aromat. Compd.*
- Nadeem, M.F., Azeem, M., Khalil, A., 2020. The locating number of hexagonal möbius ladder network. *J. Appl. Math. Comput.*
- Perc, M., Gómez-Gardeñes, J., Szolnoki, A., Flórida, L.M., Moreno, Y., 2013. Evolutionary dynamics of group interactions on structured populations: a review. *J. R. Soc. Interface* 10 (80).
- Perc, M., Szolnoki, A., 2010. Coevolutionary games—a mini review. *Biosystems* 99 (2), 109–125.
- Raza, H., Ji, Y., 2020. Computing the mixed metric dimension of a generalized Petersen graph $p(n,2)$. *Front. Phys.* 8, 211.
- Raza, H., Ji, Y., Qu, S., 2020. On mixed metric dimension of some path related graphs. *IEEE Access* 8, 188146–188153.
- Raza, H., Liu, J.B., Qu, S., 2020. On mixed metric dimension of rotationally symmetric graphs. *IEEE Access* 8, 11560–11569.
- Sebö, A., Tannier, E., 2004. On metric generators of graphs. *Math. Oper. Res.* 29, 383–393.
- Siddiqui, H.M.A., Imran, M., 2014. Computing metric and partition dimension of 2-dimensional lattices of certain nanotubes. *J. Comput. Theor. Nanosci.* 11 (12), 2419–2423.
- Siddiqui, M.K., Imran, M., 2015. Computing the metric and partition dimension of h-naphthalenic and vc5c7 nanotubes. *J. Optoelectron. Adv. Mater.* 17, 790–794.
- Siddiqui, M.K., Naeem, M., Rahman, N.A., Imran, M., 2016. Computing topological indices of certain networks. *J. Optoelectron. Adv. Mater.* 18, 9–10.
- Simonraj, F., George, A., 2015. On the metric dimension of silicate stars. In: *ARPN J. Eng. Appl. Sci.*, pp. 2187–2192.
- Slater, P.J., 1975. Leaves of trees. In: *Proceeding of the 6th Southeastern Conference on Combinatorics, Graph Theory, and Computing, Congressus Numerantium*, vol. 14, pp. 549–559.
- Söderberg, S., Shapiro, H.S., 1963. A combinatorial detection problem. *Am. Math. Monthly* 70 (10), 1066–1070.
- Yang, B., Rafiullah, M., Siddiqui, H.M.A., Ahmad, S., 2019. On resolvability parameters of some wheel-related graphs. *J. Chem.* (1–9)