Journal of King Saud University – Science 31 (2019) 1220–1226

Contents lists available at ScienceDirect

Journal of King Saud University – Science

journal homepage: www.sciencedirect.com

Original article

m-Polar fuzzy ideals of BCK/BCI-algebras

Anas Al-Masarwah *, Abd Ghafur Ahmad

School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, 43600 UKM Bangi, Selangor DE, Malaysia

ARTICLE INFO

Article history:
Received 4 August 2018
Accepted 8 October 2018
Available online 11 October 2018

2010 Mathematics Subject Classification:
06F35
03G25
03B52

Keywords:
m-Polar fuzzy set
m-Polar fuzzy subalgebra
m-Polar fuzzy ideal
m-Polar fuzzy commutative ideal
m-Polar fuzzy closed ideal

ABSTRACT

The notions of m-polar fuzzy subalgebras and m-polar fuzzy (closed, commutative) ideals are introduced, and related properties are investigated. Characterizations of m-polar fuzzy subalgebras and m-polar fuzzy (commutative) ideals are considered. Relations between m-polar fuzzy subalgebras, m-polar fuzzy ideals and m-polar fuzzy commutative ideals are discussed.

1. Introduction

After Zadeh (1965) has established the fundamental concept of fuzzy sets, numerous generalizations of fuzzy sets are discussed, for instance, interval valued fuzzy sets, intuitionistic fuzzy sets, hesitant fuzzy sets and fuzzy multisets, etc. In 1994, Zhang (1994) generalized the idea of fuzzy sets and gave the notion of bipolar fuzzy sets on a given set X as a map which associates each element of X to a real number in the interval $[-1, 1]$. In many problems, bipolar information are used, for instance, cooperation and competition, common interests and conflict of interests, friendship and hostility are the two-sided knowledge. In 2014, Chen et al. (2014) extended the concept of bipolar fuzzy sets to obtain the notion of m-polar fuzzy sets and showed that bipolar fuzzy sets and 2-polar fuzzy sets are cryptomorphic mathematical tools. The idea behind this is that multipolar information (not just bipolar information which corresponds to two valued logic) arise because data for a real life complicated problems are sometimes come from n factors ($n \geqslant 2$). For example, Malaysia is a good country. The truth value of this statement may not be a real number in [0, 1]. Being a good country may have several components: good in tourism, good in public transport system, good in political awareness, etc. Each component may be a real number in [0, 1]. If n is the number of such components under consideration, then the truth value of the fuzzy statement is a n-tuple of real numbers in [0, 1], that is, an element of $[0, 1]^n$.

The notion of logical algebras: BCK-algebras was introduced by Imai and Iséki (1966) as a generalization of both classical and non-classical propositional calculi. In the same year, Iséki (1966) introduced BCI-algebras as a super class of the class of BCK-algebras. Meng (1991) introduced the concept of commutative ideals in BCK-algebras, and investigated some important results. Xi (1991) applied the concept of fuzzy sets to BCK-algebras. Jun and Roh (1994) studied fuzzy commutative ideals in BCK-algebras. Since then, the concepts and results of BCK/BCI-algebras have been developed to the fuzzy and fuzzy soft setting frames (Al-Masarwah and Ahmad, 2018a,b,c; Jun et al., 2013, 2014, 2017a,b, 2018a,b; Kim et al., 2018; Lee, 2009; Lee et al., 2012; Muhiuddin et al., 2017, 2018; Song et al., 2017; Zhang et al., 2017).

Recently, the notion of m-polar fuzzy set theory was applied to graph theory (Akram and Sarwar, 2018), matroid theory (Sarwar and Akram, 2017), and some algebraic structures such as groups (Farooq et al., 2016), Lie subalgebras (Akram et al., 2016) and Lie

* Corresponding author.
E-mail addresses: almasarwah85@gmail.com (A. Al-Masarwah), ghafur@ukm.edu.my (A.G. Ahmad).

Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

https://doi.org/10.1016/j.jksus.2018.10.002
1018-3647/

ideals (Akram and Farooq, 2016). In this paper, we discuss the notion of m-polar fuzzy sets with an application to *BCK/BCI*-algebras. We introduce the notions of m-polar fuzzy subalgebras and m-polar fuzzy (closed, commutative) ideals, and then we investigate several properties. We give characterizations of m-polar fuzzy subalgebras and m-polar fuzzy (commutative) ideals. We discuss relations between m-polar fuzzy subalgebras, m-polar fuzzy ideals and m-polar fuzzy commutative ideals.

2. Preliminaries

We first recall some elementary aspects which are used to present the paper. Throughout this paper, X always denotes a *BCK/BCI*-algebra without any specifications.

Definition 2.1. Imai and Iséki (1966), Iséki (1966) An algebra $(X;*,0)$ of type (2, 0) is called a *BCI*-algebra if it satisfies the following axioms:

(I) $((x*y)*(x*z))*(z*y)=0$,
(II) $(x*(x*y))*y=0$,
(III) $x*x=0$,
(IV) $x*y=0$ and $y*x=0$ imply $x=y$.

for all $x,y,z\in X$. If a *BCI*-algebra X satisfies $0*x=0$ for all $x\in X$, then X is called a *BCK*-algebra. We can define a partial ordering $\leqslant$ by

$$(\forall x\in X)(x\leqslant y\Longleftrightarrow x*y=0).$$

In a *BCK/BCI*-algebra X, the following hold:

$$(\forall x\in X)(x*0=x), \tag{1}$$
$$(\forall x,y,z\in X)((x*y)*z=(x*z)*y). \tag{2}$$

A *BCK*-algebra X is said to be commutative if it satisfies the following equality:

$$(\forall x,y\in X)\left(x\overrightarrow{\wedge}y=y\overrightarrow{\wedge}x\right), \tag{3}$$

where $x\overrightarrow{\wedge}y=x*(x*y)$.

Definition 2.2. Xi (1991) A non-empty subset I of a *BCK/BCI*-algebra X is called a subalgebra of X if $x*y\in I$ for all $x,y\in I$.

Definition 2.3. Xi (1991) A non-empty subset S of a *BCK/BCI*-algebra X is called an ideal of X if it satisfies the following:

$$0\in S, \tag{4}$$
$$(\forall x,y\in X)(x*y\in S, y\in S\Rightarrow x\in S). \tag{5}$$

Definition 2.4. Meng (1991) A non-empty subset S of a *BCK*-algebra X is called a commutative ideal of X if it satisfies (4) and

$$(\forall x,y,z\in X)\left((x*y)*z\in S, z\in S\Rightarrow x*\left(y\overrightarrow{\wedge}x\right)\in S\right). \tag{6}$$

Lemma 2.5. Meng and Jun (1994) *An ideal S of a BCK-algebra X is commutative if and only if the following assertion is valid:*

$$(\forall x,y\in X)\left(x*y\in S\Rightarrow x*\left(y\overrightarrow{\wedge}x\right)\in S\right). \tag{7}$$

We refer the reader to the books (Huang, 2006; Meng and Jun, 1994) and the paper (Iséki and Tanaka, 1978) for further information regarding *BCK/BCI*-algebras.

Definition 2.6. Chen et al. (2014) An m-polar fuzzy set $\widehat{Q}$ on a non-empty set X is a mapping $\widehat{Q}:X\to[0,1]^m$. The membership value of every element $x\in X$ is denoted by

$$\widehat{Q}(x)=\left(p_1\circ\widehat{Q}(x),p_2\circ\widehat{Q}(x),\ldots,p_m\circ\widehat{Q}(x)\right)$$

where $p_i\circ\widehat{Q}:[0,1]^m\to[0,1]$ is defined the i-th projection mapping.

Note that $[0,1]^m$ (m-th-power of [0,1]) is considered as a poset with the pointwise order $\leqslant$, where m is an arbitrary ordinal number (we make an appointment that $m=\{n|n<m\}$ when $m>0$), $\leqslant$ is defined by

$$x\leqslant y\Longleftrightarrow p_i(x)\leqslant p_i(y)$$

for each $i\in m$ $(x,y\in[0,1]^m)$, and $p_i:[0,1]^m\to[0,1]$ is the i-th projection mapping $(i\in m)$. It is easy to see that $\widehat{0}=(0,0,\ldots,0)$ is the smallest value in $[0,1]^m$ and $\widehat{1}=(1,1,\ldots,1)$ is the largest value in $[0,1]^m$.

3. *m*-Polar fuzzy subalgebras and (commutative) ideals

In this section, we introduce the notions of m-polar fuzzy subalgebras, m-polar fuzzy ideals and m-polar fuzzy commutative ideals in *BCK/BCI*-algebras and investigate some of their related properties.

Definition 3.1. An m-polar fuzzy set $\widehat{Q}$ of X is called an m-polar fuzzy subalgebra if the following assertion is valid:

$$(\forall x,y\in X)\left(\widehat{Q}(x*y)\geqslant\inf\left\{\widehat{Q}(x),\widehat{Q}(y)\right\}\right). \tag{8}$$

That is,

$$(\forall x,y\in X)\left(p_i\circ\widehat{Q}(x*y)\geqslant\inf\left\{p_i\circ\widehat{Q}(x),p_i\circ\widehat{Q}(y)\right\}\right)$$

for each $i=1,2,\ldots,m$.

Example 3.2. Let $X=\{0,a,b,c\}$ be a *BCK*-algebra with the Cayley table which is appeared in Table 1.

Define a 4-polar fuzzy set $\widehat{Q}:X\to[0,1]^4$ by:

$$\widehat{Q}(x)=\begin{cases}(0.3,0.4,0.5,0.8), & \text{if } x=0\\ (0.2,0.3,0.4,0.3), & \text{if } x=a\\ (0.1,0.2,0.3,0.2), & \text{if } x=b\\ (0.2,0.3,0.5,0.5), & \text{if } x=c.\end{cases}$$

It is routine to verify that $\widehat{Q}$ is a 4-polar fuzzy subalgebra of X.

For any m-polar fuzzy set $\widehat{Q}$ on X and $\widehat{\sigma}=(\sigma_1,\sigma_2,\ldots,\sigma_m)\in[0,1]^m$, the set

$$\widehat{Q}_{\left[\widehat{\sigma}\right]}=\left\{x\in X|\widehat{Q}(x)\geqslant\widehat{\sigma}\right\}$$

is called the $\widehat{\sigma}$-level cut set of $\widehat{Q}$, and the set

$$\widehat{Q}^{s}_{\left[\widehat{\sigma}\right]}=\left\{x\in X|\widehat{Q}(x)>\widehat{\sigma}\right\}$$

is called the strong $\widehat{\sigma}$-level cut set of $\widehat{Q}$.

Table 1
Cayley table for the $*$-operation.

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Theorem 3.3. *Let $\widehat{Q}$ be an m-polar fuzzy set of X. Then $\widehat{Q}$ is an m-polar fuzzy subalgebra of X if and only if $\widehat{Q}_{[\widehat{\sigma}]} \neq \phi$ is a subalgebra of X for all $\widehat{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_m) \in [0,1]^m$.*

Proof. Assume that $\widehat{Q}$ is an m-polar fuzzy subalgebra of X and let $\widehat{\sigma} \in [0,1]^m$ be such that $\widehat{Q}_{[\widehat{\sigma}]} \neq \emptyset$. Let $x, y \in \widehat{Q}_{[\widehat{\sigma}]}$. Then $\widehat{Q}(x) \geqslant \widehat{\sigma}$ and $\widehat{Q}(y) \geqslant \widehat{\sigma}$. It follows from (8) that $\widehat{Q}(x*y) \geqslant \inf\left\{\widehat{Q}(x), \widehat{Q}(y)\right\} \geqslant \widehat{\sigma}$, so that $x*y \in \widehat{Q}_{[\widehat{\sigma}]}$. Hence, $\widehat{Q}_{[\widehat{\sigma}]}$ is a subalgebra of X.

Conversely, assume that $\widehat{Q}_{[\widehat{\sigma}]}$ is a subalgebra of X. Suppose that there exist $a, b \in X$ such that $\widehat{Q}(a*b) < \inf\left\{\widehat{Q}(a), \widehat{Q}(b)\right\}$. Then there exists $\widehat{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in [0,1]^m$ such that $\widehat{Q}(a*b) < \widehat{\alpha} \leqslant \inf\left\{\widehat{Q}(a), \widehat{Q}(b)\right\}$. It follows that $a, b \in \widehat{Q}_{[\widehat{\alpha}]}$, but $a*b \notin \widehat{Q}_{[\widehat{\alpha}]}$. This is a contradiction, and so $\widehat{Q}(x*y) \geqslant \inf\left\{\widehat{Q}(x), \widehat{Q}(y)\right\}$ for all $x, y \in X$. Therefore $\widehat{Q}$ is an m-polar fuzzy subalgebra of X. □

Corollary 3.4. *If $\widehat{Q}$ is an m-polar fuzzy subalgebra of X, then $\widehat{Q}^s_{[\widehat{\sigma}]} \neq \emptyset$ is a subalgebra of X for all $\widehat{\sigma} \in [0,1]^m$.*

Proof. Straightforward. □

Lemma 3.5. *Every m-polar fuzzy subalgebra $\widehat{Q}$ of X satisfies the following inequality:*

$$(\forall x \in X)\left(\widehat{Q}(0) \geqslant \widehat{Q}(x)\right). \tag{9}$$

Proof. Note that $x*x = 0$ for all $x \in X$. Using (8), we have $\widehat{Q}(0) = \widehat{Q}(x*x) \geqslant \inf\left\{\widehat{Q}(x), \widehat{Q}(x)\right\} = \widehat{Q}(x)$ for all $x \in X$. □

Proposition 3.6. *If every m-polar fuzzy subalgebra $\widehat{Q}$ of X satisfies the following inequality:*

$$(\forall x, y \in X)\left(\widehat{Q}(x*y) \geqslant \widehat{Q}(y)\right), \tag{10}$$

then $\widehat{Q}(x) = \widehat{Q}(0)$.

Proof. Let $x \in X$. Using (1) and (10), we have $\widehat{Q}(x) = \widehat{Q}(x*0) \geqslant \widehat{Q}(0)$. It follows from Lemma 3.5 that $\widehat{Q}(x) = \widehat{Q}(0)$. □

Definition 3.7. An m-polar fuzzy set $\widehat{Q}$ of X is called an m-polar fuzzy ideal if the following assertion is valid:

$$(\forall x, y \in X)\left(\widehat{Q}(0) \geqslant \widehat{Q}(x) \geqslant \inf\left\{\widehat{Q}(x*y), \widehat{Q}(y)\right\}\right). \tag{11}$$

That is,

$$(\forall x, y \in X)\left(p_i \circ \widehat{Q}(0) \geqslant p_i \circ \widehat{Q}(x) \geqslant \inf\left\{p_i \circ \widehat{Q}(x*y), p_i \circ \widehat{Q}(y)\right\}\right)$$

for each $i = 1, 2, \ldots, m$.

Table 2
Cayley table for the $*$-operation.

$*$	0	a	1	2	3
0	0	0	3	2	1
a	a	0	3	2	1
1	1	1	0	3	2
2	2	2	1	0	3
3	3	3	2	1	0

Example 3.8. Let $X = \{0, a, 1, 2, 3\}$ be a *BCI*-algebra with the Cayley table which is appeared in Table 2.

Define a 4-polar fuzzy set $\widehat{Q} : X \to [0,1]^4$ by:

$$\widehat{Q}(x) = \begin{cases} (0.5, 0.6, 0.6, 0.7), & \text{if } x = 0 \\ (0.4, 0.5, 0.5, 0.7), & \text{if } x = a \\ (0.2, 0.3, 0.3, 0.2), & \text{if } x = 1, 3 \\ (0.3, 0.4, 0.4, 0.5), & \text{if } x = 2. \end{cases}$$

It is routine to verify that $\widehat{Q}$ is a 4-polar fuzzy ideal of X.

Proposition 3.9. *If $\widehat{Q}$ is an m-polar fuzzy ideal of X, then*

$$(\forall x, y \in X)\left(x \leqslant y \Rightarrow \widehat{Q}(x) \geqslant \widehat{Q}(y)\right). \tag{12}$$

Proof. Let $x, y \in X$ be such that $x \leqslant y$. Then $x*y = 0$, and so $\widehat{Q}(x) \geqslant \inf\left\{\widehat{Q}(x*y), \widehat{Q}(y)\right\} = \inf\left\{\widehat{Q}(0), \widehat{Q}(y)\right\} = \widehat{Q}(y)$. This completes the proof. □

Proposition 3.10. *Let $\widehat{Q}$ be an m-polar fuzzy ideal of X. Then the following are equivalent:*

(i) $(\forall x, y \in X)\left(\widehat{Q}(x*y) \geqslant \widehat{Q}((x*y)*y)\right)$,

(ii) $(\forall x, y, z \in X)\left(\widehat{Q}((x*z)*(y*z)) \geqslant \widehat{Q}((x*y)*z)\right)$.

Proof. Assume that (i) is valid and let $x, y, z \in X$. Since

$$\begin{aligned}((x*(y*z))*z)*z =&((x*z)*(y*z))*z \\ \leqslant&(x*y)*z.\end{aligned}$$

It follows from Proposition 3.9 that $\widehat{Q}(((x*z)*(y*z))*z) \geqslant \widehat{Q}((x*y)*z)$. Using (2) and (i), we have

$$\begin{aligned}\widehat{Q}((x*z)*(y*z)) =&\widehat{Q}((x*(y*z))*z) \\ \geqslant&\widehat{Q}(((x*(y*z))*z)*z) \\ \geqslant&\widehat{Q}((x*y)*z).\end{aligned}$$

Conversely, suppose that (ii) holds. If we use z instead of y in (ii), then

$$\begin{aligned}\widehat{Q}(x*z) =&\widehat{Q}((x*z)*0) \\ =&\widehat{Q}((x*z)*(z*z)) \\ \geqslant&\widehat{Q}((x*z)*z)\end{aligned}$$

for all $x, z \in X$ by using (III) and (1). □

Theorem 3.11. *Let $\widehat{Q}$ be an m-polar fuzzy set of X. Then $\widehat{Q}$ is an m-polar fuzzy ideal of X if and only if it satisfies*

$$(\forall \widehat{\sigma} \in [0,1]^m)\left(\widehat{Q}_{[\widehat{\sigma}]} \neq \emptyset \Rightarrow \widehat{Q}_{[\widehat{\sigma}]} \text{ is an ideal of } X\right). \tag{13}$$

Proof. Assume that $\widehat{Q}$ is an m-polar fuzzy ideal of X. Let $\widehat{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_m) \in [0,1]^m$ be such that $\widehat{Q}_{[\widehat{\sigma}]} \neq \varnothing$. Obviously, $0 \in \widehat{Q}_{[\widehat{\sigma}]}$. Let $x, y \in X$ be such that $x * y \in \widehat{Q}_{[\widehat{\sigma}]}$ and $y \in \widehat{Q}_{[\widehat{\sigma}]}$. Then $\widehat{Q}(x*y) \geqslant \widehat{\sigma}$ and $\widehat{Q}(y) \geqslant \widehat{\sigma}$. It follows from (11) that $\widehat{Q}(x) \geqslant \inf\left\{\widehat{Q}(x*y), \widehat{Q}(y)\right\} \geqslant \widehat{\sigma}$, so that $x \in \widehat{Q}_{[\widehat{\sigma}]}$. Hence, $\widehat{Q}_{[\widehat{\sigma}]}$ is an ideal of X.

Conversely, suppose that (13) is valid. If there exists $h \in X$ such that $\widehat{Q}(0) < \widehat{Q}(h)$, then $\widehat{Q}(0) < \widehat{\sigma}_h \leqslant \widehat{Q}(h)$ for some $\widehat{\sigma}_h = (\sigma_{h_1}, \sigma_{h_2}, \ldots, \sigma_{h_m}) \in [0,1]^m$. Then $0 \notin \widehat{Q}_{[\widehat{\sigma}_h]}$ which is a contradiction. Hence $\widehat{Q}(0) \geqslant \widehat{Q}(x)$ for all $x \in X$. Now, assume that there exist $h, q \in X$ such that $\widehat{Q}(h) < \inf\left\{\widehat{Q}(h*q), \widehat{Q}(q)\right\}$. Then there exists $\widehat{\beta} = (\beta_1, \beta_2, \ldots, \beta_m) \in [0,1]^m$ such that $\widehat{Q}(h) < \widehat{\beta} \leqslant \inf\left\{\widehat{Q}(h*q), \widehat{Q}(q)\right\}$. It follows that $h * q \in \widehat{Q}_{[\widehat{\beta}]}$ and $q \in \widehat{Q}_{[\widehat{\beta}]}$, but $h \notin \widehat{Q}_{[\widehat{\beta}]}$. This is impossible, and so $\widehat{Q}(x) \geqslant \inf\left\{\widehat{Q}(x*y), \widehat{Q}(y)\right\}$ for all $x, y \in X$. Therefore, $\widehat{Q}$ is an m-polar fuzzy ideal of X. □

Corollary 3.12. *If $\widehat{Q}$ is an m-polar fuzzy ideal of X, then $\widehat{Q}^s_{[\widehat{\sigma}]} \neq \varnothing$ is an ideal of X for all $\widehat{\sigma} \in [0,1]^m$.*

Proof. Straightforward. □

For any element ω of X, we consider the set

$$X_\omega = \left\{x \in X | \widehat{Q}(x) \geqslant \widehat{Q}(\omega)\right\}. \tag{14}$$

Obviously, $\omega \in X_\omega$, and so X_ω is a non-empty subset of X.

Theorem 3.13. *Let ω be an element of X. If $\widehat{Q}$ is an m-polar fuzzy ideal of X, then the set X_ω is an ideal of X.*

Proof. Obviously, $0 \in X_\omega$ by (11). Let $x, y \in X$ be such that $x * y \in X_\omega$ and $y \in X_\omega$. Then $\widehat{Q}(x*y) \geqslant \widehat{Q}(\omega)$ and $\widehat{Q}(y) \geqslant \widehat{Q}(\omega)$. Since $\widehat{Q}$ is an m-polar fuzzy ideal of X, it follows from (11) that $\widehat{Q}(x) \geqslant \inf\left\{\widehat{Q}(x*y), \widehat{Q}(y)\right\} \geqslant \widehat{Q}(\omega)$, so that $x \in X_\omega$. Hence, X_ω is an ideal of X. □

Proposition 3.14. *Let $\widehat{Q}$ be an m-polar fuzzy ideal of X. If X satisfies the following assertion:*

$$(\forall x, y, z \in X)(x * y \leqslant z), \tag{15}$$

then $\widehat{Q}(x) \geqslant \inf\left\{\widehat{Q}(y), \widehat{Q}(z)\right\}$ for all $x, y, z \in X$.

Proof. Assume that (15) is valid in X. Then $\widehat{Q}(x*y) \geqslant \inf\left\{\widehat{Q}((x*y)*z), \widehat{Q}(z)\right\} = \inf\left\{\widehat{Q}(0), \widehat{Q}(z)\right\} = \widehat{Q}(z)$ for all $x, y, z \in X$. It follows that $\widehat{Q}(x) \geqslant \inf\left\{\widehat{Q}(x*y), \widehat{Q}(y)\right\} \geqslant \inf\left\{\widehat{Q}(y), \widehat{Q}(z)\right\}$ for all $x, y, z \in X$. This completes the proof. □

Theorem 3.15. *For any BCK-algebra X, every m-polar fuzzy ideal is an m-polar fuzzy subalgebra.*

Proof. Let $\widehat{Q}$ be an m-polar fuzzy ideal of a BCK-algebra X and let $x, y \in X$. Then

$$\begin{aligned}
\widehat{Q}(x*y) &\geqslant \inf\left\{\widehat{Q}((x*y)*x), \widehat{Q}(x)\right\} \\
&= \inf\left\{\widehat{Q}((x*x)*y), \widehat{Q}(x)\right\} \\
&= \inf\left\{\widehat{Q}(0*y), \widehat{Q}(x)\right\} \\
&= \inf\left\{\widehat{Q}(0), \widehat{Q}(x)\right\} \\
&\geqslant \inf\left\{\widehat{Q}(x), \widehat{Q}(y)\right\}.
\end{aligned}$$

Therefore, $\widehat{Q}$ is an m-polar fuzzy subalgebra of X. □

The converse of Theorem 3.15 is not true in general as seen in the following example.

Example 3.16. Consider a BCK-algebra $X = \{0, a, b, c\}$ which is given in Example 3.2. Define a 3-polar fuzzy set $\widehat{Q} : X \to [0,1]^3$ by:

$$\widehat{Q}(x) = \begin{cases} (0.3, 0.7, 0.8), & \text{if } x = 0, b \\ (0.1, 0.4, 0.5), & \text{if } x = a, c. \end{cases}$$

Then $\widehat{Q}$ is a 3-polar fuzzy subalgebra of X. But it is not a 3-polar fuzzy ideal of X, since $\widehat{Q}(a) = (0.1, 0.4, 0.5) < (0.3, 0.7, 0.8) = \inf\left\{\widehat{Q}(a*b), \widehat{Q}(b)\right\}$.

Theorem 3.15 is not valid in a BCI-algebra, that is, if X is a BCI-algebra, then there is an m-polar fuzzy ideal that is not an m-polar fuzzy subalgebra, as seen in the following example.

Example 3.17. Consider a BCI-algebra $X = Y \times \mathbb{Z}$, where $(Y, *, 0)$ is a BCI-algebra and $(\mathbb{Z}, -, 0)$ is the adjoint BCI-algebra of the additive group $(\mathbb{Z}, +, 0)$ of integers (see Huang, 2006). Let $A = Y \times \mathbb{N}$, where $\mathbb{N}$ is the set of nonnegative integers. Define an m-polar fuzzy set $\widehat{Q} : X \to [0,1]^m$ as follows:

$$\widehat{Q}(x) = \begin{cases} (0.7, 0.7, \ldots, 0.7), & x \in A \\ (0.2, 0.2, \ldots, 0.2), & x \notin A. \end{cases}$$

Then $\widehat{Q}$ is an m-polar fuzzy ideal of X. If we take $x = (0,0)$ and $y = (0,1)$, then $z = x * y = (0,0) * (0,1) = (0,-1)$, and so $\widehat{Q}(x*y) = \widehat{Q}(z) = (0.2, 0.2, \ldots, 0.2) < (0.7, 0.7, \ldots, 0.7) = \inf\left\{\widehat{Q}(x), \widehat{Q}(y)\right\}$. Therefore, $\widehat{Q}$ is not an m-polar fuzzy subalgebra of X.

Definition 3.18. Let X be a BCI-algebra. An m-polar fuzzy ideal $\widehat{Q}$ of X is said to be closed if it is also an m-polar fuzzy subalgebra of X.

Example 3.19. Consider a BCI-algebra $X = \{0, a, 1, 2, 3\}$ which is given in Example 3.8. Define a 4-polar fuzzy set $\widehat{Q} : X \to [0,1]^4$ by:

$$\widehat{Q}(x) = \begin{cases} (0.5, 0.6, 0.8, 0.9), & \text{if } x = 0 \\ (0.3, 0.4, 0.6, 0.7), & \text{if } x = a \\ (0.2, 0.3, 0.5, 0.6), & \text{if } x = 1, 2, 3. \end{cases}$$

Then $\widehat{Q}$ is a closed 4-polar fuzzy ideal of X.

Theorem 3.20. Let X be a BCI-algebra and let $\widehat{Q}$ be an m-polar fuzzy set of X given as follows:

$$\widehat{Q}(x) = \begin{cases} \widehat{t} = (t_1, t_2, \ldots, t_m), & \text{if } x \in X_+ \\ \widehat{s} = (s_1, s_2, \ldots, s_m), & \textit{otherwise} \end{cases}$$

where $\hat{t}, \hat{s} \in [0,1]^m$ *with* $\hat{t} > \hat{s}$ *and* $X_+ = \{x \in X | 0 \leqslant x\}$. *Then* $\hat{Q}$ *is a closed m-polar fuzzy ideal of X.*

Proof. Since $0 \in X_+$, we have $\hat{Q}(0) = \hat{t} = (t_1, t_2, \dots, t_m) \geqslant \hat{Q}(x)$ for all $x \in X$. Let $x, y \in X$. If $x \in X_+$, then

$$\hat{Q}(x) = \hat{t} = (t_1, t_2, \dots, t_m) \geqslant \inf\left\{\hat{Q}(x * y), \hat{Q}(y)\right\}.$$

Assume that $x \notin X_+$. If $x * y \in X_+$, then $y \notin X_+$; and if $y \in X_+$, then $x * y \notin X_+$. In either case, we get

$$\hat{Q}(x) = \hat{s} = (s_1, s_2, \dots, s_m) = \inf\left\{\hat{Q}(x * y), \hat{Q}(y)\right\}.$$

For any $x, y \in X$, if any one of x and y does not belong to X_+, then

$$\hat{Q}(x * y) \geqslant \hat{s} = (s_1, s_2, \dots, s_m) = \inf\left\{\hat{Q}(x), \hat{Q}(y)\right\}.$$

If $x, y \in X_+$, then $x * y \in X_+$, and so

$$\hat{Q}(x * y) = \hat{t} = (t_1, t_2, \dots, t_m) = \inf\left\{\hat{Q}(x), \hat{Q}(y)\right\}.$$

Therefore, $\hat{Q}$ is a closed m-polar fuzzy ideal of X. □

Proposition 3.21. *Every closed m-polar fuzzy ideal* $\hat{Q}$ *of a BCI-algebra X satisfies the following assertion:*

$$(\forall x \in X)\left(\hat{Q}(0 * x) \geqslant \hat{Q}(x)\right). \tag{16}$$

Proof. For any $x \in X$, we have $\hat{Q}(0 * x) \geqslant \inf\left\{\hat{Q}(0), \hat{Q}(x)\right\} \geqslant \inf\left\{\hat{Q}(x), \hat{Q}(x)\right\} = \hat{Q}(x)$. This completes the proof. □

Proposition 3.22. *Let X be a BCI-algebra. If* $\hat{Q}$ *is an m-polar fuzzy ideal of X that satisfies the condition (16), then* $\hat{Q}$ *is an m-polar fuzzy subalgebra and hence is a closed m-polar fuzzy ideal of X.*

Proof. Note that $(x * y) * x \leqslant 0 * y$ for all $x, y \in X$. Using Proposition 3.14 and the condition of Eq. (16), we have

$$\hat{Q}(x * y) \geqslant \inf\left\{\hat{Q}(x), \hat{Q}(0 * y)\right\} \geqslant \inf\left\{\hat{Q}(x), \hat{Q}(y)\right\}.$$

Hence $\hat{Q}$ is an m-polar fuzzy subalgebra of X and therefore $\hat{Q}$ is a closed m-polar fuzzy ideal of X. □

Definition 3.23. Let X be a BCK-algebra. An m-polar fuzzy set $\hat{Q}$ of X is called an m-polar fuzzy commutative ideal of X if the following assertions are valid:

$$(\forall x \in X)\left(\hat{Q}(0) \geqslant \hat{Q}(x)\right), \tag{17}$$

$$(\forall x, y, z \in X)\left(\hat{Q}\left(x * \left(y \vec{\wedge} x\right)\right) \geqslant \inf\left\{\hat{Q}((x * y) * z), \hat{Q}(z)\right\}\right). \tag{18}$$

That is,

$$(\forall x \in X)\left(p_i \circ \hat{Q}(0) \geqslant p_i \circ \hat{Q}(x)\right),$$

$$(\forall x, y, z \in X)\left(p_i \circ \hat{Q}\left(x * \left(y \vec{\wedge} x\right)\right) \geqslant \inf\left\{p_i \circ \hat{Q}((x * y) * z), p_i \circ \hat{Q}(z)\right\}\right)$$

for each $i = 1, 2, \dots, m$.

Example 3.24. Consider a BCK-algebra $X = \{0, a, b, c\}$ which is given in Example 3.2. Define an m-polar fuzzy set $\hat{Q} : X \to [0,1]^m$ by:

$$\hat{Q}(x) = \begin{cases} \hat{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_m), & \text{if } x = 0 \\ \hat{\beta} = (\beta_1, \beta_2, \dots, \beta_m), & \text{if } x = a \\ \hat{\gamma} = (\gamma_1, \gamma_2, \dots, \gamma_m), & \text{if } x = b, c, \end{cases}$$

where $\hat{\alpha}, \hat{\beta}, \hat{\gamma} \in [0,1]^m$ and $\hat{\alpha} > \hat{\beta} > \hat{\gamma}$. It is routine to verify that $\hat{Q}$ is an m-polar fuzzy commutative ideal of X.

Theorem 3.25. *Every m-polar fuzzy commutative ideal of a BCK-algebra X is an m-polar fuzzy ideal of X.*

Proof. Let $\hat{Q}$ be an m-polar fuzzy commutative ideal of a BCK-algebra X. For any $x, z \in X$, we have

$$\begin{aligned}\hat{Q}(x) &= \hat{Q}\left(x * \left(0 \vec{\wedge} x\right)\right) \\ &\geqslant \inf\left\{\hat{Q}((x * 0) * z), \hat{Q}(z)\right\} \\ &= \inf\left\{\hat{Q}(x * z), \hat{Q}(z)\right\}.\end{aligned}$$

Hence, $\hat{Q}$ is an m-polar fuzzy ideal of X. □

The converse of Theorem 3.25 is not true in general as seen in the following example.

Example 3.26. Let $X = \{0, 1, 2, 3, 4\}$ be a set with the Cayley table which is appeared in Table 3.

Then X is a BCK-algebra Meng (1991). Define an m-polar fuzzy set $\hat{Q} : X \to [0,1]^m$ by:

$$\hat{Q}(x) = \begin{cases} \hat{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_m), & \text{if } x = 0 \\ \hat{\beta} = (\beta_1, \beta_2, \dots, \beta_m), & \text{if } x = 1 \\ \hat{\gamma} = (\gamma_1, \gamma_2, \dots, \gamma_m), & \text{if } x = 2, 3, 4, \end{cases}$$

where $\hat{\alpha}, \hat{\beta}, \hat{\gamma} \in [0,1]^m$ and $\hat{\alpha} > \hat{\beta} > \hat{\gamma}$. Then $\hat{Q}$ is an m-polar fuzzy ideal of X. But it is not an m-polar fuzzy commutative ideal of X, since $\hat{Q}\left(2 * \left(3 \vec{\wedge} 2\right)\right) < \inf\left\{\hat{Q}((2 * 3) * 0), \hat{Q}(0)\right\}$.

We consider characterizations of an m-polar fuzzy commutative ideal of a BCK-algebra X.

Theorem 3.27. *Let* $\hat{Q}$ *be an m-polar fuzzy ideal of a BCK-algebra X. Then* $\hat{Q}$ *is an m-polar fuzzy commutative ideal of X if and only if the following assertion is valid:*

$$(\forall x, y \in X)\left(\hat{Q}\left(x * \left(y \vec{\wedge} x\right)\right) \geqslant \hat{Q}(x * y)\right). \tag{19}$$

Proof. Assume that $\hat{Q}$ is an m-polar fuzzy commutative ideal of a BCK-algebra X. Then assertion (19) is by taking $z = 0$ in (18) and using (1) and (17); then we get (19).

Conversely, suppose that an m-polar fuzzy ideal $\hat{Q}$ of a BCK-algebra X satisfies the condition (19). Then

$$(\forall x, y, z \in X)\left(\hat{Q}(x * y) \geqslant \inf\left\{\hat{Q}((x * y) * z), \hat{Q}(z)\right\}\right). \tag{20}$$

Table 3
Cayley table for the $*$-operation.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	3	0

It follows that the condition (18) is induced by (19) and (20). Therefore, $\widehat{Q}$ is an m-polar fuzzy commutative ideal of X. □

Theorem 3.28. *In a commutative BCK-algebra X, every m-polar fuzzy ideal is an m-polar fuzzy commutative ideal.*

Proof. Let $\widehat{Q}$ be an m-polar fuzzy ideal of a commutative BCK-algebra X. Using (I) and (2), we have

$$\begin{aligned}\left(\left(x*\left(y\overrightarrow{\wedge}x\right)\right)*((x*y)*z)\right)*z &= \left(\left(x*\left(y\overrightarrow{\wedge}x\right)\right)*z\right)*((x*y)*z)\\ &\leqslant \left(x*\left(y\overrightarrow{\wedge}x\right)\right)*(x*y)\\ &= \left(x\overrightarrow{\wedge}y\right)*\left(y\overrightarrow{\wedge}x\right)=0,\end{aligned}$$

and so $\left(\left(x*\left(y\overrightarrow{\wedge}x\right)\right)*((x*y)*z)\right)*z=0$, i.e., $\left(x*\left(y\overrightarrow{\wedge}x\right)\right)*((x*y)*z)\leqslant z$ for all $x,y,z\in X$. Since $\widehat{Q}$ is an m-polar fuzzy ideal, it follows from Proposition 3.14, $\widehat{Q}\left(x*\left(y\overrightarrow{\wedge}x\right)\right)\geqslant \inf\left\{\widehat{Q}((x*y)*z),\widehat{Q}(z)\right\}$. Hence, $\widehat{Q}$ is an m-polar fuzzy commutative ideal of X. □

Theorem 3.29. *Let $\widehat{Q}$ be an m-polar fuzzy set of a BCK-algebra X. Then $\widehat{Q}$ is an m-polar fuzzy commutative ideal of X if and only if it satisfies*

$$(\forall\widehat{\sigma}\in[0,1]^m)\left(\widehat{Q}_{\left[\widehat{\sigma}\right]}\neq\varnothing\Rightarrow\widehat{Q}_{\left[\widehat{\sigma}\right]}\text{ is a commutative ideal of }X\right). \tag{21}$$

Proof. Let $\widehat{Q}$ be an m-polar fuzzy commutative ideal of X. Then $\widehat{Q}$ is an m-polar fuzzy ideal of X, and so every non-empty σ-level cut set $\widehat{Q}_{\left[\widehat{\sigma}\right]}$ of $\widehat{Q}$ is an ideal of X. Let $x,y,z\in X$ be such that $(x*y)*z\in\widehat{Q}_{\left[\widehat{\sigma}\right]}$ and $z\in\widehat{Q}_{\left[\widehat{\sigma}\right]}$. Then $\widehat{Q}((x*y)*z)\geqslant\widehat{\sigma}$ and $\widehat{Q}(z)\geqslant\widehat{\sigma}$. It follows from (18) that

$$\widehat{Q}\left(x*\left(y\vec{\wedge}x\right)\right)\geqslant\inf\left\{\widehat{Q}((x*y)*z),\widehat{Q}(z)\right\}\geqslant\widehat{\sigma},$$

so that $x*\left(y\vec{\wedge}x\right)\in\widehat{Q}_{\left[\widehat{\sigma}\right]}$. Hence $\widehat{Q}_{\left[\widehat{\sigma}\right]}$ is a commutative ideal of X.

Conversely, suppose that (21) is valid. Obviously, $\widehat{Q}(0)\geqslant\widehat{Q}(x)$ for all $x\in X$. Let $\widehat{Q}((x*y)*z)=\widehat{\alpha}=(\alpha_1,\alpha_2,\ldots,\alpha_m)$ and $\widehat{Q}(z)=\widehat{\beta}=(\beta_1,\beta_2,\ldots,\beta_m)$ for all $x,y,z\in X$. Then $(x*y)*z\in\widehat{Q}_{\left[\widehat{\alpha}\right]}$ and $z\in\widehat{Q}_{\left[\widehat{\beta}\right]}$. Without loss of generality, we may assume that $\widehat{\alpha}\leqslant\widehat{\beta}$. Then $\widehat{Q}_{\left[\widehat{\beta}\right]}\subseteq\widehat{Q}_{\left[\widehat{\alpha}\right]}$, and so $z\in\widehat{Q}_{\left[\widehat{\alpha}\right]}$. Since $\widehat{Q}_{\left[\widehat{\alpha}\right]}$ is a commutative ideal of X by hypothesis, we obtain that $x*\left(y\overrightarrow{\wedge}x\right)\in\widehat{Q}_{\left[\widehat{\alpha}\right]}$, and so

$$\widehat{Q}\left(x*\left(y\vec{\wedge}x\right)\right)\geqslant\widehat{\alpha}=\inf\left\{\widehat{\alpha},\widehat{\beta}\right\}=\inf\left\{\widehat{Q}((x*y)*z),\widehat{Q}(z)\right\}.$$

Therefore, $\widehat{Q}$ is an m-polar fuzzy commutative ideal of X. □

Corollary 3.30. If $\widehat{Q}$ is an m-polar fuzzy commutative ideal of a BCK-algebra X, then $\widehat{Q}^s_{\left[\widehat{\sigma}\right]}\neq\varnothing$ is a commutative ideal of X for all $\widehat{\sigma}\in[0,1]^m$.

Proof. Straightforward. □

Theorem 3.31. *Let X be a BCK-algebra and let $f: X\to X$ be an injective mapping. Given an m-polar fuzzy set $\widehat{Q}$ of X, the following are equivalent:*

(1) $\widehat{Q}$ is an m-polar fuzzy commutative ideal of X, satisfying the following condition:

$$(\forall x\in X)\left(\widehat{Q}(f(x))=\widehat{Q}(x)\right). \tag{22}$$

(2) $\widehat{Q}_{\left[\widehat{\sigma}\right]}$ is a communicative ideal of $\widehat{Q}$, satisfying the following condition:

$$f\left(\widehat{Q}_{\left[\widehat{\sigma}\right]}\right)=\widehat{Q}_{\left[\widehat{\sigma}\right]}. \tag{23}$$

Proof. Let $\widehat{Q}$ be an m-polar fuzzy commutative ideal of X, satisfying condition (22). Then $\widehat{Q}_{\left[\widehat{\sigma}\right]}$ is a communicative ideal of $\widehat{Q}$ by Theorem 3.29. Let $\widehat{\sigma}=(\sigma_1,\sigma_2,\ldots,\sigma_m)\in Im\left(\widehat{Q}\right)$ and $x\in\widehat{Q}_{\left[\widehat{\sigma}\right]}$. Then $\widehat{Q}(f(x))=\widehat{Q}(x)\geqslant\widehat{\sigma}$. Thus $f(x)\in\widehat{Q}_{\left[\widehat{\sigma}\right]}$, which shows that $f\left(\widehat{Q}_{\left[\widehat{\sigma}\right]}\right)\subseteq\widehat{Q}_{\left[\widehat{\sigma}\right]}$. Let $y\in X$ be such that $f(y)=x$. Then $\widehat{Q}(y)=\widehat{Q}(f(y))=\widehat{Q}(x)\geqslant\widehat{\sigma}$, which implies that $y\in\widehat{Q}_{\left[\widehat{\sigma}\right]}$. Thus, $x=f(y)\in f\left(\widehat{Q}_{\left[\widehat{\sigma}\right]}\right)$, and so $\widehat{Q}_{\left[\widehat{\sigma}\right]}\subseteq f\left(\widehat{Q}_{\left[\widehat{\sigma}\right]}\right)$. Therefore (23) is valid.

Conversely, assume that $\widehat{Q}_{\left[\widehat{\sigma}\right]}$ is a commutative ideal of $\widehat{Q}$, satisfying the condition (23). Then $\widehat{Q}$ is an m-polar fuzzy commutative ideal of X by Theorem 3.29. Let $x\in X$ be such that $\widehat{Q}(x)=\widehat{\sigma}$. Note that

$$\widehat{Q}(x)=\widehat{\sigma}\Longleftrightarrow x\in\widehat{Q}_{\left[\widehat{\sigma}\right]}\text{ and }x\notin\widehat{Q}_{\left[\widehat{\gamma}\right]}\text{ for all }\widehat{\sigma}<\widehat{\gamma}.$$

It follows from (23) that $f(x)\in\widehat{Q}_{\left[\widehat{\sigma}\right]}$. Hence, $\widehat{Q}(f(x))\geqslant\widehat{\sigma}$. Let $\widehat{\gamma}=\widehat{Q}(f(x))$. If $\widehat{\sigma}<\widehat{\gamma}$, then $f(x)\in\widehat{Q}_{\left[\widehat{\gamma}\right]}=f\left(\widehat{Q}_{\left[\widehat{\gamma}\right]}\right)$ which implies from the injectivity of f that $x\in\widehat{Q}_{\left[\widehat{\gamma}\right]}$, a contradiction. Hence, $\widehat{Q}(f(x))=\widehat{\sigma}=\widehat{Q}(x)$. This completes the proof. □

Theorem 3.32. *Let ω be an element of a BCK-algebra X. If $\widehat{Q}$ is an m-polar fuzzy commutative ideal of X, the set X_ω in Eq. (14) is a commutative ideal of X.*

Proof. If $\widehat{Q}$ is an m-polar fuzzy commutative ideal of a BCK-algebra X, then it is an m-polar fuzzy ideal of X and so X_ω is an ideal of X by Theorem 3.13. Let $x*y\in X_\omega$ for any $x,y\in X$. Then $\widehat{Q}(x*y)\geqslant\widehat{Q}(\omega)$. It follows from Theorem 3.27 that

$$\widehat{Q}\left(x*\left(y\vec{\wedge}x\right)\right)\geqslant\widehat{Q}(x*y)\geqslant\widehat{Q}(\omega).$$

Hence, $x*\left(y\overrightarrow{\wedge}x\right)\in X_\omega$, and therefore X_ω is a commutative ideal of X by Lemma 2.5. □

Theorem 3.33. *Any commutative ideal of a BCK-algebra X can be realized as level commutative ideals of some m-polar fuzzy commutative ideal of X.*

Proof. Suppose C is a commutative ideal of BCK-algebra X and let $\widehat{Q}$ be an m-polar fuzzy set in X defined by

$$Q(x) = \begin{cases} \widehat{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m), & \text{if } x \in C \\ \widehat{0} = (0, 0, \ldots, 0), & \text{if } x \notin C, \end{cases}$$

where $\widehat{\alpha} \in (0,1)^m$. Let $x, y, z \in X$. We will divide into the following cases to verify that $\widehat{Q}$ is an m-polar fuzzy commutative ideal of X.

If $(x * y) * z \in C$ and $z \in C$, then $x * \left(y \overrightarrow{\wedge} x\right) \in C$. Thus

$$\begin{aligned} \widehat{Q}((x * y) * z) &= \widehat{Q}(z) \\ &= \widehat{Q}\left(x * \left(y \overrightarrow{\wedge} x\right)\right) \\ &= \widehat{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m), \end{aligned}$$

and so (18) is clearly verified.

If $(x * y) * z \notin C$ and $z \notin C$, then $\widehat{Q}((x * y) * z) = \widehat{Q}(z) = \widehat{0} = (0, 0, \ldots, 0)$. Hence,

$$\widehat{Q}\left(x * \left(y \overrightarrow{\wedge} x\right)\right) \geqslant \inf\left\{\widehat{Q}((x * y) * z), \widehat{Q}(z)\right\}.$$

If exactly one of $(x * y) * z$ and z belongs to C, then exactly one of $\widehat{Q}((x * y) * z)$ and $\widehat{Q}(z)$ is equal to $\widehat{0} = (0, 0, \ldots, 0)$. So

$$\widehat{Q}\left(x * \left(y \overrightarrow{\wedge} x\right)\right) \geqslant \inf\left\{\widehat{Q}((x * y) * z), \widehat{Q}(z)\right\}.$$

The results above show

$$\widehat{Q}\left(x * \left(y \overrightarrow{\wedge} x\right)\right) \geqslant \inf\left\{\widehat{Q}((x * y) * z), \widehat{Q}(z)\right\}$$

for all $x, y, z \in X$. It is clear that $\widehat{Q}(0) \geqslant \widehat{Q}(x)$ for all $x \in X$. Hence, $\widehat{Q}$ is an m-polar fuzzy commutative ideal of X and obviously $\widehat{Q}_{\left[\widehat{\alpha}\right]} = C$. This completes the proof. $\square$

4. Conclusions

An m-polar fuzzy model is a generalized form of a bipolar fuzzy model. The m-polar fuzzy models provide more precision, flexibility and compatibility to the system when more than one agreements are to be dealt with. In this article, we have discussed the ideal theory of BCK/BCI-algebras based on m-polar fuzzy sets. We have introduced the notions of m-polar fuzzy subalgebras and m-polar fuzzy (closed, commutative) ideals, and investigated several properties. We have considered characterizations of m-polar fuzzy subalgebras and m-polar fuzzy (commutative) ideals. We have also discussed relations between m-polar fuzzy subalgebras, m-polar fuzzy ideals and m-polar fuzzy commutative ideals. The concepts proposed in this article may be extended further to various kind of ideals in BCK/BCI-algebras, for example, a-ideals, (positive) implicative ideals, n-fold (positive) implicative ideals and n-fold commutative ideals. Furthermore, the work presented in this paper may be extended to several algebraic structures, for example, BCH-algebras, BCC-algebras, B-algebras, BRK-algebras, semigroups, semirings and lattice implication algebras.

References

Akram, M., Farooq, A., 2016. m-Polar fuzzy lie ideals of lie algebras. Quasigroups Relat. Syst. 24 (2), 141–150.
Akram, M., Farooq, A., Shum, K.P., 2016. On m-polar fuzzy lie subalgebras. Ital. J. Pure Appl. Math. 36, 445–454.
Akram, M., Sarwar, M., 2018. New applications of m-polar fuzzy competition graphs. New Math. Nat. Comput. 14 (2), 249–276.
Al-Masarwah, A., Ahmad, A.G., 2018a. Doubt bipolar fuzzy subalgebras and ideals in BCK/BCI-algebras. J. Math. Anal. 9 (3), 9–27.
Al-Masarwah, A., Ahmad, A.G., 2018b. Novel concepts of doubt bipolar fuzzy H-ideals of BCK/BCI-algebras. Int. J. Innovat. Comput., Inf. Control 14 (6), 2025–2041.
Al-Masarwah, A., Ahmad, A.G., 2018c. On some properties of doubt bipolar fuzzy H-ideals in BCK/BCI-algebras. Eur. J. Pure Appl. Math. 11 (3), 652–670.
Chen, J., Li, S., Ma, S., Wang, X., 2014. m-Polar fuzzy sets: an extension of bipolar fuzzy sets. Sci. World J. 2014, 8. Article Id 416530.
Farooq, A., Alia, G., Akram, M., 2016. On m-polar fuzzy groups. Int. J. Algebra Stat. 5 (2), 115–127.
Huang, Y., 2006. BCI-Algebra. Science Press, Beijing, China.
Imai, Y., Iséki, K., 1966. On axiom systems of propositional calculi. Proc. Japan. Acad. Ser. A, Math. Sci. 42, 19–21.
Iséki, K., 1966. An algebra related with a propositional calculus. Proc. Japan. Acad. Ser. A, Math. Sci. 42, 26–29.
Iséki, K., Tanaka, S., 1978. An introduction to the theory of BCK-algebras. Math. Jpn. 23, 1–26.
Jun, Y.B., Ahn, S.S., Muhiuddin, G., 2014. Hesitant fuzzy soft subalgebras and ideals in BCK/BCI-algebras. Sci World J. 2014, 7. Article ID 763929.
Jun, Y.B., Muhiuddin, G., Al-roqi, A.M., 2013. Ideal theory of BCK/BCI-algebras based on double-framed soft sets. Appl. Math. Inf. Sci. 7 (5), 1879–1887.
Jun, Y.B., Muhiuddin, G., Ozturk, M.A., Roh, E.H., 2017a. Cubic soft ideals in BCK/BCI-algebras. J. Comput. Anal. Appl. 22 (5), 929–940.
Jun, Y.B., Roh, E.H., 1994. Fuzzy commutative ideals of BCK-algebras. Fuzzy. Sets. Syst. 64, 401–405.
Jun, Y.B., Smarandache, F., Bordbar, H., 2017b. Neutrosophic N-structures applied to BCK/BCI-algebras. Information 8 (4), 128.
Jun, Y.B., Smarandache, F., Ozturk, M.A., 2018a. Commutative falling neutrosophic ideals in BCK-algebras. Neutrosophic Sets Syst. 20, 44–53.
Jun, Y.B., Song, S.Z., Kim, S.J., 2018b. Cubic interval-valued intuitionistic fuzzy sets and their application in BCK/BCI-algebras. Axioms 7 (1), 7.
Kim, S.J., Song, S.Z., Jun, Y.B., 2018. Generalizations of neutrosophic subalgebras in BCK/BCI-algebras based on neutrosophic points. Neutrosophic Sets Syst. 20, 26–35.
Lee, K.J., 2009. Bipolar fuzzy subalgebras and bipolar fuzzy ideals of BCK/BCI-algebras. Bull. Korean. Math. Soc. 32 (3), 361–373.
Lee, K.J., Jun, Y.B., Zhang, X., 2012. N-subalgebras of type $(\in, \in \vee q)$ based on point N-structures in BCK/BCI-algebras. Commun. Korean. Math. Soc. 27 (3), 431–439.
Meng, J., 1991. Commutative ideals in BCK-algebras. Pure Appl. Math. 9, 49–53.
Meng, J., Jun, Y.B., 1994. BCK-Algebras. Kyungmoon Sa Co., Seoul, Korea.
Muhiuddin, G., Bordbar, H., Smarandache, F., Jun, Y.B., 2018. Further results on $(\in, \in)$-neutrosophic subalgebras and ideals in BCK/BCI-algebras. Neutrosophic Sets Syst. 20, 36–43.
Muhiuddin, G., Kim, H.S., Song, S.Z., Jun, Y.B., 2017. Hesitant fuzzy translations and extensions of subalgebras and ideals in BCK/BCI-algebras. J. Intell. Fuzzy Syst. 32 (1), 43–48.
Sarwar, M., Akram, M., 2017. New applications of m-polar fuzzy matroids. Symmetry 9 (12), 319.
Song, S.Z., Khan, M., Smarandache, F., Jun, Y.B., 2017. Interval neutrosophic sets applied to ideals in BCK/BCI-algebras. Neutrosophic Sets Syst. 18, 16–26.
Xi, O.G., 1991. Fuzzy BCK-algebras. Math. Jpn. 24, 935–942.
Zadeh, L.A., 1965. Fuzzy sets. Inf. Control. 8, 338–353.
Zhang, W.R., 1994. Bipolar fuzzy sets and relations: a computational framework for cognitive and modeling and multiagent decision analysis. Proc. of IEEE Conf., 305–309
Zhang, X., Ma, Y., Smarandache, F., 2017. Neutrosophic regular filters and fuzzy regular filters in pseudo-BCI algebras. Neutrosophic Sets Syst. 17, 10–15.