Journal of King Saud University – Science 35 (2023) 102526

Contents lists available at ScienceDirect

Journal of King Saud University – Science

journal homepage: www.sciencedirect.com

Original article

A study on curvature relations of conformal generic submersions

Fahad Sikander [a], Tanveer Fatima [b,*], Sana Abdulkream Alharbi [b]

[a] Department of Basic Sciences, College of Science and Theoretical Studies, Saudi Electronic University, Jeddah Branch, Saudi Arabia
[b] Department of Mathematics and Statistics, College of Sciences, Taibah University, Yanbu-41911, Saudi Arabia

ARTICLE INFO

Article history:
Received 3 October 2022
Revised 30 November 2022
Accepted 23 December 2022
Available online 28 December 2022

Keywords:
Riemannian submersions
Conformal Riemannian submersions
Generic Riemannian submersions, Almost product manifold

sectional curvatures

ABSTRACT

The present paper enlighten us with the study of the conformal generic submersion whose total space is locally product Riemannian manifold. We investigate;the geometry of the foliations arisen from the definition of conformal generic submersion and provided some decomposition theorem for the total space of the submersion. Meanwhile, harmonicity of conformal generic submersion is also discussed. We mainly established relations between the sectional curvatures of fibres, total space and base manifold and discussed the related consequences. A non-trivial example have also been discussed to make the content wrth.

1. Introduction

Throughout paper, we will use some abbreviations as follows:

Riemannian Manifold	RS
Riemannian Submersion	RS
Conformal Generic Submersions	CGS
Locally Product Riemannian	l.p.R.
grad	$\mathbb{G}$

One of the most popular areas of study in differential geometry is submanifold theory, and establishing proper smooth maps between two manifolds is one of the quickest ways to compare them and transfer certain structures from one manifold to another. If there are two manifolds, then a differential map is said to be an immersion (submersion) if its rank coincide with the dimension of the source manifold (target manifold). Moreover, these are called isometric immersions (isometric submersions) if the maps defined between manifolds are isometric. O'Neill (1966) and Gray (1997) were the first to address the idea of Riemannian submersions between Riemannian manifolds. The Riemannian submersions are being used extensively in both mathematics and physics. Particularly in the context of the Yang-Mills theory (Bourguignon and Lawson, 1981; Baird and Wood, 2003), Kalauza-Klien theory (Bourguignon, 1990), super-gravity and super-string theories (Ianus and Visinescu, 1991; Ianus and Visinescu, 1987), redundant robotic chains (Altafini, 2004) etc.

Riemannian submersions eventually developed into a suitable approach for describing the geometry of RMs with differential structures. The first study of RS between RMs equipped with an additional structure of almost complex type was executed by Watson (1976). In most instances, Watson established that the structure of the base manifold and each fiber is the same as that of the total space by defining an almost Hermitian submersion between almost Hermitian manifolds. In this case, the RS is also a complex mapping and consequently, the vertical and horizontal distributions are invariant with respect to the almost complex structure of the total manifold of the submersion. Almost Hermitian submersions have been extended to the almost contact manifolds (Chinea, 1985), locally conformal Kaheler manifolds (Marrero and Rocha, 1994) and QR manifolds (Ianus et al., 2008). Escobales (Escobales, 1978) studied RS from complex projective space onto a RM under the assumption that the fibres are connected, complex, totally geodesic submanifolds. In fact this

* Corresponding author at: Department of Mathematics and Statistics, College of Sciences, Taibah University, Yanbu-41911, Saudi Arabia.

E-mail addresses: f.sikander@seu.edu.sa (F. Sikander), tansari@taibahu.edu.sa (T. Fatima), saaharbi@taibahu.edu.sa (S.A. Alharbi).

Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

https://doi.org/10.1016/j.jksus.2022.102526
1018-3647/

assumption also implies that the vertical distribution is invariant with respect to the almost complex structure.

The vertical and horizontal distributions are invariant, which is a characteristic shared by all the submersions mentioned above. Then, in 2010, B. Sahin presented the notion of anti-invariant RS, which is referenced in (Şahin, 2010). Anti-invariant RS are the RS from almost Hermitian manifolds to RMs such that the vertical distributions (or, for that matter the fibres) are anti-invariant under the almost complex structure of total manifold. Such submersions are demonstrated to exhibit a wealth of geometrical characteristics and are helpful for assessing the geometry of the total manifold of the submersion. The vertical and horizontal distributions are reversed by the almost complex structure of the total manifold in a Lagrangian submersion, which is a special case of an anti-invariant RS. (Şahin, 2010; Şahin, 2011; Tastan, 2014). Following that, many RS have been investigated such as Semi-invariant submersion (Şahin, 2011), Slant submersion (Şahin, 2011), Semi-slant submersion (Park and Prasad, 2013), Generic submersion (Ali and Fatima, 2013) etc.

On the other hand, horizontally conformal submersions caught attention in 1992 and studied by Gudmundsson and Wood (1997). Horizontally conformal submersions are the generalization of RS and unlikely to the RS, horizontally conformal submersions do not preserve the distance between the points but they preserve the angles between the vector fields. This property allows one to transfer specific properties of a manifold to another manifold by deforming such properties. Recently, M. A. Akyol and B. Sahin defined conformal anti-invariant submersion (Akyol and Şahin, 2016) and later on conformal semi-invariant, conformal slant, conformal semi-slant and CGS are studied by M. A Akyol and others (we refer to Akyol and Şahin, 2017; Akyol, 2017; Akyol and Şahin, 2019; Akyol, 2017; Akyol, 2021 Özdemir et al., 2017).

The current article's objective is to investigate CGS from l.p.R manifolds onto RMs. The manuscript is structured as follows: Section 2 reviews the fundamental ideas of RS and horizontally conformal submersions and Section 3 explains the requirement for a l.p.R manifold. In Section 4, the definition and an example of CGS are covered. Then, several fundamental conclusions are reached, including equivalent criteria for the distributions' integrability and conditions that are required and adequate for the distributions to describe completely geodesic foliations. Section 4 also lists the criteria for CGS to be a harmonic map. The sectional curvature relations of the total manifold, fibers and manifolds are the only focus of Section 5.

2. Conformal Riemannian Submersions

We shall review the concept of conformal submersions, which are one of a large class of conformal maps, but in this study, we won't look at these maps.

Definition 1. (Baird and Wood, 2003) " Let $(\mathbb{M}, g)$ and $(\mathbb{N}, h)$ be two RMs with dimensions m and n, respectively, and let $\Pi : (\mathbb{M}, g) \to (\mathbb{N}, h)$ be a $C^\infty-$ differentiable map between them. If either

(i) $\Pi_{*p} = 0$, or

(ii) Π_{*p} maps horizontal space $\mathscr{H}_p = (ker\Pi_{*p})^\perp$ conformally onto $T_{\Pi(p)}\mathbb{N}$, i.e., Π_{*p} is surjective and there exists a number $\Lambda(p) \neq 0$ such that

$$h(\Pi_{*p}\mathbb{X}, \Pi_{*p}\mathbb{Y}) = \Lambda(p) g(\mathbb{X}, \mathbb{Y}), \tag{1}$$

for any $\mathbb{X}, \mathbb{Y} \in \Gamma(ker\Pi_*)^\perp$.

Then, Π is called horizontally weakly conformal or semi conformal at $p \in \mathbb{M}$. ".

In the definition above, if "(i) is met, we say that p is a *critical point* of Π and if (ii) is met we refer point p as a *regular point*. At a critical point, Π_{*p} has rank 0 where as at a regular point Π_{*p} has rank n and represents submersion. The number $\Lambda(x)$, which is necessarily non-negative, is called the *square dilation* of Π at p. The square root $\lambda(p) = \sqrt{\Lambda(p)}$ is called the dilation of Π at p. The map Π is called *horizontally weakly conformal* or *semi conformal* on M if it is horizontally weakly conformal at every point of M. It is clear that we refer to a *horizontally conformal submersion* if Π has no critical points. Let $\Pi : \mathbb{M} \to \mathbb{N}$ be a submersion. A vector field E on $\mathbb{M}$ is said to be projectable if there exists a vector field $\check{E}$ on $\mathbb{N}$, such that $\Pi_*(E_p) = \check{E}_{\Pi(p)}$ for all $p \in \mathbb{M}$. In this case E and $\check{E}$ are called $\Pi-$ related. A horizontal vector field $\mathbb{Y}$ on $(\mathbb{M}, g)$ is called basic, if it is projectable. It is well known fact that if $\check{Z}$ is a vector field on $\mathbb{N}$, then there exists a unique basic vector field Z on $\mathbb{M}$, such that Z and $\check{Z}$ are $\Pi-$ related. The vector field Z is called the horizontal lift of $\check{Z}$ (O'Neill, 1966).".

In (O'Neill, 1966), the basic tensors of submersions were presented. They function similar to the second fundamental form of immersion. More precisely, O'Neill's tensors $\mathscr{T}$ and $\mathscr{A}$ defined for any vector fields $E_1, E_2 \in \Gamma(TM)$ by

$$\mathscr{A}_{E_1}E_2 = \mathscr{V}\nabla_{\mathscr{H}E_1}\mathscr{H}E_2 + \mathscr{H}\nabla_{\mathscr{H}E_1}\mathscr{V}E_2 \tag{2}$$

$$\mathscr{T}_{E_1}E_2 = \mathscr{H}\nabla_{\mathscr{V}E_1}\mathscr{V}E_2 + \mathscr{V}\nabla_{\mathscr{V}E_1}\mathscr{H}E_1, \tag{3}$$

where $\mathscr{V}$ and $\mathscr{H}$ are the vertical and horizontal projections of E_1 and E_2 (see Falcitelli et al., 2004). On the other hand, from (2) and (3), we have

$$\nabla_V\mathbb{W} = \mathscr{T}_V\mathbb{W} + \hat{\nabla}_V\mathbb{W} \tag{4}$$

$$\nabla_{\mathbb{V}}\mathbb{X} = \mathscr{H}\nabla_{\mathbb{V}}\mathbb{X} + \mathscr{T}_{\mathbb{V}}\mathbb{X} \tag{5}$$

$$\nabla_{\mathbb{X}}\mathbb{V} = \mathscr{A}_{\mathbb{X}}\mathbb{V} + \mathscr{V}\nabla_{\mathbb{X}}\mathbb{V} \tag{6}$$

$$\nabla_{\mathbb{X}}\mathbb{Y} = \mathscr{H}\nabla_{\mathbb{X}}\mathbb{Y} + \mathscr{A}_{\mathbb{X}}\mathbb{Y} \tag{7}$$

for any $\mathbb{X}, \mathbb{Y} \in \Gamma(ker\Pi_*)^\perp$ and $\mathbb{V}, \mathbb{W} \in \Gamma(ker\Pi_*)$, where $\hat{\nabla}_{\mathbb{V}}\mathbb{W} = \mathscr{V}\nabla_{\mathbb{V}}\mathbb{W}$. If $\mathbb{X}$ is basic, then $\mathscr{H}\nabla_{\mathbb{V}}\mathbb{X} = \mathscr{A}_{\mathbb{X}}\mathbb{V}$. It is easily seen that for any $\mathbb{X} \in \Gamma(ker\Pi_*)^\perp$ and $\mathbb{V} \in \Gamma(ker\Pi_*)$, the linear operators $\mathscr{T}_{\mathbb{V}}, \mathscr{A}_{\mathbb{X}} : T_p\mathbb{M} \to T_p\mathbb{M}$ are skew-symmetric, that is

$$g(\mathscr{T}_{\mathbb{V}}E_1, E_2) = -g(E_1, \mathscr{T}_{\mathbb{V}}E_2) \text{ and } g(\mathscr{A}_{\mathbb{X}}E_1, E_2) = -g(E_1, \mathscr{A}_{\mathbb{X}}E_2)$$

for all $E_1, E_2 \in \Gamma(T_p\mathbb{M})$ and $p \in \mathbb{M}$. The restriction of $\mathscr{T}$ to the vertical distribution $\mathscr{T}|_{\mathbb{V}\times\mathbb{V}}$ is precisely the same as the second fundamental form of the fibers, as can also be seen. We conclude that Π has totally geodesic fibres if and only if $\mathscr{T} \equiv 0$ because $\mathscr{T}_{\mathbb{V}}$ is skew-symmetric. The following are the results for the horizontal conformal submersion:

Proposition 1. *(Gudmundsson, 1992) "Let $\Pi : (\mathbb{M}, g) \to (\mathbb{N}, h)$ be a horizontally conformal submersion with dilation λ $\mathbb{X}, \mathbb{Y}$ be horizontal vectors fields, then*

$$\mathscr{A}_{\mathbb{X}}\mathbb{Y} = \frac{1}{2}\left\{\mathscr{V}[\mathbb{X}, \mathbb{Y}] - \lambda^2 g(\mathbb{X}, \mathbb{Y})\mathbb{G}_{\mathscr{V}}\left(\frac{1}{\lambda^2}\right)\right\}. \tag{8}$$

We see that the skew-symmetric part of $\mathscr{A}|_{(ker\Pi_*)^\perp\times(ker\Pi_*)^\perp}$ measures the obstruction integrability of the horizontal distribution $(ker\Pi_*)^\perp$.

The following curvature relations for horizontally conformal submersion are now brought to mind from (Gromoll et al., 1975 and Gudmundsson, 1992).

Theorem 1. *"Let $(\mathbb{M}^m, g, \nabla, R)$ and $(\mathbb{N}^n, h, \nabla, R^*)$ be two RMs, where $m > n \geqslant 2$. Let R and R^* be the curvature tensors on $\mathbb{M}$ and $\mathbb{N}$, respectively. Let $\Pi : (\mathbb{M}, g) \to (\mathbb{N}, h)$ be a horizontally conformal submersion, with dilation $\lambda : \mathbb{M} \to \mathbb{R}^+$ and let $\hat{R}$ be the curvature tensor of the fibres of the submersion. If $\mathbb{X}, \mathbb{Y}, \mathbb{Z}, \mathbb{H}$ are horizontal $\mathbb{U}, \mathbb{V}, \mathbb{W}, \mathbb{G}$ vertical vectors, then*

$$g(R(\mathbb{U},\mathbb{V})\mathbb{W},\mathbb{G}) = g\left(\hat{R}(\mathbb{U},\mathbb{V})\mathbb{W},\mathbb{G}\right) + g(\mathscr{T}_{\mathbb{U}}\mathbb{W},\mathscr{T}_{\mathbb{V}}\mathbb{G}) - g(\mathscr{T}_{\mathbb{V}}\mathbb{W},\mathscr{T}_{\mathbb{U}}\mathbb{G}), \tag{9}$$

$$g(R(\mathbb{U},\mathbb{V})\mathbb{W},\mathbb{X}) = \mathbb{g}\left((\nabla_{\mathbb{U}}\mathbb{T})_{\mathbb{V}}\mathbb{W},\mathbb{X}\right) - \mathbb{g}\left((\nabla_{\mathbb{V}}\mathbb{T})_{\mathbb{U}}\mathbb{W},\mathbb{X}\right), \tag{10}$$

$$g(\mathbb{R}(\mathbb{U},\mathbb{X})\mathbb{Y},\mathbb{V}) = g\left((\nabla_{\mathbb{U}}A)_{\mathbb{X}}\mathbb{Y},\mathbb{V}\right) + g(A_{\mathbb{X}}\mathbb{U},A_{\mathbb{Y}}\mathbb{V}) - g\left((\nabla_{\mathbb{X}}T)_{\mathbb{U}}\mathbb{Y},\mathbb{V}\right) - g(\mathscr{T}_{\mathbb{V}}\mathbb{Y},\mathscr{T}_{\mathbb{U}}\mathbb{X}) + \lambda^2 g(A_{\mathbb{X}}\mathbb{Y},\mathbb{U})g\left(\mathbb{V},\mathbb{G}_{\mathscr{V}}\left(\tfrac{1}{\lambda^2}\right)\right), \tag{11}$$

$$\begin{aligned} g(R(\mathbb{X},\mathbb{Y})\mathbb{Z},\mathbb{H}) &= \frac{1}{\lambda^2}h(R^*(\mathbb{X}_*,\mathbb{Y}_*),\mathbb{Z}_*,\mathbb{H}_*) + \frac{1}{4}\{g(\mathscr{V}[\mathbb{X},\mathbb{Z}],\mathscr{V}[\mathbb{Y},H]) \\ &\quad - g(\mathscr{V}[\mathbb{Y},\mathbb{Z}],\mathscr{V}[\mathbb{X},\mathbb{H}]) + 2g(\mathscr{V}[\mathbb{X},\mathbb{Y}],\mathscr{V}[\mathbb{Z},\mathbb{H}])\} \\ &\quad + \tfrac{\lambda^2}{2}\left\{g(\mathbb{X},\mathbb{Z})g\left(\nabla_{\mathbb{Y}}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),H\right) - g(\mathbb{Y},\mathbb{Z})g\left(\nabla_{\mathbb{X}}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),\mathbb{H}\right)\right. \\ &\quad \left. + g(\mathbb{Y},\mathbb{H})g\left(\nabla_{\mathbb{X}}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),\mathbb{Z}\right) - g(\mathbb{X},\mathbb{H})g\left(\nabla_{\mathbb{Y}}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),\mathbb{Z}\right)\right\} \\ &\quad + \tfrac{\lambda^4}{4}\left\{(g(\mathbb{X},\mathbb{H})g(\mathbb{Y},\mathbb{Z}) - g(\mathbb{Y},\mathbb{H})g(\mathbb{X},\mathbb{Z}))\|\mathbb{G}\left(\tfrac{1}{\lambda^2}\right)\|^2\right. \\ &\quad \left. + g\left(\mathbb{X}\left(\tfrac{1}{\lambda^2}\right)\mathbb{Y} - \mathbb{Y}\left(\tfrac{1}{\lambda^2}\right)\mathbb{X}, \mathbb{H}\left(\tfrac{1}{\lambda^2}\right)\mathbb{Z} - \mathbb{Z}\left(\tfrac{1}{\lambda^2}\right)H\right)\right\}. \end{aligned} \tag{12}$$

"Let $(\mathbb{M}, g)$ and $(\mathbb{N}, g)$ be RMs and suppose that $\varphi : \mathbb{M} \to \mathbb{N}$ is a smooth map between them. The differential of φ_* of φ can be viewed a section of the bundle $Hom(T\mathbb{M}, \varphi^{-1}T\mathbb{N}) \to \mathbb{M}$, where $\varphi^{-1}T\mathbb{N}$ is the pullback bundle which has fibres $(\varphi^{-1}T\mathbb{N})_p = \mathscr{T}_{\varphi(p)}\mathbb{N}$, $p \in \mathbb{M}$. $Hom(T\mathbb{M}, \varphi^{-1}T\mathbb{N})$ has a connection ∇ induced from the Levi–Civita connection $\nabla^{\mathbb{M}}$ and the pullback connection. Then, the second fundamental form of φ is given by

$$\nabla\varphi_* : T\mathbb{M} \times T\mathbb{M} \to T\mathbb{N}$$

defined by

$$(\nabla\varphi_*)(\mathbb{X},\mathbb{Y}) = \nabla^{\varphi}_{\mathbb{X}}\varphi_*(\mathbb{Y}) - \varphi_*(\nabla^{\mathbb{M}}_{\mathbb{X}}\mathbb{Y}) \tag{13}$$

for $\mathbb{X}, \mathbb{Y} \in \Gamma(T\mathbb{M})$, where ∇^{φ} is the pullback connection." The symmetry of the second fundamental form is well recognised. "A smooth map $\varphi : (\mathbb{M}, g_{\mathbb{M}}) \to (\mathbb{N}, g_{\mathbb{N}})$ is said to be harmonic if $trace(\nabla\varphi_*) = 0$. On the other hand, the tension field of φ is the section $\tau(\varphi)$ of $\Gamma(\varphi^{-1}TN)$ defined by

$$\tau(\varphi) = \mathrm{div}\varphi_* = \sum_{i=1}^{m}(\nabla\varphi_*)(e_i, e_i), \tag{14}$$

where $\{e_1, \ldots, e_m\}$ is the orthonormal frame on $\mathbb{M}$. Thus, it follows that $\tau(\varphi) = 0$ is the neccesary and sufficient condition under which φ is harmonic." For more information, see (Baird and Wood, 2003).

Lemma 1. *(Urakawa, 1993) "Consider that $\varphi : \mathbb{M} \to \mathbb{N}$ is a smooth map between the RMs $(\mathbb{M}, g)$ and $(\mathbb{N}, h)$. Then*

$$\nabla^{\varphi}_{\mathbb{X}}\varphi_*(\mathbb{Y}) - \nabla^{\varphi}_{\mathbb{Y}}\varphi_*(\mathbb{X}) - \varphi_*([\mathbb{X},\mathbb{Y}]) = 0, \tag{15}$$

for any $\mathbb{X}, \mathbb{Y} \in \Gamma(T\mathbb{M})$."

Last but not least, consider the following (Baird and Wood, 2003).

Lemma 2. *"Assume that the submersion $\Pi : \mathbb{M} \to \mathbb{N}$ is horizontally conformal. Then, for any vertical fields $\mathbb{V}, \mathbb{W}$ and horizontal vector fields $\mathbb{X}, \mathbb{Y}$,*

(i) $(\nabla\Pi_*)(\mathbb{X},\mathbb{Y}) = \mathbb{X}(\ln\lambda)\Pi_*\mathbb{Y} + \mathbb{Y}(\ln\lambda)\Pi_*\mathbb{X} - g(\mathbb{X},\mathbb{Y})\Pi_*(\mathbb{G}\ln\lambda)$;
(ii) $(\nabla\Pi_*)(\mathbb{V},\mathbb{W}) = -\Pi_*(\mathscr{T}_{\mathbb{V}}\mathbb{W})$;
(iii) $(\nabla\Pi_*)(\mathbb{X},\mathbb{V}) = -\Pi_*(\nabla_{\mathbb{X}}\mathbb{V}) = -\Pi_*(A_{\mathbb{X}}\mathbb{V})$."

3. Locally Product Riemannian manifolds

"Assume that $\mathbb{M}$ is an m-dimensional manifold with a tensor $\mathscr{F}$ of type (1,1) such that $\mathscr{F}^2 = I, (\mathscr{F} \neq I)$. Then, we assert that $\mathbb{M}$ is an almost product manifold with $\mathscr{F}$ almost product structure. We place

$$\mathscr{P} = \frac{1}{2}(I + \mathscr{F}), \quad \mathscr{Q} = \frac{1}{2}(I - \mathscr{F}).$$

It's simple to observe that

$$\mathscr{P} + \mathscr{Q} = I, \quad \mathscr{P}^2 = \mathscr{P}, \quad \mathscr{Q}^2 = \mathscr{Q}, \quad \mathscr{P}\mathscr{Q} = \mathscr{Q}\mathscr{P} = 0, \quad \mathscr{F} = \mathscr{P} - \mathscr{Q}.$$

As a result $\mathscr{P}$ and $\mathscr{Q}$ define two complimentary distributions. We note that the $\mathscr{F}$'s eigenvalues are either +1 or -1. If $\mathbb{M}$, an almost product manifold, admits g, a Riemannian metric, then

$$g(\mathscr{F}E_1, \mathscr{F}E_2) = g(E_1, E_2), \tag{16}$$

for any vector fields E_1 and E_2 on $\mathbb{M}$, then $\mathbb{M}$ is called an almost product RM, denoted by $(\mathbb{M}, g, F)$ and M is called a l.p.R. manifold if $\mathscr{F}$ is parallel with respect to ∇ i.e.,

$$(\nabla_{E_1}\mathscr{F})E_2 = 0, \quad E \in \Gamma(TM), \tag{17}$$

where ∇ denotes the Levi–Civita connection on $\mathbb{M}$ with respect to g (Yano and Kon, 1984)."

4. Conformal generic submersions

This section assesses the study of CGS as the total space of the submersions is l.p.R. manifold. We define the CGS and provide a non-trivial example to assure the existence of such submersion.

Definition 2. Let $(\mathbb{M}, g, \mathscr{F})$ be a l.p.R. manifold with the product structure $\mathscr{F}$ and $(\mathbb{N}, h)$ be a RM. Consider a horizontally conformal submersion $\Pi : (\mathbb{M}, g, \mathscr{F}) \to (\mathbb{N}, h)$. Then Π is called CGS if there are two orthogonal complementary distributions $\mathbb{D}$ and $\mathbb{D}_1$ of $\ker\Pi_*$ such that

$$ker\Pi_* = \mathbb{D} \oplus \mathbb{D}_1, \quad F(\mathbb{D}) = \mathbb{D}, \tag{18}$$

where $(\mathbb{D}_1)_p = ker\Pi_* \cap F(ker\Pi_*), p \in \mathbb{M}$, a complex subspace of a vertical space $\mathscr{V}_p$, has a constant dimension along $\mathbb{M}$ and defines a differentiable distribution on $\mathbb{M}$. The distribution $\mathbb{D}_1$ is called the purely real distribution.

As observed that the vertical distribution $ker\Pi_*$ is integrable. Therefore, this defintion simply implies that integral manifolds, $\Pi^{-1}(q), q \in \mathbb{N}$ of the submersion are generic submanifold of M. For generic submersions, we refer to (Chen, 1981).

Let Π be a CGS from a l.p.R. manifold $(\mathbb{M}, g, \mathscr{F})$ onto a RM $(\mathbb{N}, h)$. For any $U \in \Gamma(ker\Pi_*)$,

$$F\mathbb{U} = \alpha\mathbb{U} + \beta\mathbb{U}, \tag{19}$$

where $\alpha\mathbb{U} \in \Gamma(ker\Pi_*)$ and $\beta\mathbb{U} \in \Gamma(ker\Pi_*)^{\perp}$. Also for any $\mathbb{X} \in \Gamma(ker\Pi_*)^{\perp}$,

$$F\mathbb{X} = B\mathbb{X} + C\mathbb{X}, \tag{20}$$

where $B\mathbb{X} \in \Gamma(\mathbb{D}_1)$ and $C\mathbb{X} \in \Gamma(v)$. Then, $(ker\Pi_*)^{\perp}$ is decomposed as

$$(ker\Pi_*)^{\perp} = \beta\mathbb{D}_1 \oplus v, \tag{21}$$

where ν denotes the orthogonal complement of $\beta\mathbb{D}_1$ in $(ker\Pi_*)^\perp$ and is invariant under the almost product structure $\mathscr{F}$.

We now present a non-trivial example of CGS whose total space is an almost product manifold.

Consider an Euclidean space $\mathbb{R}^8$ with coordinates $(a^1, a^2, \dots, a^8)$. We denote by $\mathscr{F}$ the compatible almost product structure on $\mathbb{R}^8$ as follows;

$$\mathscr{F}(a^1, a^2, \dots, a^8) = \tfrac{1}{\sqrt{2}}(-a^2 - a^8, -a^1 - a^7, -a^4 + a^6, -a^3 + a^5, a^6 + a^4, a^5 + a^3, \ a^8 - a^2, a^7 - a^1).$$

Example 1. Let $\Pi : \mathbb{R}^8 \to \mathbb{R}^2$ be a submersion defined by

$$\Pi(u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8) = e^7\left(\frac{1}{\sqrt{2}}(u_3 + u_7), \frac{1}{\sqrt{2}}(u_3 - u_7)\right).$$

Then it follows that

$$ker\Pi_* = \left\langle V_1 = \frac{\partial}{\partial x_1}, V_2 = \frac{\partial}{\partial x_2}, \ V_3 = \frac{\partial}{\partial x_4}, V_4 = \frac{\partial}{\partial x_5}, V_5 = \frac{\partial}{\partial x_6}, V_6 = \frac{\partial}{\partial x_8}\right\rangle$$

and

$$(ker\Pi_*)^\perp = \left\langle \mathbb{X}_1 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_7}\right), \mathbb{X}_2 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_7}\right)\right\rangle,$$

Moreover,

$$\begin{aligned}
\mathscr{F}V_1 &= \tfrac{1}{\sqrt{2}}V_2 - \tfrac{1}{\sqrt{2}}V_6, \mathscr{F}V_2 &&= -\tfrac{1}{\sqrt{2}}V_1 - \tfrac{1}{2}\mathbb{X}_1 + \tfrac{1}{2}\mathbb{X}_2,\\
\mathscr{F}V_3 &= \tfrac{1}{\sqrt{2}}V_4 - \tfrac{1}{2}\mathbb{X}_1 - \tfrac{1}{2}\mathbb{X}_2, FV_4 &&= -\tfrac{1}{\sqrt{2}}V_3 + \tfrac{1}{\sqrt{2}}V_5,\\
\mathscr{F}V_5 &= \tfrac{1}{\sqrt{2}}V_4 + \tfrac{1}{2}\mathbb{X}_1 + \tfrac{1}{2}\mathbb{X}_2, \mathscr{F}V_6 &&= -\tfrac{1}{\sqrt{2}}V_1 + \tfrac{1}{2}\mathbb{X}_1 - \tfrac{1}{2}\mathbb{X}_2.
\end{aligned}$$

Hence, $\mathbb{D} = V_1, V_4$ and $\mathbb{D}_1 = V_2, V_3, V_5, V_6$. Also by direct computations, we obtain

$$g_{\mathbb{R}^8}(\mathbb{X}_1, \mathbb{X}_1) = (e^7)^{-2} g_{\mathbb{R}^2}(\Pi_*\mathbb{X}_1, \Pi_*\mathbb{X}_1) \text{ and}$$
$$g_{\mathbb{R}^8}(\mathbb{X}_2, \mathbb{X}_2) = (e^7)^{-2} g_{\mathbb{R}^2}(\Pi_*\mathbb{X}_2, \Pi_*\mathbb{X}_2),$$

where $g_{\mathbb{R}^8}$ and $g_{\mathbb{R}^2}$ denote the standard metrices on $\mathbb{R}^8$ and $\mathbb{R}^2$, respectively. Thus Π is CGS with $\lambda = e^7$.

We start with the preliminary results of CGS.

Proposition 2. *Let $\Pi : (\mathbb{M}, g, \mathscr{F}) \to (\mathbb{N}, h)$ be a CGS from a l.p.R. manifold $(\mathbb{M}, g, \mathscr{F})$ onto a RM $(\mathbb{N}, h)$. Then*

$(i)\ \alpha\mathbb{D} = \mathbb{D}$, $(ii)\ \beta\mathbb{D}_1 = 0$, $(iii)\ \alpha\mathbb{D}_1 \subset \mathbb{D}_1$, $(iv)\ B(ker\Pi_*)^\perp = \mathbb{D}_1$,
$(v)\ \alpha^2 + B\beta = id$, $(vi)\ C^2 + \beta B = id$, $(vii)\ \beta\alpha + C\beta = 0$, $(viii)\ BC + \alpha B = 0$.

Proof. These can easily be obtained with the help of (19)–(21).

Using (4), (5), (19) and (20), we have the covariant derivative of α and β as follows;

$$(\nabla_U\alpha)V = B\mathscr{T}_UV - \mathscr{T}_U\beta V \tag{22}$$
$$(\nabla_U\beta)V = C\mathscr{T}_UV - \mathscr{T}_U\alpha V \tag{23}$$
$$(\nabla_U\alpha)V = \hat{\nabla}_U\alpha V - \alpha\hat{\nabla}_UV \tag{24}$$
$$(\nabla_U\beta)V = \mathscr{A}_{\beta V}U - \beta\hat{\nabla}_UV, \tag{25}$$

for any $U, V \in \Gamma(ker\Pi_*)$.

In view of (4)–(7), (19) and (20), we have.

Lemma 3. *Let $\Pi : (\mathbb{M}, g, \mathscr{F}) \to (B, h)$ be a CGS from a l.p.R. manifold $(\mathbb{M}, g, \mathscr{F})$ onto a RM (B, h). Then*

(a) $\mathscr{A}_{\mathbb{X}}B\mathbb{Y} + \mathscr{H}\nabla_{\mathbb{X}}C\mathbb{Y} = C\mathscr{H}\nabla_{\mathbb{X}}\mathbb{Y} + \beta\mathscr{A}_{\mathbb{X}}\mathbb{Y}$
$\mathscr{V}\nabla_{\mathbb{X}}B\mathbb{Y} + \mathscr{A}_{\mathbb{X}}C\mathbb{Y} = B\mathscr{H}\nabla_{\mathbb{X}}\mathbb{Y} + \alpha\mathscr{A}_{\mathbb{X}}\mathbb{Y}$,

(b) $\mathscr{T}_U\alpha V + \mathscr{A}_{\beta V}U = C\mathscr{T}_UV + \beta\hat{\nabla}_UV$
$\hat{\nabla}_U\alpha V + \mathscr{T}_U\beta V = B\mathscr{T}_UV + \alpha\hat{\nabla}_UV$,

(c) $\mathscr{A}_{\mathbb{X}}\alpha U + \mathscr{H}\nabla_U\beta V = C\mathscr{A}_{\mathbb{X}}U + \beta\mathscr{V}\nabla_{\mathbb{X}}U$
$\mathscr{V}\nabla_{\mathbb{X}}\alpha U + \mathscr{A}_{\mathbb{X}}\beta U = B\mathscr{A}_{\mathbb{X}}U + \alpha\mathscr{V}\nabla_{\mathbb{X}}U$,

for any $U, V \in \Gamma(ker\Pi_*)$ and $\mathbb{X}, \mathbb{Y} \in \Gamma(ker\Pi_*)^\perp$.

We proceed now to the main results of this section where first we prove the equivalent conditions for the integrability of the distributions $\mathbb{D}$ and $\mathbb{D}_1$. As vertical distribution $(ker\Pi_*)$ is integrable so far, we give the necessary and sufficient condition for the horizontal distribution $(ker\Pi_*)^\perp$ to be integrable. Also we study the geometry of foliations of all the distributions.

Theorem 2. *Let $\Pi : (\mathbb{M}, g, \mathscr{F}) \to (B, h)$ be a CGS from l.p.R. manifold $(M, g, \mathscr{F})$ onto a RM (B, h). Then the f.a.e;*

(a) The distribution $\mathbb{D}$ is integrable.
(b) $h((\nabla\Pi_*)(U_1, \mathscr{F}U_2) - (\nabla\Pi_*)(U_2, \mathscr{F}U_1), \Pi_*\beta V) = \lambda^2 g\left(\alpha\left(\hat{\nabla}_{U_1}\mathscr{F}U_2 - \hat{\nabla}_{U_2}\mathscr{F}U_1\right), V\right)$,
(c) $(\mathscr{T}_{U_2}\mathscr{F}U_1 - \mathscr{T}_{U_1}\mathscr{F}U_2) \in \Gamma(\mu)$ and $\left(\hat{\nabla}_{U_1}\mathscr{F}U_2 - \hat{\nabla}_{U_2}\mathscr{F}U_1\right) \in \Gamma(\mathbb{D})$,

for any $U_1, U_2 \in \Gamma(\mathbb{D})$ and $V \in \Gamma(\mathbb{D}_1)$.

Proof. For any $U_1, U_2 \in \Gamma(\mathbb{D}), V \in \Gamma(\mathbb{D}_1)$, applying (16), (17), (4) and (19), we get

$$\begin{aligned}
g([U_1, U_2], V) &= g(\mathscr{H}\nabla_{U_1}\mathscr{F}U_2, \beta V) + g\left(\hat{\nabla}_{U_1}\mathscr{F}U_2, \alpha V\right)\\
&\quad - g(\mathscr{H}\nabla_{U_2}\mathscr{F}U_1, \beta V) - g\left(\hat{\nabla}_{U_2}\mathscr{F}U_1, \alpha V\right).
\end{aligned}$$

Since Π is a CGS, using (19) and Lemma 2, we get

$$\begin{aligned}
g([U_1, U_2], V) &= \lambda^{-2}h\left(-(\nabla\Pi_*)(U_1, \mathscr{F}U_2) + \nabla^{\Pi}_U\Pi_*\mathscr{F}U_2, \Pi_*\beta V\right)\\
&\quad - \lambda^{-2}h\left(-(\nabla\Pi_*)(U_2, \mathscr{F}U_1) \ + \nabla^{\Pi}_V\Pi_*\mathscr{F}U_1, \Pi_*\beta V\right)\\
&\quad + g\left(\mathscr{F}\left(\hat{\nabla}_{U_1}\mathscr{F}U_2 - \hat{\nabla}_{U_2}\mathscr{F}U_1\right), V\right)\\
&= \lambda^{-2}h((\nabla\Pi_*)(U_2, \mathscr{F}U_1) - (\nabla\Pi_*)(U_1, \mathscr{F}U_2), \Pi_*\beta V)\\
&\quad + g\left(\alpha\left(\hat{\nabla}_{U_1}\mathscr{F}U_2 - \hat{\nabla}_{U_2}\mathscr{F}U_1\right), V\right).
\end{aligned} \tag{26}$$

We note that $g([U_1, U_2], W) = 0$, for any $W \in \Gamma(ker\Pi_*)$ as the distribution $ker\Pi_*$ is always integrable. Therefore, $(a) \iff (b)$ follows. Moreover, with Lemma 2 and (1), (26) reduces to

$$\begin{aligned}
g([U_1, U_2], V) &= -g(\mathscr{T}_{U_2}\mathscr{F}U_1 - \mathscr{T}_{U_1}\mathscr{F}U_2, \beta V)\\
&\quad + g\left(\alpha\left(\hat{\nabla}_{U_1}\alpha U_2 - \hat{\nabla}_{U_2}\alpha U_1\right), V\right)
\end{aligned}$$

By using (23) and Proposition 2, we obtain $(a) \iff (c)$.

Theorem 3. *Let $(\mathbb{M}, g, \mathscr{F})$ be a l.p.R. manifold and (B, h), a RM and $\Pi : (\mathbb{M}, g, \mathscr{F}) \to (B, h)$ be a CGS. Then the f.a.e;*

(a) The distribution $\mathbb{D}_1$ is integrable.
(b) $h((\nabla\Pi_*)(V_1, \alpha V_2) - (\nabla\Pi_*)(V_2, \alpha V_1), U) = \lambda^2 g(\mathscr{T}_{V_1}\beta V_2 - \mathscr{T}_{V_2}\beta V_1, U)$,
(c) $\hat{\nabla}_{V_1}\alpha V_2 - \hat{\nabla}_{V_2}\alpha V_1 + \mathscr{T}_{V_1}\beta V_2 - \mathscr{T}_{V_2}\beta V_1 \in \Gamma(\mathbb{D}_2)$,

for any $U \in \Gamma(\mathbb{D})$ and $V_1, V_2 \in \Gamma(\mathbb{D}_1)$.

Proof. The distribution $\mathbb{D}_1$ to be integrable iff $g([V_1,V_2],\mathscr{F}U)=0$, as well as $g([V_1,V_2],W)=0$, for any $V_1,V_2\in\Gamma(\mathbb{D}_1)$, $U\in\Gamma(\mathbb{D})$ and $W\in\Gamma(\ker\Pi_*)^{\perp}$. As $\ker\Pi_*$ is integrable, then we can easily get $g([V_1,V_2],W)=0$.

Moreover, by using (16), (17), (19) and (13), we obtain

$$\begin{aligned} g([V_1,V_2],FU) &= g(\nabla_{V_1}FV_2-\nabla_{V_2}FV_1,FU)\\ &= g(\nabla_{V_1}\alpha V_2,U)+g(\nabla_{V_1}\beta V_2,U)-g(\nabla_{V_2}\alpha V_1,U)-g(\nabla_{V_2}\beta V_1,U).\\ &= g\left(\hat{\nabla}_{V_1}\alpha V_2-\hat{\nabla}_{V_2}\alpha V_1,U\right)+g(\mathscr{T}_{V_1}\beta V_2-\mathscr{T}_{V_2}\beta V_1,U),\end{aligned}$$

which shows $(a)\iff(c)$.

On the other side, applying (13), we arrive at

$$\begin{aligned} g([V_1,V_2],\mathscr{F}U) = \lambda^{-2}h(-(\nabla\Pi_*)(V_1,\alpha V_2)+(\nabla\Pi_*)(V_2,\alpha V_1),U)\\ +g(\mathscr{T}_{V_1}\beta V_2-\mathscr{T}_{V_2}\beta V_1,U).\end{aligned}$$

Hence, $(a)\iff(b)$.

Theorem 4. *Let Π be a CGS from a l.p.R manifold $(\mathbb{M},g,\mathscr{F})$ to a RM $(\mathbb{N},h)$ i.e., $\Pi:(\mathbb{M},g,\mathscr{F})\to(\mathbb{N},h)$. Then, the distribution $(ker\Pi_*)^{\perp}$ is integrable iff*

(i) There is no component of $\mathscr{V}(\nabla_{\mathbb{X}}B\mathbb{Y}-\nabla_{\mathbb{Y}}B\mathbb{X})+\mathscr{A}_{\mathbb{X}}C\mathbb{Y}-\mathscr{A}_{\mathbb{Y}}C\mathbb{X}$ in $\Gamma(\mathbb{D})$.

(ii) $$\begin{aligned}\lambda^2 g(\mathbb{Y}(\ln\lambda)C-\mathbb{X}(\ln\lambda)C\mathbb{Y}-\mathscr{C}\mathbb{Y}(\ln\lambda)\mathbb{X}+C\mathbb{X}(\ln\lambda)\mathbb{Y}+2g(\mathbb{X},C\mathbb{Y})\nabla\ln\lambda,\beta U_2)\\ +g(-\alpha(\mathscr{V}\nabla_{\mathbb{X}}B\mathbb{Y}-\mathscr{V}\nabla_{\mathbb{Y}}B\mathbb{X}+\mathscr{A}_{\mathbb{X}}C\mathbb{Y}-\mathscr{A}_{\mathbb{Y}}C\mathbb{X}),U_2)\\ =h\left((\nabla\Pi_*)(\mathbb{X},B\mathbb{Y})-(\nabla\Pi_*)(\mathbb{Y},B\mathbb{X})-\nabla^{\Pi}_{\mathbb{X}}\Pi_*(C\mathbb{Y})+\nabla^{\Pi}_{\mathbb{Y}}\Pi_*(C\mathbb{X}),\Pi_*(\beta U_2)\right),\end{aligned}$$

for $\mathbb{X},\mathbb{Y}\in\Gamma ker\Pi_*)^{\perp},U_1\in\Gamma(\mathbb{D})$ and $U_2\in\Gamma(\mathbb{D}_1)$.

Proof. Since the horizontal distribution is integrable if and only if $g([\mathbb{X},\mathbb{Y}],W)=0$, for any $\mathbb{X},\mathbb{Y}\in\Gamma(ker\Pi_*)^{\perp}$ and $W\in\Gamma(ker\Pi_*)$, where $W=U_1+U_2$ such that $U_1\in\mathbb{D}$ and $U_2\in\mathbb{D}_1$. This implies that $g([\mathbb{X},\mathbb{Y}],U_1)=0$ and $g([\mathbb{X},\mathbb{Y}],U_2)=0$.

We omit the proof as it easily follows by the use of the Eqs. (16), (17), (6), (7), (20), (13)in addition to the fact that Π is a horizontal conformal submersion, using Lemma 2.

Theorem 5. *Let Π be a CGS from a l.p.R manifold $(M,g,\mathscr{F})$ onto a RM (B,h). Then,*

(i) The distribution $\mathbb{D}$ defines a totally geodesic foliation on M iff

$$h((\nabla\Pi_*)(U_1,\alpha V_1),\Pi_*\beta U_2)=\lambda^2 g\left(\hat{\nabla}_{U_1}\alpha V_1,\alpha U_2\right)$$

and

$$h((\nabla\Pi_*)(U_1,\alpha V_1),\Pi_*C\mathbb{X})=-\lambda^2 g\left(\hat{\nabla}_{U_1}\alpha B\mathbb{X}+\mathscr{T}_{U_1}\beta B\mathbb{X},V_1\right).$$

(ii) The distribution $\mathbb{D}_2$ defines a totally geodesic foliation on M iff

$$h((\nabla\Pi_*)(U_2,\mathscr{F}U_1),\Pi_*(\beta V_2))=\lambda^2 g\left(\hat{\nabla}_{U_2}\mathscr{F}U_1,\alpha V_2\right)$$

and

$$h((\nabla\Pi_*)(U_2,V_2),\Pi_*(\mathscr{F}C\mathbb{X}))=\lambda^2\left\{g(\mathscr{T}_{U_2}B\mathbb{X},\beta V_2)-g\left(\hat{\nabla}_{U_2}\alpha V_2,B\mathbb{X}\right)\right\},$$

for any $U_1,V_1\in\Gamma(\mathbb{D}),U_2\in\Gamma(\mathbb{D}_1)$ and $\mathbb{X}\in\Gamma(ker\Pi_*)^{\perp}$.

Proof. (i) For any $U_1,V_1\in\Gamma(\mathbb{D}),U_2\in\Gamma(\mathbb{D}_1)$ and $Z\in\Gamma(ker\Pi_*)^{\perp}$, using (16) and (17) and Proposition (2), we have

$$\begin{aligned} g(\nabla_{U_1}V_1,U_2) &= g(\nabla_{U_1}\mathscr{F}V_1,\alpha U_2)+g(\nabla_{U_1}\mathscr{F}V_1,\beta U_2)\\ &= g\left(\hat{\nabla}_{U_1}\alpha V_1,\alpha U_2\right)+g(\mathscr{H}\nabla_{U_1}\alpha V_1,\beta U_2).\end{aligned}\tag{27}$$

Furthermore, using (13), we get

$$g(\nabla_{U_1}V_1,U_2)=g\left(\hat{\nabla}_{U_1}\alpha V_1,\alpha U_2\right)-\lambda^{-2}h((\nabla\Pi_*)(U_1,\alpha V_1),\Pi_*\beta U_2).\tag{28}$$

Taking into account that Π is a CGS and using (5), (13), we obtain

$$\begin{aligned} g(\nabla_{U_1}V_1,Z) &= -g\left(V_1,\hat{\nabla}_{U_1}\alpha BZ\right)-g(V_1,\mathscr{T}_{U_1}\beta BZ)\\ &\quad -\lambda^{-2}h((\nabla\Pi_*)(U_1,\alpha V_1),\Pi_*CZ).\end{aligned}\tag{29}$$

Therefore, by the virtue of (28) and (29), we obtain (i).

(ii) For any vector fields $U_1\in\Gamma\mathbb{D})$ and $U_2,V_2\in\Gamma(\mathbb{D}_1)$, using (16)(17)

$$\begin{aligned} g(\nabla_{U_2}V_2,U_1) &= g(\mathscr{F}\nabla_{U_2}V_2,\mathscr{F}U_1)\\ &= -g(\nabla_{U_2}\mathscr{F}U_1,\mathscr{F}V_2)\\ &= -g\left(\hat{\nabla}_{U_2}\mathscr{F}U_1,\alpha V_2\right)-g\left(\hat{\nabla}_{U_2}\mathscr{F}U_1,\beta V_2\right)\\ &= -g\left(\hat{\nabla}_{U_2}\mathscr{F}U_1,\alpha V_2\right)+g((\nabla\Pi_*)(U_2,\mathscr{F}U_1),\Pi_*\beta V_2).\end{aligned}\tag{30}$$

Moreover, for any $\mathbb{X}\in\Gamma(ker\Pi_*)^{\perp}$

$$\begin{aligned} g(\nabla_{U_2}V_2,\mathbb{X}) &= g(\mathscr{F}\nabla_{U_2}V_2,\mathscr{F}\mathbb{X})\\ &= g(\nabla_{U_2}\alpha V_2,B\mathbb{X})+g(\nabla_{U_2}\beta V_2,B\mathbb{X})+g(\nabla_{U_2}\mathscr{F}V_2,C\mathbb{X})\\ &= g\left(\hat{\nabla}_{U_2}\alpha V_2,B\mathbb{X}\right)-g(\mathscr{T}_{U_2}B\mathbb{X},\beta V_2)-g((\nabla\Pi_*)(U_2,V_2),\Pi_*(\mathscr{F}C\mathbb{X})).\end{aligned}\tag{31}$$

Therefore, with (30) and (31), we arrive at (ii).

In view of Theorem 5, we have;

Corollary 1. Let $\Pi:(\mathbb{M},g,\mathscr{F})\to(\mathbb{N},h)$ be CGS from a l.p.R. manifold $(\mathbb{M},g,\mathscr{F})$ to a RM $(\mathbb{N},h)$. Then, the necessary and sufficient conditions for the fibres of Π to be a l.P. manifold of the form $M_{\mathbb{D}}\times M_{\mathbb{D}_1}$ are

(i) $h((\nabla\Pi_*)(U_1,\alpha V_1),\Pi_*\beta U_2)=\lambda^2 g\left(\hat{\nabla}_{U_1}\alpha V_1,\alpha U_2\right)$ and $h((\nabla\Pi_*)(U_1,\alpha V_1),\Pi_*C\mathbb{X})=-\lambda^2 g\left(\hat{\nabla}_{U_1}\alpha B\mathbb{X}+\mathscr{T}_{U_1}\beta B\mathbb{X},V_1\right)$,

(ii) $h((\nabla\Pi_*)(U_2,FU_1),\Pi_*(\beta V_2))=\lambda^2 g\left(\hat{\nabla}_{U_2}FU_1,\alpha V_2\right)$ and $h((\nabla\Pi_*)(U_2,V_2),\Pi_*(FC\mathbb{X}))=\lambda^2\left\{g(\mathscr{T}_{U_2}B\mathbb{X},\beta V_2)-g\left(\hat{\nabla}_{U_2}\alpha V_2,B\mathbb{X}\right)\right\}$,

for any $U_1,V_1\in\Gamma(\mathbb{D}),U_2\in\Gamma(\mathbb{D}_1)$ and $\mathbb{X}\in\Gamma(ker\Pi_*)^{\perp},M_{\mathbb{D}}$ and $M_{\mathbb{D}_1}$ are the integral manifolds of the distribution $\mathbb{D}$ and $\mathbb{D}_1$, respectively.

Proofs of the following Theorem 6 and Theorem 7 can easily be obtained from Theorem 3.20 and Theorem 3.21 Akyol (Akyol, 2021), respectively.

Theorem 6. *Let $\Pi:(\mathbb{M},g,\mathscr{F})\to(\mathbb{N},h)$ be a CGS from a l.p.R manifold $(\mathbb{M},g,\mathscr{F})$ onto a RM $(\mathbb{N},h)$. Then, the horizontal distribution $(ker\Pi_*)^{\perp}$ defines a totally geodesic foliation on $\mathbb{M}$ iff*

(i) $h((\nabla\Pi_*)(\mathbb{X},\mathscr{F}V_1),\Pi_*\mathbb{Y})=\lambda^2 g(\mathbb{Y},\mathscr{V}\nabla_{\mathbb{X}}\mathscr{F}V_1)$.

(ii) $h\left(\nabla^{\Pi}_{\mathbb{X}}\Pi_*(\beta V_2),\Pi_*(C\mathbb{Y})\right)=-\lambda^2\{g(\alpha(\mathscr{A}_{\mathbb{X}}C\mathbb{Y}+\mathscr{V}\nabla_{\mathbb{X}}B\mathbb{Y}),V_2)+g(\mathscr{A}_{\mathbb{X}}B\mathbb{Y}-\mathbb{X}(\ln\lambda)C\mathbb{Y}-C\mathbb{Y}(\ln\lambda)\mathbb{X}+g(\mathbb{X},C\mathbb{Y})\nabla\ln\lambda,\beta V_2)\}$,

for any $\mathbb{X},\mathbb{Y}\in\Gamma(ker\Pi_*)^{\perp},V_1\in\Gamma(\mathbb{D})$ and $V_2\in\Gamma(\mathbb{D}_1)$.

Theorem 7. *Let $\Pi : (\mathbb{M}, g, \mathscr{F}) \to (\mathbb{N}, h)$ be a CGS from a l.p.R manifold $(M, g, \mathscr{F})$ to a RM $(\mathbb{N}, h)$. Then, the vertical distribution $(ker\Pi_*)$ defines a totally geodesic foliation on M iff*

$h((\nabla\Pi_*)(\mathbb{U}, \beta\mathbb{V}), \Pi_*(C\mathbb{X})) = \lambda^2\left\{g\left(\hat{\nabla}_{\mathbb{U}}\alpha\mathbb{V}, B\mathbb{X}\right) - g(\beta\mathbb{V}, \mathscr{T}_{\mathbb{U}}B\mathbb{X}) + g(\mathscr{T}_{\mathbb{U}}\beta\mathbb{V}, C\mathbb{X})\right\}$, *for any $\mathbb{U}, \mathbb{V} \in \Gamma(ker\Pi_*)$ and $\mathbb{X} \in \Gamma(ker\Pi_*)^\perp$.*

We recall that a horizontal conformal map $\varphi : (M_1, g_1) \to (M_2, g_2)$ between two RMs is called horizontally homothetic if and only if $\mathscr{H}(\mathbb{G}\lambda) = 0$.

We now imitate Theorem 4 and Theorem 6 as follows;

Theorem 8. *Let Π be a CGS from a l.p.R manifold $(\mathbb{M}, g, \mathscr{F})$ to a RM $(\mathbb{N}, h)$ i.e., $\Pi : (\mathbb{M}, g, \mathscr{F}) \to (\mathbb{N}, h)$ and additionally Π is horizontally homotheic map. Then, the distribution $(ker\Pi_*)^\perp$ is integrable iff*

(i) There is no component of $\mathscr{V}(\nabla_{\mathbb{X}}B\mathbb{Y} - \nabla_{\mathbb{Y}}B\mathbb{X}) + \mathscr{A}_{\mathbb{X}}C\mathbb{Y} - \mathscr{A}_{\mathbb{Y}}C\mathbb{X}$ in $\Gamma(\mathbb{D})$.

(ii)
$$\lambda^2 g(\alpha(\mathscr{V}\nabla_{\mathbb{X}}B\mathbb{Y} + \mathscr{V}\nabla_{\mathbb{Y}}B\mathbb{X} - \mathscr{A}_{\mathbb{X}}C\mathbb{Y} + \mathscr{A}_{\mathbb{Y}}C\mathbb{X}), U_2) = h\Big((\nabla\Pi_*)(\mathbb{Y}, B\mathbb{X}) - (\nabla\Pi_*)(\mathbb{X}, B\mathbb{Y}) + \nabla^{\Pi}_{\mathbb{X}}\Pi_*(C\mathbb{Y}) - \nabla^{\Pi}_{\mathbb{Y}}\Pi_*(C\mathbb{X}), \Pi_*(\beta U_2)\Big),$$

for any $\mathbb{X}, \mathbb{Y} \in \Gamma ker\Pi_*)^\perp, U_1 \in \Gamma(\mathbb{D})$ and $U_2 \in \Gamma(\mathbb{D}_1)$.

Theorem 9. *Let $\Pi : (\mathbb{M}, g, \mathscr{F}) \to (\mathbb{N}, h)$ be a CGS from a l.p.R manifold $(\mathbb{M}, g, \mathscr{F})$ to a RM $(\mathbb{N}, h)$ and Π is horizontally homothetic. Then, the horizontal distribution $(ker\Pi_*)^\perp$ defines a totally geodesic foliation on $\mathbb{M}$ iff*

(i) $h((\nabla\Pi_*)(\mathbb{X}, \mathscr{F}V_1), \Pi_*\mathbb{Y}) = \lambda^2 g(\mathbb{Y}, \mathscr{V}\nabla_{\mathbb{X}}\mathscr{F}V_1)$.

(ii) $h\left(\nabla^{\Pi}_{\mathbb{X}}\Pi_*(\beta V_2), \Pi_*(C\mathbb{Y})\right) = \lambda^2\{-g(\alpha(\mathscr{A}_{\mathbb{X}}C\mathbb{Y} + \mathscr{V}\nabla_{\mathbb{X}}B\mathbb{Y}), V_2) + g(\mathscr{A}_{\mathbb{X}}B\mathbb{Y}, \beta V_2)\}$,

for $\mathbb{X}, \mathbb{Y} \in \Gamma(ker\Pi_*)^\perp, V_1 \in \Gamma(\mathbb{D})$ and $V_2 \in \Gamma(\mathbb{D}_1)$.

From Theorems 6 and 7, we deduce the following:

Theorem 10. *Let $\Pi : (\mathbb{M}, g, \mathscr{F}) \to (\mathbb{N}, h)$ be CGS from a l.p.R manifold $(\mathbb{M}, g, \mathscr{F})$ onto a RM $(\mathbb{N}, h)$. Then, the total space M is a generic product RM $\mathbb{M} = \mathbb{M}_{(ker\Pi_*)} \times \mathbb{M}_{(ker\Pi_*)^\perp}$ iff*

$$h((\nabla\Pi_*)(\mathbb{X}, \mathscr{F}V_1), \Pi_*\mathbb{Y}) = \lambda^2 g(\mathbb{Y}, \mathscr{V}\nabla_{\mathbb{X}}\mathscr{F}V_1),$$

$$h\left(\nabla^{\Pi}_{\mathbb{X}}\Pi_*(\beta V_2), \Pi_*(C\mathbb{Y})\right) = \lambda^2\{-g(\alpha(\mathscr{A}_{\mathbb{X}}C\mathbb{Y} + \mathscr{V}\nabla_{\mathbb{X}}B\mathbb{Y}), V_2) + g(\mathscr{A}_{\mathbb{X}}B\mathbb{Y} - \mathbb{X}(\ln\lambda)C\mathbb{Y} - C\mathbb{Y}(\ln\lambda)\mathbb{X} + g(\mathbb{X}, C\mathbb{Y})\nabla\ln\lambda, \beta V_2)\}$$

and

$$h((\nabla\Pi_*)(U, \beta V), \Pi_*(C\mathbb{X})) = \lambda^2\left\{g\left(\hat{\nabla}_U\alpha V, B\mathbb{X}\right) - g(\beta V, \mathscr{T}_U B\mathbb{X}) + g(\mathscr{T}_U\alpha V, C\mathbb{X})\right\},$$

for any $\mathbb{X}, \mathbb{Y} \in \Gamma(ker\Pi_*)^\perp, U, V \in \Gamma(ker\Pi), V_1 \in \Gamma(\mathbb{D})$ and $V_2 \in \Gamma(\mathbb{D}_1)$, where $\mathbb{M}_{(ker\Pi_*)}$ and $\mathbb{M}_{(ker\Pi_*)^\perp}$ are leaves of the distributions $ker\Pi_*$ and $(ker\Pi_*)^\perp$, respectively.

We now conclude this section with the necessary and sufficient condition for CGS to be harmonic map.

Theorem 11. *Let $\Pi : (\mathbb{M}, g, \mathscr{F}) \to (\mathbb{N}, h)$ be a CGS, where $(\mathbb{M}, g, \mathscr{F})$ is a l.p.R. manifold and $(\mathbb{N}, h)$ is a RM. Then, Π is harmonic if and only if*

$$trace|_{(\mathbb{D})}\Pi_*\left(C\mathscr{T}_{\mathscr{F}U}U + \beta\hat{\nabla}_{\mathscr{F}U}U\right) + trace|_{(\mathbb{D}_1)}\Pi_*\left(C\mathscr{T}_{(V)}\alpha V + \beta\hat{\nabla}_V\alpha(V) + \beta\mathscr{T}_V\beta V + C\mathscr{H}\nabla^M_V\beta V\right)$$
$$-trace|_{(ker\Pi_*)^\perp}\left(\nabla^{\Pi}_{\mathbb{X}}\Pi_*\left(C^2\mathbb{X} + \beta\mathscr{B}\mathbb{X}\right) + \Pi_*\left(C\mathscr{A}_{\mathbb{X}}B\mathbb{X} + C\mathscr{H}\nabla^M_{\mathbb{X}}C\mathbb{X} + \beta\mathscr{A}_{\mathbb{X}}C\mathbb{X} + \beta\mathscr{V}\nabla^M_{\mathbb{X}}C\mathbb{X}\right)\right) = 0.$$

Proof. For any vertical vector field $U \in \Gamma(\mathbb{D}), V \in \Gamma(\mathbb{D}_1)$ and $\mathbb{X} \in \Gamma(ker\Pi_*)^\perp$, using (16), (13), (19), (20) and Proposition (2), we have

$$(\nabla\Pi_*)(\mathscr{F}U, \mathscr{F}U) + (\nabla\Pi_*)(V, V) + (\nabla\Pi_*)(\mathbb{X}, \mathbb{X}) = -\Pi_*\left(\mathscr{F}\nabla^M_{\mathscr{F}U}U\right)$$
$$-\Pi_*\left(\mathscr{F}\left(\nabla^M_V\alpha V + \nabla^M_V\beta V\right)\right) + \nabla^{\Pi}_{\mathbb{X}}\Pi_*\left(C^2\mathbb{X} + \beta B\mathbb{X}\right) - \Pi_*\left(\mathscr{F}\left(\nabla^M_{\mathbb{X}}B\mathbb{X} + \nabla^M_{\mathbb{X}}C\mathbb{X}\right)\right).$$

In view of the Eqs. 19,20, we get

$$(\nabla\Pi_*)(\mathscr{F}U, \mathscr{F}U) + (\nabla\Pi_*)(V, V) + (\nabla\Pi_*)(\mathbb{X}, \mathbb{X}) = -\Pi_*\left(C\mathscr{T}_{\mathscr{F}U}U + \beta\hat{\nabla}_{\mathscr{F}U}U\right)$$
$$-\Pi_*\left(C\mathscr{T}_V\alpha V + \beta\hat{\nabla}_V\alpha V + \beta\mathscr{T}_V\beta V + C\mathscr{H}\nabla^M_V\beta V\right)$$
$$+\nabla^{\Pi}_{\mathbb{X}}\Pi_*\left(C^2\mathbb{X} + \beta B\mathbb{X}\right) - \Pi_*\left(C\mathscr{A}_{\mathbb{X}}B\mathbb{X} + C\mathscr{H}\nabla^M_{\mathbb{X}}C\mathbb{X} + \beta\mathscr{A}_{\mathbb{X}}C\mathbb{X} + \beta\mathscr{V}\nabla^M_{\mathbb{X}}C\mathbb{X}\right).$$

Hence, the assertion follows directly.

5. Curvature relations on conformal generic submersions

This section investigates the sectional curvatures of the fibres of a CGS as well as the total space and base manifold. Let $\Pi : (\mathbb{M}, g, \mathscr{F}) \to (\mathbb{N}, h)$ be a CGS whose total space $(\mathbb{M}, g, \mathscr{F})$ is a l. p.R. manifold and the base space $(\mathbb{N}, h)$ be a RM. We denote the Riemannian curvature tensors by $\widehat{R}, R_{\mathbb{M}}$ and R^* for any fibre $\Pi^{-1}(p), \mathbb{M}$ and $\mathbb{N}$, respectively. The sectional curvature denoted by K, defined as follows;

$$K(\mathbb{X}, \mathbb{Y}) = \frac{R(\mathbb{X}, \mathbb{Y}, \mathbb{Y}, \mathbb{X})}{\|\mathbb{X}\|^2\|\mathbb{Y}\|^2}, \tag{32}$$

where $\mathbb{X}$ and $\mathbb{Y}$ are the pair of non-zero orthogonal tangent vectors on M.

Theorem 12. *Let Π be a CGS from a l.p.R. manifold $(M, g, \mathscr{F})$ to a RM (N, h). Then, for any horizontal vector field $\mathbb{X}, \mathbb{Y}$ and vertical vector fields $\mathbb{U}, V$*

$$\begin{aligned} K(\mathbb{U}, V) = {} & \widehat{K}(\alpha\mathbb{U}, \alpha V) + \|\mathscr{T}_{\alpha\mathbb{U}}\alpha V\|^2 - g(\mathscr{T}_{\alpha V}\alpha V, \mathscr{T}_{\alpha\mathbb{U}}\alpha\mathbb{U}) + \tfrac{1}{\lambda^2}K^*(\Pi_*\beta\mathbb{U}, \Pi_*\beta V) - \tfrac{3}{4}\|\mathscr{V}[\beta\mathbb{U}, \beta V]\|^2 \\ & + \tfrac{\lambda^2}{2}\left\{g(\beta\mathbb{U}, \beta V)g\left(\nabla_{\beta V}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right), \beta\mathbb{U}\right) - g(\beta V, \beta V)g\left(\nabla_{\beta\mathbb{U}}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right), \beta\mathbb{U}\right) + g(\beta V, \beta\mathbb{U})g\left(\nabla_{\beta\mathbb{U}}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right), \beta V\right) - g(\beta\mathbb{U}, \beta\mathbb{U})g\left(\nabla_{\beta V}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right), \beta V\right)\right\} \\ & + \tfrac{\lambda^4}{4}\left\{(g(\beta\mathbb{U}, \beta\mathbb{U})g(\beta V, \beta V) - g(\beta V, \beta\mathbb{U})g(\beta\mathbb{U}, \beta V))\|\mathbb{G}\left(\tfrac{1}{\lambda^2}\right)\|^2 + \|\beta\mathbb{U}\left(\tfrac{1}{\lambda^2}\right)\beta V - \beta V\left(\tfrac{1}{\lambda^2}\right)\beta\mathbb{U}\|^2\right\} + g\left((\nabla_{\alpha\mathbb{U}}A)_{\beta V}\beta V, \alpha\mathbb{U}\right) + \|\mathscr{A}_{\beta V}\alpha\mathbb{U}\|^2 \\ & - g\left(\left(\nabla_{\beta V}\mathscr{T}\right)_{\alpha\mathbb{U}}\beta V, \alpha\mathbb{U}\right) - \|\mathscr{T}_{\alpha\mathbb{U}}\beta V\|^2 + g\left((\nabla_{\alpha V}A)_{\beta\mathbb{U}}\beta\mathbb{U}, \alpha V\right) \\ & + \|\mathscr{A}_{\beta\mathbb{U}}\alpha V\|^2 - g\left(\left(\nabla_{\beta\mathbb{U}}\mathscr{T}\right)_{\alpha V}\beta\mathbb{U}, \alpha V\right) - \|\mathscr{T}_{\alpha V}\beta\mathbb{U}\|^2, \end{aligned} \tag{33}$$

$$\begin{aligned} K(\mathbb{X}, \mathbb{Y}) = {} & \widehat{K}(B\mathbb{X}, B\mathbb{Y}) + \|\mathscr{T}_{B\mathbb{X}}B\mathbb{Y}\|^2 - g(\mathscr{T}_{B\mathbb{Y}}B\mathbb{Y}, \mathscr{T}_{B\mathbb{X}}B\mathbb{X}) + \tfrac{1}{\lambda^2}K^*(F_*C\mathbb{X}, F_*C\mathbb{Y}) - \tfrac{3}{4}\|\mathscr{V}[C\mathbb{X}, C\mathbb{Y}]\|^2 \\ & + \tfrac{\lambda^2}{2}\left\{g(C\mathbb{X}, C\mathbb{Y})g\left(\nabla_{C\mathbb{Y}}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right), C\mathbb{X}\right) - g(C\mathbb{Y}, C\mathbb{Y})g\left(\nabla_{C\mathbb{X}}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right), C\mathbb{X}\right) + g(C\mathbb{Y}, C\mathbb{X})g\left(\nabla_{C\mathbb{X}}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right), C\mathbb{Y}\right) - g(C\mathbb{X}, C\mathbb{X})g\left(\nabla_{C\mathbb{Y}}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right), C\mathbb{Y}\right)\right\} \\ & + \tfrac{\lambda^4}{4}\left\{(g(C\mathbb{X}, C\mathbb{X})g(C\mathbb{Y}, C\mathbb{Y}) - g(C\mathbb{Y}, C\mathbb{X})g(C\mathbb{X}, C\mathbb{Y}))\|\mathbb{G}\left(\tfrac{1}{\lambda^2}\right)\|^2 + \|C\mathbb{X}\left(\tfrac{1}{\lambda^2}\right)C\mathbb{Y} - C\mathbb{Y}\left(\tfrac{1}{\lambda^2}\right)C\mathbb{X}\|^2\right\} + g((\nabla_{B\mathbb{X}}\mathscr{A})_{C\mathbb{Y}}C\mathbb{Y}, B\mathbb{X}) + \|\mathscr{A}_{C\mathbb{Y}}B\mathbb{X}\|^2 \\ & - g((\nabla_{C\mathbb{Y}}T)_{B\mathbb{X}}C\mathbb{Y}, B\mathbb{X}) - \|\mathscr{T}_{B\mathbb{X}}C\mathbb{Y}\|^2 + g((\nabla_{B\mathbb{Y}}\mathscr{A})_{C\mathbb{X}}C\mathbb{X}, B\mathbb{Y}) + g((\nabla_{B\mathbb{Y}}\mathscr{A})_{C\mathbb{X}}C\mathbb{X}, B\mathbb{Y}) \\ & + \|\mathscr{A}_{C\mathbb{X}}B\mathbb{Y}\|^2 - g((\nabla_{C\mathbb{X}}T)_{B\mathbb{Y}}C\mathbb{X}, B\mathbb{Y}) - \|\mathscr{T}_{B\mathbb{Y}}C\mathbb{X}\|^2, \end{aligned} \tag{34}$$

$$\begin{aligned} K(\mathbb{X},\mathbb{U}) &= \widehat{K}(B\mathbb{X},\alpha\mathbb{U}) + \|\mathscr{T}_{B\mathbb{X}}\alpha\mathbb{U}\|^2 - g(\mathscr{T}_{\alpha\mathbb{U}}\alpha\mathbb{U},\mathscr{T}_{B\mathbb{X}}B\mathbb{X})\\ &+g\Big((\nabla_{B\mathbb{X}}\mathscr{A})_{\beta\mathbb{U}}\beta\mathbb{U},B\mathbb{X}\Big) + \|\mathscr{A}_{\beta\mathbb{U}}B\mathbb{X}\|^2\\ &-g\big((\nabla_{\beta\mathbb{U}}\mathscr{T})_{B\mathbb{X}}\beta\mathbb{U},B\mathbb{X}\big) - \|\mathscr{T}_{B\mathbb{X}}\beta\mathbb{U}\|^2 + g\big((\nabla_{\alpha\mathbb{U}}A)_{C\mathbb{X}}C\mathbb{X},\alpha V\big)\\ &+\|\mathscr{A}_{C\mathbb{X}}\alpha\mathbb{U}\|^2 - g\big((\nabla_{C\mathbb{X}}T\mathscr{T})_{\alpha\mathbb{U}}C\mathbb{X},\alpha\mathbb{U}\big) - \|\mathscr{T}_{\alpha\mathbb{U}}C\mathbb{X}\|^2\\ &+\tfrac{1}{\lambda^2}h(R^*(\Pi_*C\mathbb{X},\Pi_*\beta\mathbb{U})\Pi_*\beta\mathbb{U},\Pi_*C\mathbb{X})\\ &-\tfrac{3}{4}\|\mathscr{V}[C\mathbb{X},\beta\mathbb{U}]\|^2 + 2g(\mathscr{V}[C\mathbb{X},\beta\mathbb{U}],\mathscr{V}[\beta\mathbb{U},C\mathbb{X}])\}\\ &+\tfrac{\lambda^2}{2}\Big\{g(C\mathbb{X},\beta\mathbb{U})g\Big(\nabla_{\beta\mathbb{U}}\mathbb{G}\Big(\tfrac{1}{\lambda^2}\Big),C\mathbb{X}\Big) - g(\beta\mathbb{U},\beta\mathbb{U})g\Big(\nabla_{C\mathbb{X}}\mathbb{G}\Big(\tfrac{1}{\lambda^2}\Big),C\mathbb{X}\Big) + g(\beta\mathbb{U},C\mathbb{X})g\Big(\nabla_{C\mathbb{X}}\mathbb{G}\Big(\tfrac{1}{\lambda^2}\Big),\beta\mathbb{U}\Big) - g(C\mathbb{X},C\mathbb{X})g\Big(\nabla_{\beta\mathbb{U}}\mathbb{G}\Big(\tfrac{1}{\lambda^2}\Big),\beta\mathbb{U}\Big)\Big\}\\ &+\tfrac{\lambda^4}{4}\Big\{(g(C\mathbb{X},C\mathbb{X})g(\beta\mathbb{U},\beta\mathbb{U}) - g(\beta\mathbb{U},C\mathbb{X})g(C\mathbb{X},\beta\mathbb{U}))\|\mathbb{G}\Big(\tfrac{1}{\lambda^2}\Big)\|^2 + \|C\mathbb{X}\Big(\tfrac{1}{\lambda^2}\Big)\beta\mathbb{U} - \beta\mathbb{U}\Big(\tfrac{1}{\lambda^2}\Big)C\mathbb{X}\|^2\Big\}, \end{aligned} \tag{35}$$

where K, K^* and $\widehat{K}$ be the sectional curvatures of the total space $\mathbb{M}$, the base space $\mathbb{N}$ and fibers, respectively.

Proof. Since Π is a CGS and $\mathbb{M}$ is a l.p.R. manifold, using (19), we obtain

$$\begin{aligned} K_M(\mathbb{U},V) &= K(\mathscr{F}\mathbb{U},\mathscr{F}V) = K(\alpha\mathbb{U}+\beta\mathbb{U},\alpha V+\beta V)\\ &= K(\alpha\mathbb{U},\alpha V) + K(\alpha\mathbb{U},\beta V) + K(\beta\mathbb{U},\alpha V) + K(\beta\mathbb{U},\beta V). \end{aligned} \tag{37}$$

Using (32) and (9), we arrive at

$$\begin{aligned} &K(\alpha\mathbb{U},\alpha V) = g(R(\alpha\mathbb{U},a\alpha V)\alpha V,\alpha\mathbb{U}) = g\Big(\widehat{R}(\alpha\mathbb{U},\alpha V)B\mathbb{Y},B\mathbb{X}\Big)\\ &+g(\mathscr{T}_{\alpha\mathbb{U}}\alpha V,\mathscr{T}_{\alpha V}\alpha\mathbb{U}) - g(\mathscr{T}_{\alpha V}\alpha V,\mathscr{T}_{\alpha\mathbb{U}}\alpha\mathbb{U})\\ &= \widehat{K}(\alpha\mathbb{U},\alpha V) + \|\mathscr{T}_{\alpha\mathbb{U}}\alpha V\|^2 - g(\mathscr{T}_{\alpha V}\alpha V,\mathscr{T}_{\alpha\mathbb{U}}\alpha\mathbb{U}). \end{aligned} \tag{38}$$

Similarly, using (12), we arrive at

$$\begin{aligned} &K(\beta\mathbb{U},\beta V) = g(R(\beta\mathbb{U},\beta V)\beta V,\beta\mathbb{U}) = \tfrac{1}{\lambda^2}h(R^*(\Pi_*\beta\mathbb{U},\Pi_*\beta V)\Pi_*\beta V,\Pi_*\beta\mathbb{U})\\ &+\tfrac{1}{4}\{g(\mathscr{V}[\beta\mathbb{U},\beta V],\mathscr{V}[\beta V,\beta\mathbb{U}]) - g(\mathscr{V}[\beta V,\beta V],\mathscr{V}[\beta\mathbb{U},\beta\mathbb{U}]) + 2g(\mathscr{V}[\beta\mathbb{U},\beta V],\mathscr{V}[\beta V,\beta\mathbb{U}])\}\\ &+\tfrac{\lambda^2}{2}\Big\{g(\beta\mathbb{U},\beta V)g\Big(\nabla_{\beta V}\mathbb{G}\Big(\tfrac{1}{\lambda^2}\Big),\beta\mathbb{U}\Big) - g(\beta V,\beta V)g\Big(\nabla_{\beta\mathbb{U}}\mathbb{G}\Big(\tfrac{1}{\lambda^2}\Big),\beta\mathbb{U}\Big) + g(\beta V,\beta\mathbb{U})g\Big(\nabla_{\beta\mathbb{U}}\mathbb{G}\Big(\tfrac{1}{\lambda^2}\Big),\beta V\Big) - g(\beta\mathbb{U},\beta\mathbb{U})g\Big(\nabla_{\beta V}\mathbb{G}\Big(\tfrac{1}{\lambda^2}\Big),\beta V\Big)\Big\}\\ &+\tfrac{\lambda^4}{4}\Big\{(g(\beta\mathbb{U},\beta\mathbb{U})g(\beta V,\beta V) - g(\beta V,\beta\mathbb{U})g(\beta\mathbb{U},\beta V))\|\mathbb{G}\Big(\tfrac{1}{\lambda^2}\Big)\|^2 + g\Big(\beta\mathbb{U}\Big(\tfrac{1}{\lambda^2}\Big)\beta V - \beta V\Big(\tfrac{1}{\lambda^2}\Big)\beta\mathbb{U},\beta\mathbb{U}\Big(\tfrac{1}{\lambda^2}\Big)\beta V - \beta V\Big(\tfrac{1}{\lambda^2}\Big)\beta\mathbb{U}\Big)\Big\}. \end{aligned}$$

Also by direct calculations, we obtain

$$\begin{aligned} &K(\beta\mathbb{U},\beta V) = \tfrac{1}{\lambda^2}K^*(\Pi_*\beta\mathbb{U},\Pi_*\beta V) - \tfrac{3}{4}\|\mathscr{V}[\beta\mathbb{U},\beta V]\|^2\\ &+\tfrac{\lambda^2}{2}\Big\{g(\beta\mathbb{U},\beta V)g\Big(\nabla_{\beta V}\mathbb{G}\Big(\tfrac{1}{\lambda^2}\Big),\beta\mathbb{U}\Big) - g(\beta V,\beta V)g\Big(\nabla_{\beta\mathbb{U}}\mathbb{G}\Big(\tfrac{1}{\lambda^2}\Big),\beta\mathbb{U}\Big) + g(\beta V,\beta\mathbb{U})g\Big(\nabla_{\beta\mathbb{U}}\mathbb{G}\Big(\tfrac{1}{\lambda^2}\Big),\beta V\Big) - g(\beta\mathbb{U},\beta\mathbb{U})g\Big(\nabla_{\beta V}\mathbb{G}\Big(\tfrac{1}{\lambda^2}\Big),\beta V\Big)\Big\}\\ &+\tfrac{\lambda^4}{4}\Big\{(g(\beta\mathbb{U},\beta\mathbb{U})g(\beta V,\beta V) - g(\beta V,\beta\mathbb{U})g(\beta\mathbb{U},\beta V))\|\mathbb{G}\Big(\tfrac{1}{\lambda^2}\Big)\|^2 + \|\beta\mathbb{U}\Big(\tfrac{1}{\lambda^2}\Big)\beta V - \beta V\Big(\tfrac{1}{\lambda^2}\Big)\beta\mathbb{U}\|^2\Big\}. \end{aligned} \tag{39}$$

Similarly, using (11) we have

$$\begin{aligned} K(\alpha\mathbb{U},\beta V) &= g(R(\alpha\mathbb{U},\beta V)\beta V,\alpha\mathbb{U}) = g\Big((\nabla_{\alpha\mathbb{U}}\mathscr{A})_{\beta V}\beta V,\alpha\mathbb{U}\Big) + \|\mathscr{A}_{\beta V}\alpha\mathbb{U}\|^2\\ &-g\Big((\nabla_{\beta V}T)_{\alpha\mathbb{U}}\beta V,\alpha\mathbb{U}\Big) - \|\mathscr{T}_{\alpha\mathbb{U}}\beta V\|^2. \end{aligned} \tag{40}$$

On using (11), we obtain

$$\begin{aligned} K(\beta\mathbb{U},\alpha V) &= K(\alpha V,\beta\mathbb{U}) = g(R(\alpha V,\beta\mathbb{U})\beta\mathbb{U},\alpha V) = g\Big((\nabla_{\alpha V}A)_{\beta\mathbb{U}}\beta\mathbb{U},\alpha V\Big)\\ &+\|\mathscr{A}_{\beta\mathbb{U}}\alpha V\|^2 - g\big((\nabla_{\beta\mathbb{U}}T)_{\alpha V}\beta\mathbb{U},\alpha V\big) - \|\mathscr{T}_{\alpha V}\beta\mathbb{U}\|^2. \end{aligned} \tag{41}$$

In view of (38), (39), (40), (41) and (37), we get (35).

As M is a l.p.R. manifold, for unit vector fields $\mathbb{X}$ and $\mathbb{U}$, using (19) and (20) we have

$$K(\mathbb{X},\mathbb{U}) = K(\mathscr{F}\mathbb{X},\mathscr{F}\mathbb{U}) = K(B\mathbb{X},\alpha\mathbb{U}) + K(C\mathbb{X},\alpha\mathbb{U}) + K(B\mathbb{X},\beta\mathbb{U}) + K(C\mathbb{X},\beta\mathbb{U}). \tag{42}$$

With the help of (32) and (9), we obtain

$$\begin{aligned} &K(B\mathbb{X},\alpha\mathbb{U}) = g(R(B\mathbb{X},\alpha\mathbb{U})\alpha\mathbb{U},B\mathbb{X}) = g\Big(\widehat{R}(B\mathbb{X},\alpha\mathbb{U})\alpha\mathbb{U},B\mathbb{X}\Big)\\ &+g(\mathscr{T}_{B\mathbb{X}}\alpha\mathbb{U},\mathscr{T}_{\alpha\mathbb{U}}B\mathbb{X}) - g(\mathscr{T}_{\alpha\mathbb{U}}\alpha\mathbb{U},\mathscr{T}_{B\mathbb{X}}B\mathbb{X})\\ &= \widehat{K}(B\mathbb{X},\alpha\mathbb{U}) + \|\mathscr{T}_{B\mathbb{X}}\alpha\mathbb{U}\|^2 - g(\mathscr{T}_{\alpha\mathbb{U}}\alpha\mathbb{U},\mathscr{T}_{B\mathbb{X}}B\mathbb{X}). \end{aligned} \tag{43}$$

Using (11), we have

$$\begin{aligned} K(B\mathbb{X},\beta\mathbb{U}) &= g(R(B\mathbb{X},\beta\mathbb{U})\beta\mathbb{U},B\mathbb{X}) = g\Big((\nabla_{B\mathbb{X}}\mathscr{A})_{\beta\mathbb{U}}\beta\mathbb{U},B\mathbb{X}\Big) + \|\mathscr{A}_{\beta\mathbb{U}}B\mathbb{X}\|^2\\ &-g\big((\nabla_{\beta\mathbb{U}}\mathscr{T})_{B\mathbb{X}}\beta\mathbb{U},B\mathbb{X}\big) - \|\mathscr{T}_{B\mathbb{X}}\beta\mathbb{U}\|^2. \end{aligned} \tag{44}$$

Lastly, using (11) we get

$$\begin{aligned} K(C\mathbb{X},\alpha\mathbb{U}) &= K(\alpha\mathbb{U},C\mathbb{X}) = g(R(\alpha\mathbb{U},C\mathbb{X})C\mathbb{X},\alpha\mathbb{U}) = g\big((\nabla_{\alpha\mathbb{U}}A)_{C\mathbb{X}}C\mathbb{X},\alpha V\big)\\ &+\|\mathscr{A}_{C\mathbb{X}}\alpha\mathbb{U}\|^2 - g\big((\nabla_{C\mathbb{X}}T)_{\alpha\mathbb{U}}C\mathbb{X},\alpha\mathbb{U}\big) - \|\mathscr{T}_{\alpha\mathbb{U}}C\mathbb{X}\|^2. \end{aligned} \tag{45}$$

Similarly, from (12) we have

$$\begin{aligned} &K(C\mathbb{X},\beta\mathbb{U}) = g(R(C\mathbb{X},\beta\mathbb{U})\beta\mathbb{U},C\mathbb{X}) = \tfrac{1}{\lambda^2}h(R^*(\Pi_*C\mathbb{X},\Pi_*\beta\mathbb{U})\Pi_*\beta\mathbb{U},\Pi_*C\mathbb{X})\\ &-\tfrac{3}{4}\|\mathscr{V}[C\mathbb{X},\beta\mathbb{U}]\|^2 + \tfrac{\lambda^2}{2}\Big\{g(C\mathbb{X},\beta\mathbb{U})g\Big(\nabla_{\beta\mathbb{U}}\mathbb{G}\Big(\tfrac{1}{\lambda^2}\Big),C\mathbb{X}\Big) - g(\beta\mathbb{U},\beta\mathbb{U})g\Big(\nabla_{C\mathbb{X}}\mathbb{G}\Big(\tfrac{1}{\lambda^2}\Big),C\mathbb{X}\Big) + g(\beta\mathbb{U},C\mathbb{X})g\Big(\nabla_{C\mathbb{X}}\mathbb{G}\Big(\tfrac{1}{\lambda^2}\Big),\beta\mathbb{U}\Big) - g(C\mathbb{X},C\mathbb{X})g\Big(\nabla_{\beta\mathbb{U}}\mathbb{G}\Big(\tfrac{1}{\lambda^2}\Big),\beta\mathbb{U}\Big)\Big\}\\ &+\tfrac{\lambda^4}{4}\Big\{(g(C\mathbb{X},C\mathbb{X})g(\beta\mathbb{U},\beta\mathbb{U}) - g(\beta\mathbb{U},C\mathbb{X})g(C\mathbb{X},\beta\mathbb{U}))\|\mathbb{G}\Big(\tfrac{1}{\lambda^2}\Big)\|^2 + \|C\mathbb{X}\Big(\tfrac{1}{\lambda^2}\Big)\beta\mathbb{U} - \beta\mathbb{U}\Big(\tfrac{1}{\lambda^2}\Big)C\mathbb{X}\|^2\Big\}. \end{aligned} \tag{46}$$

Hence, (36) follows by (42), (44) and (43) and (46).

Since M is a l.p.R. manifold, using (17) and (19) for unit vector fields $\mathbb{X}$ and $\mathbb{Y}$, we have

$$K(\mathbb{X},\mathbb{Y}) = K(\mathscr{F}\mathbb{X},\mathscr{F}\mathbb{Y}) = K(B\mathbb{X},B\mathbb{Y}) + K(C\mathbb{X},C\mathbb{Y}) + K(B\mathbb{X},C\mathbb{Y}) + K(C\mathbb{X},B\mathbb{Y}). \quad (47)$$

Using (9), we derive

$$K(B\mathbb{X},B\mathbb{Y}) = g(R(B\mathbb{X},B\mathbb{Y})B\mathbb{Y},B\mathbb{X}) = g\left(\widehat{R}(B\mathbb{X},B\mathbb{Y})B\mathbb{Y},B\mathbb{X}\right) + g(\mathscr{T}_{B\mathbb{X}}B\mathbb{Y},\mathscr{T}_{B\mathbb{Y}}B\mathbb{X}) - g(\mathscr{T}_{B\mathbb{Y}}B\mathbb{Y},\mathscr{T}_{B\mathbb{X}}B\mathbb{X}) = \widehat{K}(B\mathbb{X},B\mathbb{Y}) + \|\mathscr{T}_{B\mathbb{X}}B\mathbb{Y}\|^2 - g(\mathscr{T}_{B\mathbb{Y}}B\mathbb{Y},\mathscr{T}_{B\mathbb{X}}B\mathbb{X}). \quad (48)$$

Similarly, using (12), we get

$$K(C\mathbb{X},C\mathbb{Y}) = g(R(C\mathbb{X},C\mathbb{Y})C\mathbb{Y},C\mathbb{X}) = \tfrac{1}{\lambda^2}h(R^*(\Pi_*C\mathbb{X},\Pi_*C\mathbb{Y})\Pi_*C\mathbb{Y},\Pi_*C\mathbb{X}) + \tfrac{1}{4}\{g(\mathscr{V}[C\mathbb{X},C\mathbb{Y}],\mathscr{V}[C\mathbb{Y},C\mathbb{X}]) - g(\mathscr{V}[C\mathbb{Y},C\mathbb{Y}],\mathscr{V}[C\mathbb{X},C\mathbb{X}]) + 2g(\mathscr{V}[C\mathbb{X},C\mathbb{Y}],\mathscr{V}[C\mathbb{Y},C\mathbb{X}])\} + \tfrac{\lambda^2}{2}\left\{g(C\mathbb{X},C\mathbb{Y})g\left(\nabla_{C\mathbb{Y}}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),C\mathbb{X}\right) - g(C\mathbb{Y},C\mathbb{Y})g\left(\nabla_{C\mathbb{X}}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),C\mathbb{X}\right) + g(C\mathbb{Y},C\mathbb{X})g\left(\nabla_{C\mathbb{X}}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),C\mathbb{Y}\right) - g(C\mathbb{X},C\mathbb{X})g\left(\nabla_{C\mathbb{Y}}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),C\mathbb{Y}\right)\right\} + \tfrac{\lambda^4}{4}\left\{(g(C\mathbb{X},C\mathbb{X})g(C\mathbb{Y},C\mathbb{Y}) - g(C\mathbb{Y},C\mathbb{X})g(C\mathbb{X},C\mathbb{Y}))\|\mathbb{G}\left(\tfrac{1}{\lambda^2}\right)\|^2 + g\left(C\mathbb{X}\left(\tfrac{1}{\lambda^2}\right)C\mathbb{Y} - C\mathbb{Y}\left(\tfrac{1}{\lambda^2}\right)C\mathbb{X}, C\mathbb{X}\left(\tfrac{1}{\lambda^2}\right)C\mathbb{Y} - C\mathbb{Y}\left(\tfrac{1}{\lambda^2}\right)C\mathbb{X}\right)\right\}.$$

Also by direct calculations, we obtain

$$K(C\mathbb{X},C\mathbb{Y}) = \tfrac{1}{\lambda^2}K^*(\Pi_*C\mathbb{X},\Pi_*C\mathbb{Y}) - \tfrac{3}{4}\|\mathscr{V}[C\mathbb{X},C\mathbb{Y}]\|^2 + \tfrac{\lambda^2}{2}\left\{g(C\mathbb{X},C\mathbb{Y})g\left(\nabla_{C\mathbb{Y}}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),C\mathbb{X}\right) - g(C\mathbb{Y},C\mathbb{Y})g\left(\nabla_{C\mathbb{X}}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),C\mathbb{X}\right) + g(C\mathbb{Y},C\mathbb{X})g\left(\nabla_{C\mathbb{X}}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),C\mathbb{Y}\right) - g(C\mathbb{X},C\mathbb{X})g\left(\nabla_{C\mathbb{Y}}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),C\mathbb{Y}\right)\right\} + \tfrac{\lambda^4}{4}\left\{(g(C\mathbb{X},C\mathbb{X})g(C\mathbb{Y},C\mathbb{Y}) - g(C\mathbb{Y},C\mathbb{X})g(C\mathbb{X},C\mathbb{Y}))\|\mathbb{G}\left(\tfrac{1}{\lambda^2}\right)\|^2 + \|C\mathbb{X}\left(\tfrac{1}{\lambda^2}\right)C\mathbb{Y} - C\mathbb{Y}\left(\tfrac{1}{\lambda^2}\right)C\mathbb{X}\|^2\right\}. \quad (49)$$

In a similar way, using (11) we have

$$K(B\mathbb{X},C\mathbb{Y}) = g(R(B\mathbb{X},C\mathbb{Y})C\mathbb{Y},B\mathbb{X}) = g((\nabla_{B\mathbb{X}}A)_{C\mathbb{Y}}C\mathbb{Y},B\mathbb{X}) + \|\mathscr{A}_{C\mathbb{Y}}B\mathbb{X}\|^2 - g((\nabla_{C\mathbb{Y}}\mathscr{T})_{B\mathbb{X}}C\mathbb{Y},B\mathbb{X}) - \|\mathscr{T}_{B\mathbb{X}}C\mathbb{Y}\|^2. \quad (50)$$

Lastly, using (11) we have

$$K(C\mathbb{X},B\mathbb{Y}) = K(B\mathbb{Y},C\mathbb{X}) = g(R(B\mathbb{Y},C\mathbb{X})C\mathbb{X},B\mathbb{Y}) = g((\nabla_{B\mathbb{Y}}\mathscr{A})_{C\mathbb{X}}C\mathbb{X},B\mathbb{Y}) + \|\mathscr{A}_{C\mathbb{X}}B\mathbb{Y}\|^2 - g((\nabla_{C\mathbb{X}}T)_{B\mathbb{Y}}C\mathbb{X},B\mathbb{Y}) - \|\mathscr{T}_{B\mathbb{Y}}C\mathbb{X}\|^2. \quad (51)$$

Writing (48), (49), (50) and (51) in (47), we get (35).

Theorem 12 has the following direct consequences, which we state below:

Corollary 2. Let $\Pi : (\mathbb{M},g,\mathscr{F}) \to (\mathbb{N},h)$ be a CGS from a l.p.R. manifold $(\mathbb{M},g,\mathscr{F})$ to a RM$(\mathbb{N},h)$. Then, for any $U,V \in \Gamma(\ker F_*)$

$$\hat{K}_M(U,V) \leqslant \widehat{K}(\alpha U,\alpha V) + \tfrac{1}{\lambda^2}K^*(\Pi_*\beta U,\Pi_*\beta V) + \|\mathscr{T}_{\alpha U}\alpha V\|^2 - g(\mathscr{T}_{\alpha V}\alpha V,\mathscr{T}_{\alpha U}\alpha U) - \tfrac{3}{4}\|\mathscr{V}[\beta U,\beta V]\|^2 + \tfrac{\lambda^2}{2}\left\{g(\beta U,\beta V)g\left(\nabla_{\beta V}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),\beta U\right) - g(\beta V,\beta V)g\left(\nabla_{\beta U}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),\beta U\right) + g(\beta V,\beta U)g\left(\nabla_{\beta U}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),\beta V\right) - g(\beta U,\beta U)g\left(\nabla_{\beta V}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),\beta V\right)\right\} + \tfrac{\lambda^4}{4}\left\{(g(\beta U,\beta U)g(\beta V,\beta V) - g(\beta V,\beta U)g(\beta U,\beta V))\|\mathbb{G}\left(\tfrac{1}{\lambda^2}\right)\|^2 + \|\beta U\left(\tfrac{1}{\lambda^2}\right)\beta V - \beta V\left(\tfrac{1}{\lambda^2}\right)\beta U\|^2\right\} + g\left((\nabla_{\alpha U}A)_{\beta V}\beta V,\alpha U\right) + \|\mathscr{A}_{\beta V}\alpha U\|^2 - g((\nabla_{\beta V}T)_{\alpha U}\beta V,\alpha U) - \|\mathscr{T}_{\alpha U}\beta V\|^2 + g\left((\nabla_{\alpha V}A)_{\beta U}\beta U,\alpha V\right) + \|\mathscr{A}_{\beta U}\alpha V\|^2 - g((\nabla_{\beta U}T)_{\alpha V}\beta U,\alpha V) - \|\mathscr{T}_{\alpha V}\beta U\|^2 + g(\mathscr{T}_V V,\mathscr{T}_U U).$$

The equality holds iff the fibers are totally geodesic and $\beta\ker\Pi_*$ is integrable.

Proof. Using Corollary 1, (O'Neill, 1966), Eq. (34) can be rewritten as

$$\begin{aligned}\widehat{K}(U,V) \geqslant\ & \widehat{K}(\alpha U,\alpha V)+\tfrac{1}{\lambda^2}K^*(\Pi_*\beta U,\Pi_*\beta V)+\|\mathscr{T}_{\alpha U}\alpha V\|^2-g(\mathscr{T}_{\alpha V}\alpha V,\mathscr{T}_{\alpha U}\alpha U)-\tfrac{3}{4}\|\mathscr{V}[\beta U,\beta V]\|^2\\&+\tfrac{\lambda^2}{2}\left\{g(\beta U,\beta V)g\left(\nabla_{\beta V}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),\beta U\right)-g(\beta V,\beta V)g\left(\nabla_{\beta U}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),\beta U\right)+g(\beta V,\beta U)g\left(\nabla_{\beta U}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),\beta V\right)-g(\beta U,\beta U)g\left(\nabla_{\beta V}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),\beta V\right)\right\}\\&+g\left((\nabla_{\alpha U}\mathscr{A})_{\beta V}\beta V,\alpha U\right)+\|\mathscr{A}_{\beta V}\alpha U\|^2\\&-g\left((\nabla_{\beta V}\mathscr{T})_{\alpha U}\beta V,\alpha U\right)-\|\mathscr{T}_{\alpha U}\beta V\|^2+g\left((\nabla_{\alpha V}A)_{\beta U}\beta U,\alpha V\right)\\&+\|\mathscr{A}_{\beta U}\alpha V\|^2-g\left((\nabla_{\beta U}T)_{\alpha V}\beta U,\alpha V\right)-\|\mathscr{T}_{\alpha V}\beta U\|^2-\|\mathscr{T}_U V\|^2+g(\mathscr{T}_V V,\mathscr{T}_U U),\end{aligned}$$

which proves our claim.

Furthermore,

Corollary 3. Let Π be a CGS from a l.p.R. manifold $(\mathbb{M},g,\mathscr{F}$ to a RM $(\mathbb{N},h)$ i.e., $\Pi: M\to N$. Then,

$$\begin{aligned}\widehat{K}(U,V) \geqslant\ & \widehat{K}(\alpha U,\alpha V)+\tfrac{1}{\lambda^2}K^*(\Pi_*\beta U,\Pi_*\beta V)+\|\mathscr{T}_{\alpha U}\alpha V\|^2-g(\mathscr{T}_{\alpha V}\alpha V,\mathscr{T}_{\alpha U}\alpha U)-\tfrac{3}{4}\|\mathscr{V}[\beta U,\beta V]\|^2\\&+\tfrac{\lambda^2}{2}\left\{g(\beta U,\beta V)g\left(\nabla_{\beta V}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),\beta U\right)-g(\beta V,\beta V)g\left(\nabla_{\beta U}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),\beta U\right)+g(\beta V,\beta U)g\left(\nabla_{\beta U}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),\beta V\right)-g(\beta U,\beta U)g\left(\nabla_{\beta V}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),\beta V\right)\right\}\\&+g\left((\nabla_{\alpha U}\mathscr{A})_{\beta V}\beta V,\alpha U\right)+\|\mathscr{A}_{\beta V}\alpha U\|^2\\&-g\left((\nabla_{\beta V}\mathscr{T})_{\alpha U}\beta V,\alpha U\right)-\|\mathscr{T}_{\alpha U}\beta V\|^2+g\left((\nabla_{\alpha V}A)_{\beta U}\beta U,\alpha V\right)\\&+\|\mathscr{A}_{\beta U}\alpha V\|^2-g\left((\nabla_{\beta U}T)_{\alpha V}\beta U,\alpha V\right)-\|\mathscr{T}_{\alpha V}\beta U\|^2-\|\mathscr{T}_U V\|^2+g(\mathscr{T}_V V,\mathscr{T}_U U),\end{aligned}$$

for $U,V\in\Gamma(\ker\Pi_*)$. In above expression equality is satisfied iff Π is a homothetic submersion.

Corollary 4. Let $\Pi:(\mathbb{M},g,\mathscr{F})\to(\mathbb{N},h)$ be a CGS from a l.p.R. manifold $(\mathbb{M},g,\mathscr{F})$ to a RM $(\mathbb{N},h)$. Then,

$$\begin{aligned}K(\mathbb{X},\mathbb{Y}) \geqslant\ & \widehat{K}(B\mathbb{X},B\mathbb{Y})+\|\mathscr{T}_{B\mathbb{X}}B\mathbb{Y}\|^2-g(\mathscr{T}_{B\mathbb{Y}}B\mathbb{Y},\mathscr{T}_{B\mathbb{X}}B\mathbb{X})+\tfrac{1}{\lambda^2}K^*(F_*C\mathbb{X},F_*C\mathbb{Y})-\tfrac{3}{4}\|\mathscr{V}[C\mathbb{X},C\mathbb{Y}]\|^2\\&+\tfrac{\lambda^2}{2}\left\{g(C\mathbb{X},C\mathbb{Y})g\left(\nabla_{C\mathbb{Y}}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),C\mathbb{X}\right)-g(C\mathbb{Y},C\mathbb{Y})g\left(\nabla_{C\mathbb{X}}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),C\mathbb{X}\right)+g(C\mathbb{Y},C\mathbb{X})g\left(\nabla_{C\mathbb{X}}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),C\mathbb{Y}\right)-g(C\mathbb{X},C\mathbb{X})g\left(\nabla_{C\mathbb{Y}}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),C\mathbb{Y}\right)\right\}\\&+\tfrac{\lambda^4}{4}\left\{(g(C\mathbb{X},C\mathbb{X})g(C\mathbb{Y},C\mathbb{Y})-g(C\mathbb{Y},C\mathbb{X})g(C\mathbb{X},C\mathbb{Y}))\|\mathbb{G}\left(\tfrac{1}{\lambda^2}\right)\|^2+\|C\mathbb{X}\left(\tfrac{1}{\lambda^2}\right)C\mathbb{Y}-C\mathbb{Y}\left(\tfrac{1}{\lambda^2}\right)C\mathbb{X}\|^2\right\}+g((\nabla_{B\mathbb{X}}A)_{C\mathbb{Y}}C\mathbb{Y},B\mathbb{X})+\|\mathscr{A}_{C\mathbb{Y}}B\mathbb{X}\|^2\\&-g((\nabla_{C\mathbb{Y}}T)_{B\mathbb{X}}C\mathbb{Y},B\mathbb{X})-\|\mathscr{T}_{B\mathbb{X}}C\mathbb{Y}\|^2+g((\nabla_{B\mathbb{Y}}A)_{C\mathbb{X}}C\mathbb{X},B\mathbb{Y})+g((\nabla_{B\mathbb{Y}}A)_{C\mathbb{X}}C\mathbb{X},B\mathbb{Y})\\&+\|\mathscr{A}_{C\mathbb{X}}B\mathbb{Y}\|^2-g((\nabla_{C\mathbb{X}}T)_{B\mathbb{Y}}C\mathbb{X},B\mathbb{Y})-\|\mathscr{T}_{B\mathbb{Y}}C\mathbb{X}\|^2,\end{aligned}$$

for $\mathbb{X},\mathbb{Y}\in\Gamma\left((\ker\Pi_*)^\perp\right)$. The equality holds iff for any $\mathbb{X}$, $\mathbb{Y}\in\Gamma\left((\ker F_*)^\perp\right)$, $\mathscr{T}_{B\mathbb{X}}B\mathbb{Y}=0$, $\mathscr{A}_{C\mathbb{Y}}B\mathbb{X}=0$ and either $rank\mu=1$ or $\mathbb{G}\lambda|_\mu=0$.

Proof. On using (35) we have,

$$\begin{aligned}K(\mathbb{X},\mathbb{Y})\ & -\|\mathscr{T}_{B\mathbb{X}}B\mathbb{Y}\|^2-\|\mathscr{A}_{C\mathbb{Y}}B\mathbb{X}\|^2-\|C\mathbb{X}\left(\tfrac{1}{\lambda^2}\right)C\mathbb{Y}-C\mathbb{Y}\left(\tfrac{1}{\lambda^2}\right)C\mathbb{X}\|^2\\&=\widehat{K}(B\mathbb{X},B\mathbb{Y})+\|\mathscr{T}_{B\mathbb{X}}B\mathbb{Y}\|^2-g(\mathscr{T}_{B\mathbb{Y}}B\mathbb{Y},\mathscr{T}_{B\mathbb{X}}B\mathbb{X})+\tfrac{1}{\lambda^2}K^*(F_*C\mathbb{X},F_*C\mathbb{Y})-\tfrac{3}{4}\|\mathscr{V}[C\mathbb{X},C\mathbb{Y}]\|^2\\&+\tfrac{\lambda^2}{2}\left\{g(C\mathbb{X},C\mathbb{Y})g\left(\nabla_{C\mathbb{Y}}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),C\mathbb{X}\right)-g(C\mathbb{Y},C\mathbb{Y})g\left(\nabla_{C\mathbb{X}}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),C\mathbb{X}\right)+g(C\mathbb{Y},C\mathbb{X})g\left(\nabla_{C\mathbb{X}}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),C\mathbb{Y}\right)-g(C\mathbb{X},C\mathbb{X})g\left(\nabla_{C\mathbb{Y}}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),C\mathbb{Y}\right)\right\}\\&+\tfrac{\lambda^4}{4}\left\{(g(C\mathbb{X},C\mathbb{X})g(C\mathbb{Y},C\mathbb{Y})-g(C\mathbb{Y},C\mathbb{X})g(C\mathbb{X},C\mathbb{Y}))\|\mathbb{G}\left(\tfrac{1}{\lambda^2}\right)\|^2+\|C\mathbb{X}\left(\tfrac{1}{\lambda^2}\right)C\mathbb{Y}-C\mathbb{Y}\left(\tfrac{1}{\lambda^2}\right)C\mathbb{X}\|^2\right\}+g((\nabla_{B\mathbb{X}}\mathscr{A})_{C\mathbb{Y}}C\mathbb{Y},B\mathbb{X})+\|\mathscr{A}_{C\mathbb{Y}}B\mathbb{X}\|^2\\&-g((\nabla_{C\mathbb{Y}}\mathscr{T})_{B\mathbb{X}}C\mathbb{Y},B\mathbb{X})-\|\mathscr{T}_{B\mathbb{X}}C\mathbb{Y}\|^2+g((\nabla_{B\mathbb{Y}}A)_{C\mathbb{X}}C\mathbb{X},B\mathbb{Y})+g((\nabla_{B\mathbb{Y}}A)_{C\mathbb{X}}C\mathbb{X},B\mathbb{Y})\\&+\|\mathscr{A}_{C\mathbb{X}}B\mathbb{Y}\|^2-g((\nabla_{C\mathbb{X}}T)_{B\mathbb{Y}}C\mathbb{X},B\mathbb{Y})-\|\mathscr{T}_{B\mathbb{Y}}C\mathbb{X}\|^2,\end{aligned}$$

which proves the assertion. The necessary and sufficient condition for the equality is

$$\|\mathscr{T}_{B\mathbb{X}}B\mathbb{Y}\|^2+\|\mathscr{A}_{C\mathbb{Y}}B\mathbb{X}\|^2+\|C\mathbb{X}\left(\frac{1}{\lambda^2}\right)C\mathbb{Y}-C\mathbb{Y}\left(\frac{1}{\lambda^2}\right)C\mathbb{X}\|^2=0.$$

Hence, we obtain $\mathscr{T}_{B\mathbb{X}}B\mathbb{Y}=0$, $\mathscr{A}_{C\mathbb{Y}}B\mathbb{X}=0$, for any $\mathbb{X},\mathbb{Y}\in\Gamma\left((\ker F_*)^\perp\right)$ and either $rank\mu=1$ or $\mathbb{G}\lambda|_\mu=0$.

Moreover,

Corollary 5. Let $\Pi:(\mathbb{M},g,\mathscr{F})\to(\mathbb{N},h)$ be a CGS from a l.p.R. manifold $(\mathbb{M},g,\mathscr{F})$ to a RM $(\mathbb{N},h)$. Then,

$$\begin{aligned} K(\mathbb{X},\mathbb{Y}) \quad & \leqslant \widehat{K}(B\mathbb{X},B\mathbb{Y}) + \|\mathscr{T}_{B\mathbb{X}}B\mathbb{Y}\|^2 - g(\mathscr{T}_{B\mathbb{Y}}B\mathbb{Y},\mathscr{T}_{B\mathbb{X}}B\mathbb{X}) + \tfrac{1}{\lambda^2}K^*(F_*C\mathbb{X},F_*C\mathbb{Y}) \\ & + \tfrac{\lambda^2}{2}\left\{g(C\mathbb{X},C\mathbb{Y})g\left(\nabla_{C\mathbb{Y}}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),C\mathbb{X}\right) - g(C\mathbb{Y},C\mathbb{Y})g\left(\nabla_{C\mathbb{X}}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),C\mathbb{X}\right) + g(C\mathbb{Y},C\mathbb{X})g\left(\nabla_{C\mathbb{X}}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),C\mathbb{Y}\right) - g(C\mathbb{X},C\mathbb{X})g\left(\nabla_{C\mathbb{Y}}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),C\mathbb{Y}\right)\right\} \\ & + \tfrac{\lambda^4}{4}\left\{(g(C\mathbb{X},C\mathbb{X})g(C\mathbb{Y},C\mathbb{Y}) - g(C\mathbb{Y},C\mathbb{X})g(C\mathbb{X},C\mathbb{Y}))\|\mathbb{G}\left(\tfrac{1}{\lambda^2}\right)\|^2 + \|C\mathbb{X}\left(\tfrac{1}{\lambda^2}\right)C\mathbb{Y} - C\mathbb{Y}\left(\tfrac{1}{\lambda^2}\right)C\mathbb{X}\|^2\right\} + g\left((\nabla_{B\mathbb{X}}A)_{C\mathbb{Y}}C\mathbb{Y},B\mathbb{X}\right) + \|\mathscr{A}_{C\mathbb{Y}}B\mathbb{X}\|^2 \\ & - g\left((\nabla_{C\mathbb{Y}}T)_{B\mathbb{X}}C\mathbb{Y},B\mathbb{X}\right) + g\left((\nabla_{B\mathbb{Y}}A)_{C\mathbb{X}}C\mathbb{X},B\mathbb{Y}\right) + g\left((\nabla_{B\mathbb{Y}}A)_{C\mathbb{X}}C\mathbb{X},B\mathbb{Y}\right) \\ & + \|A_{C\mathbb{X}}B\mathbb{Y}\|^2 - g\left((\nabla\mathscr{A}_{C\mathbb{X}}T)_{B\mathbb{Y}}C\mathbb{X},B\mathbb{Y}\right) - \|\mathscr{T}_{B\mathbb{Y}}C\mathbb{X}\|^2, \end{aligned}$$

for $\mathbb{X},\mathbb{Y} \in \Gamma(\ker\Pi_*)^{\perp}$. The equality holds iff $\mathscr{T}_{B\mathbb{X}}C\mathbb{Y} = 0$ and $[C\mathbb{X},C\mathbb{Y}] \in \Gamma(\mathscr{H})$.

Corollary 6. Let $\Pi : (\mathbb{M},g,\mathscr{F}) \to (\mathbb{N},h)$ be a CGS from a l.p.R. manifold $(\mathbb{M},g,\mathscr{F})$ to a RM $(\mathbb{N},h)$. Then,

$$\begin{aligned} K(\mathbb{X},U) \quad & \geqslant \widehat{K}(B\mathbb{X},\alpha U) + \|\mathscr{T}_{B\mathbb{X}}\alpha U\|^2 - g(\mathscr{T}_{\alpha U}\alpha U,\mathscr{T}_{B\mathbb{X}}B\mathbb{X}) \\ & + g\left((\nabla\mathscr{A}_{B\mathbb{X}}A)_{\beta U}\beta U,B\mathbb{X}\right) + \|\mathscr{A}_{\beta U}B\mathbb{X}\|^2 \\ & - g\left((\nabla_{\beta U}T)_{B\mathbb{X}}\beta U,B\mathbb{X}\right) - \|\mathscr{T}_{B\mathbb{X}}\beta U\|^2 + g\left((\nabla_{\alpha U}A)_{C\mathbb{X}}C\mathbb{X},\alpha V\right) \\ & + \|\mathscr{A}_{C\mathbb{X}}\alpha U\|^2 - g\left((\nabla_{C\mathbb{X}}T)_{\alpha U}C\mathbb{X},\alpha U\right) - \|\mathscr{T}_{\alpha U}C\mathbb{X}\|^2 \\ & + \tfrac{1}{\lambda^2}h(R^*(\Pi_*C\mathbb{X},\Pi_*\beta U)\Pi_*\beta U,\Pi_*C\mathbb{X}) \\ & - \tfrac{3}{4}\|\mathscr{V}[C\mathbb{X},\beta U]\|^2 + 2g(\mathscr{V}[C\mathbb{X},\beta U],\mathscr{V}[\beta U,C\mathbb{X}])\} \\ & + \tfrac{\lambda^2}{2}\left\{g(C\mathbb{X},\beta U)g\left(\nabla_{\beta U}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),C\mathbb{X}\right) - g(\beta U,\beta U)g\left(\nabla_{C\mathbb{X}}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),C\mathbb{X}\right) + g(\beta U,C\mathbb{X})g\left(\nabla_{C\mathbb{X}}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),\beta U\right) - g(C\mathbb{X},C\mathbb{X})g\left(\nabla_{\beta U}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),\beta U\right)\right\} \\ & + \tfrac{\lambda^4}{4}\left\{(g(C\mathbb{X},C\mathbb{X})g(\beta U,\beta U) - g(\beta U,C\mathbb{X})g(C\mathbb{X},\beta U))\|\mathbb{G}\left(\tfrac{1}{\lambda^2}\right)\|^2 + \|C\mathbb{X}\left(\tfrac{1}{\lambda^2}\right)\beta U - \beta U\left(\tfrac{1}{\lambda^2}\right)C\mathbb{X}\|^2\right\}, \end{aligned}$$

for $\mathbb{X} \in \Gamma\left((\ker F_*)^{\perp}\right)$ and $U \in \Gamma(\ker F_*)$. The equality holds iff $\mathscr{A}_{\mathscr{F}U}B\mathbb{X} = 0, \mathbb{G}\left(\frac{1}{\lambda^2}\right) = 0$ and $\mathscr{F}$ horizontally homothetic submersion.

Proof. On Using (36) we have,

$$\begin{aligned} K_M(\mathbb{X},U) \quad & - \|\mathscr{A}_{\beta U}B\mathbb{X}\|^2 - \tfrac{\lambda^4}{4}\left\{g(C\mathbb{X},C\mathbb{X})\|\mathbb{G}\left(\tfrac{1}{\lambda^2}\right)\|^2 - \|C\mathbb{X}\left(\tfrac{1}{\lambda^2}\right)\beta U - \beta U\left(\tfrac{1}{\lambda^2}\right)C\mathbb{X}\|^2\right\} \\ & = \widehat{K}(B\mathbb{X},\alpha U) + \|\mathscr{T}_{B\mathbb{X}}\alpha U\|^2 - g(\mathscr{T}_{\alpha U}\alpha U,\mathscr{T}_{B\mathbb{X}}B\mathbb{X}) \\ & + g\left((\nabla_{B\mathbb{X}}\mathscr{A})_{\beta U}\beta U,B\mathbb{X}\right) + \|\mathscr{A}_{\beta U}B\mathbb{X}\|^2 \\ & - g\left((\nabla_{\beta U}\mathscr{T})_{B\mathbb{X}}\beta U,B\mathbb{X}\right) - \|\mathscr{T}_{B\mathbb{X}}\beta U\|^2 + g\left((\nabla_{\alpha U}A)_{C\mathbb{X}}C\mathbb{X},\alpha V\right) \\ & + \|\mathscr{A}_{C\mathbb{X}}\alpha U\|^2 - g\left((\nabla_{C\mathbb{X}}T)_{\alpha U}C\mathbb{X},\alpha U\right) - \|\mathscr{T}_{\alpha U}C\mathbb{X}\|^2 \\ & + \tfrac{1}{\lambda^2}h(R^*(\Pi_*C\mathbb{X},\Pi_*\beta U)\Pi_*\beta U,\Pi_*C\mathbb{X}) \\ & - \tfrac{3}{4}\|\mathscr{V}[C\mathbb{X},\beta U]\|^2 + 2g(\mathscr{V}[C\mathbb{X},\beta U],\mathscr{V}[\beta U,C\mathbb{X}])\} \\ & + \tfrac{\lambda^2}{2}\left\{g(C\mathbb{X},\beta U)g\left(\nabla_{\beta U}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),C\mathbb{X}\right) - g(\beta U,\beta U)g\left(\nabla_{C\mathbb{X}}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),C\mathbb{X}\right) + g(\beta U,C\mathbb{X})g\left(\nabla_{C\mathbb{X}}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),\beta U\right) - g(C\mathbb{X},C\mathbb{X})g\left(\nabla_{\beta U}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),\beta U\right)\right\} \\ & + \tfrac{\lambda^4}{4}\left\{(g(C\mathbb{X},C\mathbb{X})g(\beta U,\beta U) - g(\beta U,C\mathbb{X})g(C\mathbb{X},\beta U))\|\mathbb{G}\left(\tfrac{1}{\lambda^2}\right)\|^2 + \|C\mathbb{X}\left(\tfrac{1}{\lambda^2}\right)\beta U - \beta U\left(\tfrac{1}{\lambda^2}\right)C\mathbb{X}\|^2\right\}, \end{aligned}$$

This follows the inequality. For the equality case,

$$\|\mathscr{A}_{\beta U}B\mathbb{X}\|^2 + \frac{\lambda^4}{4}\left\{g(C\mathbb{X},C\mathbb{X})\|\mathbb{G}\left(\frac{1}{\lambda^2}\right)\|^2 + \|C\mathbb{X}\left(\frac{1}{\lambda^2}\right)\beta U - \beta U\left(\frac{1}{\lambda^2}\right)C\mathbb{X}\|^2\right\} = 0.$$

Thus, the eqality follows if and only if $\mathscr{A}_{\beta U}B\mathbb{X} = 0$ and $\mathbb{G}\left(\frac{1}{\lambda^2}\right) = 0$,
$C\mathbb{X}\left(\frac{1}{\lambda^2}\right)\beta U - \beta U\left(\frac{1}{\lambda^2}\right)C\mathbb{X} = 0$ which ensures that Π is horizontally homothetic.

We now conclude this section with the following result;

Corollary 7. Let $\Pi : (M,g,\mathscr{F}) \to (N,h)$ be a CGS from a l.p.R. manifold $(M,g,\mathscr{F})$ to a RM (N,h). Then,

$$\begin{aligned} K_M(\mathbb{X},U) \quad & \leqslant \widehat{K}(B\mathbb{X},\alpha U) + \|\mathscr{T}_{B\mathbb{X}}\alpha U\|^2 - g(\mathscr{T}_{\alpha U}\alpha U,\mathscr{T}_{B\mathbb{X}}B\mathbb{X}) \\ & + g\left((\nabla_{B\mathbb{X}}\mathscr{A})_{\beta U}\beta U,B\mathbb{X}\right) + \|\mathscr{A}_{\beta U}B\mathbb{X}\|^2 \\ & - g\left((\nabla_{\beta U}\mathscr{T})_{B\mathbb{X}}\beta U,B\mathbb{X}\right) + g\left((\nabla_{\alpha U}A)_{C\mathbb{X}}C\mathbb{X},\alpha V\right) \\ & + \|\mathscr{A}_{C\mathbb{X}}\alpha U\|^2 - g\left((\nabla_{C\mathbb{X}}\mathscr{T})_{\alpha U}C\mathbb{X},\alpha U\right) - \|\mathscr{T}_{\alpha U}C\mathbb{X}\|^2 \\ & + \tfrac{1}{\lambda^2}h(R^*(\Pi_*C\mathbb{X},\Pi_*\beta U)\Pi_*\beta U,\Pi_*C\mathbb{X}) + 2g(\mathscr{V}[C\mathbb{X},\beta U],\mathscr{V}[\beta U,C\mathbb{X}])\} \\ & + \|\mathscr{A}_{\beta U}B\mathbb{X}\|^2 + \tfrac{\lambda^4}{4}\left\{g(C\mathbb{X},C\mathbb{X})\|\mathbb{G}\left(\tfrac{1}{\lambda^2}\right)\|^2 + \|C\mathbb{X}\left(\tfrac{1}{\lambda^2}\right)\beta U - \beta U\left(\tfrac{1}{\lambda^2}\right)C\mathbb{X}\|^2\right\} \\ & + \tfrac{\lambda^2}{2}\left\{g(C\mathbb{X},\beta U)g\left(\nabla_{\beta U}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),C\mathbb{X}\right) - g(\beta U,\beta U)g\left(\nabla_{C\mathbb{X}}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),C\mathbb{X}\right) + g(\beta U,C\mathbb{X})g\left(\nabla_{C\mathbb{X}}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),\beta U\right) - g(C\mathbb{X},C\mathbb{X})g\left(\nabla_{\beta U}\mathbb{G}\left(\tfrac{1}{\lambda^2}\right),\beta U\right)\right\} \\ & + \tfrac{\lambda^4}{4}\left\{(g(C\mathbb{X},C\mathbb{X})g(\beta U,\beta U) - g(\beta U,C\mathbb{X})g(C\mathbb{X},\beta U))\|\mathbb{G}\left(\tfrac{1}{\lambda^2}\right)\|^2 + \|C\mathbb{X}\left(\tfrac{1}{\lambda^2}\right)\beta U - \beta U\left(\tfrac{1}{\lambda^2}\right)C\mathbb{X}\|^2\right\}, \end{aligned}$$

for $\mathbb{X} \in \Gamma(\ker F_*)^{\perp}$ and $U \in \Gamma(\ker F_*)$. The equality case follows iff $\mathscr{T}_{B\mathbb{X}}\beta U = 0$ and $[C\mathbb{X},\beta U] \in \Gamma(\mathscr{H})$.

Data Availability Statement

My manuscript has no associated data set.

Author Contributions

All authors contributed to the study conception and design. All authors read and approved the final manuscript.

Funding

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

References

Akyol, M.A., 2017. Conformal semi-slant submersions. Int. J. Geometric Methods Modern Phys. 14 (7), 1750114.

Akyol, M.A., 2017. Conformal semi-invariant submersions from almost product Riemannian manifolds. Acta Mathematica Vietnamica 42 (3), 491–507. https://doi.org/10.1007/s40306-016-0193-9.

Akyol, M.A., 2021. Conformal generic submersion. Turk. J. Math. 45, 201–219. https://doi.org/10.3906/mat-2005-40.

Akyol, M.A., Şahin, B., 2016. Conformal anti-invariant submersions from almost Hermitian manifolds. Turkish J. Mathe. 40 (1), 43–70.

Akyol, M.A., Şahin, B., 2017. Conformal semi-invariant submersions. Commun. Contemp. Mathe. 19 (2), 1650011. https://doi.org/10.1142/S0219199716500115.

Akyol, M.A., Şahin, B., 2019. Conformal slant submersions. Hacettepe J. Mathe. Stat. 48 (1), 28–44.

Ali, S., Fatima, T., 2013. Generic Riemannian submersions. Tamkang J. Mathe. 44 (4), 395–409.

Altafini, C., 2004. Redundant Robotic chains on Riemannian submersions. IEEE Trans. Robot. Automat. 20(20), 335–340.

Baird, P., Wood, J.C., 2003. Harmonic Morphisms Between Riemannian Manifolds, London Mathematical Society Monographs, 29. Oxford University Press, The Clarendon Press, Oxford.

Bourguignon, J.P., 1990. A mathematian's visit to Klauza-Klein theory, Rend. Sem. Mat. Univ. Politec. Torino 1989, Special Issue, 143–163.

Bourguignon, J.P., Lawson, Jr. H.B., 1981. Stability and isolation phenomena for Yang-Mills fields, Comm. Math. Phys. 79(2), 189–230.

Chen, B.Y., 1981. Differential geometry of real submanifolds in a Kaehler manifold. Monatshefte für Mathematik 91, 257–274.

Chinea, D., 1985. Almost contact metric submersions. Rend. Circ. Mat. Plaeermo, 34 (1), 89–104.

Escobales, R.H. Jr., Riemannian submersions from complex projective space, J. Differential Geometry 13(1), 93–107.

Falcitelli, M., Ianus, S., Pastore, A.M., 2004. Riemannian submersions and Related Topics. World Scientific, River Edge, NJ.

Gray, A., 1997. Pseudo-Riemannian almost product manifolds and submersions. J. Appl. Mathe. Mech. 16, 715–737.

Gromoll, D., Klingenberg, W., Meyer, W., 1975. Riemannsche Geometrie im Groβen. In: Lecture Notes in Mathematics 55. Springer.

Gudmundsson, S., 1992. The Geometry of Harmonic Morphisms. University of Leeds. Ph.D. Thesis.

Gudmundsson, S., Wood, J.C., 1997. Harmonic Morphisms between almost Hermitian manifolds. Bollettino dell'Unione Matematica Italiana 11 (2), 185–197.

Ianus, S., Visinescu, M., 1987. Kaluza-Klein theory with scalar filds and generalised Hopf manifolds. Classical Quantum Gravity 4(5), 1317–1325.

Ianus, S., and Visinescu, M., 1991. Space-time compactification and Riemannian submersions in The mathematical mheritage of C, F. Gauss, 358–371, world Sci. Publ., River Edge, NJ.

Ianus, S., Mazzocco, R., Vilcu, G.E., 2008. Riemannian submersions from quaternionic manifolds. Acta Appl. Mathe. 104 (1), 83–89.

Marrero, J.C., Rocha, J., 1994. Locally conformal Kähler submersions. Geom. Dedicata. 52 (3), 271–289.

O'Neill, B., 1966. The fundamental equations of a submersion. Michigan Mathe. J. 13, 458–469.

Özdemir, F., Sayar, C., Taştan, H.M., 2017. Semi-invariant submersions whose total manifolds are locally product Riemannian. Quaest. Mathe. 40 (7), 909–929.

Park, K.S., Prasad, R., 2013. Semi-slant submersions. Bull. Korean Mathe. Soc. 50 (3), 951–962.

Şahin, B., 2010. Anti-invariant Riemannian submersions from almost Hermitian manifolds. Central Eur. J. Mathe. 3, 437–447.

Şahin, B., 2011. Semi-invariant Riemannian submersions from almost Hermitian manifolds. Can. Mathe. Bull. 56, 173–182.

Şahin, B., 2011. Slant submersions from almost Hermitian manifolds. Bull. Mathe. Societe des Sciences Mathe. Roumanie 54 (1), 93–105.

Tastan, H.M., 2014. On Langrangian submersions Hacettepe. J. Math. Stat. 43 (6), 993–1000.

Urakawa, H., 1993. Calculus of Variations and Harmonic Maps. American Mathematical Society, Providence, RI.

Watson, B., 1976. Almost Hermitian submersions. J. Diff. Geometry 11 (1), 147–165.

Yano, K., Kon, M., 1984. Structures on Manifolds. World Scientific, Singapore.