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ORIGINAL ARTICLE

Generalization of some overdetermined systems of complex partial differential equations

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Abstract In this paper, we explore new generalization for solution of overdetermined systems of complex partial differential equations. And relation between analytic functions and solutions of quasi-linear systems is discussed in the paper.

1. Introduction

Extension of some properties of analytic functions on several complex variables during the past century is the main goal for many a researcher, such as Vekua (1962) who studied the system

$$\frac{\partial W}{\partial \bar{z}} = A(z)W + B(z)\overline{W} \tag{1}$$

and extended many properties of functions analytic in $z \in C$ to generalized solutions of (1), under suitable hypotheses of coefficients $A(z)$ and $B(z)$. Therefore the generalized solutions of (1) are called generalized analytic functions (see Michailov, 1976; Gilbert and Buchanan, 1983; Hoermander, 1967; Koohara, 1971; Son, 1979). In this paper we consider the system

$$\frac{\partial W}{\partial \bar{z}_{js}} = A_{js}(z_{1s}, z_{2s}, \ldots, z_{ns})Ws, \quad j = 1, 2, \ldots, n \tag{2}$$

And some properties of the system (2) are considered, as the necessary and sufficient condition for the existence of a non-vanished solution, the Cauchy integral formula, the Principle of maximum modulus and Liouville's Theorem. It is proved that there exists a global "l-l" correspondence between the space of regular solutions of (2) and the space of analytic functions, while the same result in Tutschke (1977) is only locally.

2. Methodology

In this section at first we suppose that coefficients

$$A_{js}(z_{1s}, z_{2s}, \ldots, z_{ns}) = A_{js}(z_s) \in c^{\infty}(G); \quad s, j = 1, 2, \ldots, n$$

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where G is a domain of holomorphy in C^n.

Theorem 2.1. *The necessary and sufficient condition for the existence of a non-vanished solution of the system (2) is:*

$$\frac{\partial A_{js}}{\partial \bar{z}_{ks}} = \frac{\partial A_{ks}}{\partial \bar{z}_{js}}, \quad s,j,k = 1,2,\ldots,n \tag{3}$$

Proof. Let $W(z_{1s}, z_{2s}, \ldots, z_{ns}) = W(z_s) \in c^2(G)$ be a non-vanished solution of (2) then

$$\frac{\partial^2 W}{\partial \bar{z}_{js} \partial \bar{z}_{ks}} = \frac{\partial A_{js}}{\partial \bar{z}_{ks}} W + A_{js} A_{ks} W \tag{4}$$

and

$$\frac{\partial^2 W}{\partial \bar{z}_{ks} \partial \bar{z}_{js}} = \frac{\partial A_{ks}}{\partial \bar{z}_{js}} W + A_{ks} A_{js} W \tag{5}$$

Since $W \neq 0$, by comparing (4) and (5) we have (3).

Now suppose that (3) holds. Consider the system:

$$\frac{\partial W}{\partial \bar{z}_{js}} = A_{js}(z_{1s}, z_{2s}, \ldots, z_{ns}) = A_{js}(z_s) \quad s,j = 1,2,\ldots,n \tag{6}$$

in G. Since G is a domain of holomorphy, this system is solvable in $c^\infty(G)$ (see Tutschke, 1977). Take a fixed solution $w_0(z_s) \in c^\infty(G), z_s = (z_{1s}, z_{2s}, \ldots, z_{ns})$ of (6) and consider the function:

$$W(z_s) = ce^{-w_0(z_s)}; \quad c \neq 0, \ s = 1,2,\ldots,n \tag{7}$$

Then we get a solution $W(z_s) \neq 0$ for all $z_s \in G$ And we suppose that (6) always holds in sequel. Let Ω be an open subset of G and $W \in c^\infty(\Omega)$ be a solution of (2). Set:

$$\Phi(z_s) = W(z_s) e^{-w_0(z_s)}, \quad s = 1,2,\ldots,n \tag{8}$$

where $w_0(z_s)$ is a solution of (6) as above. It may easily be verified that Φ is analytic in Ω. From (8) it follows:

$$W(z_s) = \Phi(z_s) e^{w_0(z_s)}, \quad s = 1,2,\ldots,n \tag{9}$$

Conversely, let Φ be a given analytic function in Ω, then it follows immediately that the function $W(z_s)$ defined by formula (9) is a c^∞-solution of (2) in Ω. Thus we obtain. □

Theorem 2.2. *There is a "1-1" correspondence between the set $v(\Omega)$ of all c^∞-solutions (of system (2)) and the set $\hbar(\Omega)$ of analytic functions in Ω. Now let*

$$D = D_1 \times D_2 \times \cdots \times D_n \subset G$$

be a polycylinder, where D_j, are domains in $C(z_{js})$ with piece-smooth boundaries and

$$W(z_s) \in v(D) \cap c(\overline{D}), \quad s = 1,2,\ldots,n$$

By applying the Cauchy integral Formula we have

$$\Phi(z_s) = \frac{1}{(2\pi i)^n} \int_{\partial D_1} \int_{\partial D_2} \cdots \int_{\partial D_n} \frac{\Phi(\xi_1 \cdots \xi_n)}{(\xi_1 - z_{1s}) \cdots (\xi_n - z_{ns})} d\xi_1 \cdots d\xi_n, \quad s = 1,2,\ldots,n \tag{10}$$

$$z_s = (z_{1s}, z_{2s}, \ldots, z_{ns}), \quad s = 1,2,\ldots,n$$

From (9) and (10) it follows

$$W(z_s) = \int_{\Gamma(D)} F(\xi, z_s) W(\xi) d\xi_1 \ldots d\xi_n, \quad s = 1,2,\ldots,n \tag{11}$$

For $z_s \in D$ where

$$F(\xi, z_s) = \frac{1}{(2\pi i)^n} \frac{e^{w_0(z_s) - w_0(\xi)}}{(\xi_1 - z_{1s}) \cdots (\xi_n - z_{ns})}$$

and

$$\Gamma(D) = \partial D_1 \times \partial D_2 \times \cdots \times \partial D_n$$

Thus we obtain.

Theorem 2.3. *If D is a polycylinder in G and*

$$W(z_s) \in v(D) \cap c(\overline{D}), \quad s = 1,\ldots,n$$

Then $W(z_s)$ can be represented by formula (11) for $z_s \in D$. In the following, by applying Theorems 2.2 and 2.3 we can prove some properties of generalized analytic functions on several variables.

Theorem 2.4 (Principle of Maximum Modulus Theorem). *If*

$$W(z_s) \in v(D) \cap c(\overline{D}), \quad s = 1,\ldots,n$$

Then

$$|W(z_s)| \leqslant M max_{\xi \in \partial D} |W(\xi)|, \quad s = 1,\ldots,n$$

For $z_s \in D$ where M is a positive constant depending only on the coefficient $A_{js}(z_s)$ of the system and on the domain D.

Proof. Let $W_0(z_s) \in c^\infty(G)$ be a solution of (4) then $W_0(z_s) \in c(\overline{D})$. Set

$$m_0 = min_{\xi \in \partial D} Re|w_0(\xi)|$$

and

$$M_0 = max_{\xi \in \partial D} Re|w_0(\xi)|$$

From (8) and the Principle of maximum modulus for analytic functions it follows:

$$|\Phi(z_s)| = |W(z_s)| e^{-Re w_0(z_s)}, \quad s = 1,\ldots,n \tag{12}$$

and

$$|\Phi(z_s)| \leqslant max_{\xi \in \partial D} |\Phi(\xi)| = max_{\xi \in \partial D} |W(\xi)| e^{-Re w_0(\xi)} \leqslant e^{-m_0} max_{\xi \in \partial D} |W(\xi)| \tag{13}$$

By comparing (12) and (13) we have

$$|W(z_s)| \leqslant e^{-m_0} e^{Re w_0(z_s)} max_{\xi \in \partial D} |W(\xi)| \leqslant e^{M_0 - m_0} max_{\xi \in \partial D} |W(\xi)| = M \cdot max_{\xi \in \partial D} |W(\xi)|$$

where $M = e^{M_0 - m_0}$. By definition, M depends only on $A_{js}(z_s)$ and on D. This completes the proof. □

Note that $M_0 - m_0 \geqslant 0$, hence $M \geqslant 1$. Now suppose that $G = C^n$ and there exists a solution $w_0(z_s)$ of (4), which is bounded in the whole of C^n. Then we have:

Theorem 2.5 (Liouvllle's Theorem). *If a generalized analytic function $W(z_s)$, $s = 1,2,\ldots,n$ is continuous, bounded in C^n and vanishes at a point $z_0 \in c^n$ (in particular it may occur that $z_0 = \infty$), then $W(z_s) = 0$ everywhere.*

Proof. From (6) it follows that $\Phi(z_s)$ is analytic, bounded in C_n and vanishes at z_0. By virtue of Liouvilie's Theorem in

Complex Analysis we have $\Phi(z_s) = 0$. Take the equality (9) into account we have $W(z_s) \equiv 0$ everywhere. If $W \in v(C_n)$ and bounded in c^n, then because of (8) we have $\Phi(z_s) = c$ const. Hence it follows: □

Theorem 2.6. *Every continuous and bounded generalized analytic function $W(z_s)$ in the whole of C^n has the form:*

$$W(z_s) = c \cdot e^{w_0(z_s)}; \quad c = const.$$

where $w_0(z_s)$ is a solution of (6).

3. Conclusions

In this paper, we have seen that generalization of systems of partial differential equations on several complex variables to solutions of overdetermined systems of complex partial differential equations. The generalization of these systems have many potential applications in partial differential equations.

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