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Traveling wave solutions of conformable time-fractional Klien-Fock-Gordon equation by the improved $\tan(\Psi(\zeta)/2)$-expansion method

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ABSTRACT

This paper considers the improved $\tan(\Psi(\zeta)/2)$-expansion method to retrieve the traveling wave solutions of the time-fractional Klien-Fock-Gordon equation. The quadratic case has been discussed for the proposed model. The conformable derivative has been applied on the proposed model for investigating the fractional effects. As a result, abundant traveling wave solutions are found. The method contributes four kinds of solutions which are exponential, hyperbolic, trigonometric and rational function solutions for the aforementioned model. The obtained results are also explained graphically for some particular choices of fractional order $\alpha, 0 < \alpha < 1$. The fractional effects on solutions have been illustrated through 3D plots and line graphs.

1. Introduction

The Klien-Fock-Gordon equation (KFG) models various physical phenomena in quantum field theory, condensed matter physics, nonlinear equations, correlation of solitons, collisionless plasma and theory of relativity etc. It was established by Osker Klien, Viladimir Fock and Walter Gordon for determining relativistic electrons. This equation is calibrated version of the relativistic energy–momentum correspondence. It is associated with Schrödinger equation and also referred to as relativistic wave equation.

In recent times, various research scholars of science and telecommunication have taken fractional order mathematical models under consideration. Fractional calculus has become a preeminent tool to analyze and characterize the nonlinear complex phenomena. Many partial differential equations (PDEs) that exhibit different physical phenomena are remodeled in terms of fractional derivative and their behavior is observed along with their solutions by developed methods (Akgül, 2018; Mamun et al., 2021). Fractional differential equations are widely studied in engineering mathematics and their fractional effects are observed for particular choices of fractional order parameters (Shahen et al., 2020; Shahen et al., 2021; Mamun et al., 2021; Hosseini et al., 2020).

This study considers the conformable time-fractional form of the Klein-Fock-Gordon equation (Alam et al., 2021)

$$q_{tt}(x,t) + \omega_1 q_{xx}(x,t) + \omega_2 q(x,t) + \omega_3 q^r(x,t) = 0 \quad (1.1)$$

is investigated for its quadratic state $r = 2$. Many efforts have been made to extract closed form and approximate solutions of KFG equation by employing different mathematical techniques till now. The homotopy analysis method is applied to obtain approximate analytical solutions in Khan and Rasheed (2015), the differential transform method is implemented in Kanth and Aruna (2009) for special cases of KFG model ($\omega_2 = 0, \omega_3 \neq 0$ and $\omega_2 \neq 0, \omega_3 = 0$). Many true invariant solutions are found in Álvarez et al. (2012) by Lie-symmetry method. Similarly, many other techniques are used for solving KFG equation, such as q-homotopy analysis transform method (q-HATM), modified (G'/G)-expansion method, Kudryashov-expansion method, He's variational iteration method and Nikiforov-Uvarov method etc. Veeresha et al. (2020), Aero et al. (2013), Aruna and Kanth (2014), Kragh (1984) and Asaduzzaman et al. (2020).

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Keeping in view the importance of fractional effects, the conformable time-fractional Klien-Fock-Gordon equation is solved to construct solitary wave solutions by $\tan(\Psi(\zeta)/2)$ method for quadratic state. The conformable fractional derivative is an extension of Caputo derivative and provides a new designated memory length. It coincides with classic definition of first derivative for $\alpha = 1$. Furthermore, the conformable derivative satisfies the property of product rule, chain rule and quotient rule of two functions which is not usually satisfied by all the fractional derivative (Zheng et al., 2019). The real applications of the conformable time-fractional form of the Klein-Fock-Gordon equation include the study of different varieties of matter, spread of deviation in crystals, properties of fundamental particles and the effects and preservation of inherited memory.

Various mathematical techniques have been developed to construct the traveling wave solutions of nonlinear evolution equations (NLEEs). Some recently proposed methods include exp-$(\phi(\xi))$-expansion method (Elboree, 2020), generalized projective Riccati equations method (Akram et al., 2021), modified auxiliary equation mapping method (Sajid and Akram, 2021), Legendre-homotopy analysis method (Sadaf and Akram, 2020), first integral method (Lu, 2012), extended rational sine–cosine method (Mahak and Akram, 2019), ϕ^6-expansion method (Sajid and Akram, 2020), modified extended tanh-function method (Mamun et al., 2020; Alam et al., 2021) and several others. Various nonlinear partial differential equations have been solved to extract traveling wave solutions for studying real world phenomena (Hosseini et al., 2021; Hosseini et al., 2021; Mamun et al., 2018). From the past few decades, a variety of NLDEs have been considered for modeling complex nonlinear physical problems and their solutions have diverse applications in different fields of nonlinear sciences (Mamun et al., 2021; Ananna and Mamun, 2020; Mamun et al., 2019; Mamun and Asaduzzaman, 2019).

This article appraises the $\tan(\Psi(\zeta)/2)$-expansion method that was first introduced by Manafian and Lakestani (2015) in 2015. This is a powerful method and has been used to obtain closed form solution of many nonlinear PDEs with accuracy. The method possesses more novel solutions with wider range of implementation to deal with NLEEs. The method has been applied to a variety of NLEEs for finding their closed form solutions. A few of which are the time-fractional Kuramoto–Sivashinsky equation (Manafian and Foroutan, 2017), $(2+1)$-dimensional KP-BBM wave equation (Khan et al., 2018), Biswas-Milovic equation (Manafian and Lakestani, 2016), Gerdjikov-Ivanov equation (Manafian and Lakestani, 2016) and generalised Hirota–Satsuma coupled KdV (HScKdV) equation (Özkan and Yaşar, 2020).

This paper is coordinated as follows: In Section 2, the proposed model is introduced. In Section 3, the improved $\tan(\Psi(\zeta)/2)$-expansion method is described. In Section 4, mathematical analysis is applied on quadratic KFG equation with conformable derivative. In Section 5, graphical illustration of some solutions is given. The conclusion is presented in Section 6.

2. Governing model

The quadratic conformable time-fractional KFG equation is considered, as

$$D_t^{2\alpha}q(x,t) + \omega_1 q_{xx}(x,t) + \omega_2 q(x,t) + \omega_3 q^2(x,t) = 0,\ 0<\alpha<1, \tag{2.1}$$

where D_t^{α} is a factional derivative of conformable type. The variables x and t shows spatial and temporal evolution respectively and ω_1, ω_2 and ω_3 are non-zero constants.

Conformable Derivative A new definition of conformable fractional derivative was proposed by Khalil et al. (2014) in 2014.

The conformable derivative of function $g:[0,\infty)$ of fractional order α, where $g = g(t)$ is defined on the positive half plane is expressed, as

$$D_t^{\alpha}g(t) = \lim_{\epsilon\to 0}\frac{g(t+\epsilon t^{1-\alpha}) - g(t)}{\epsilon} \tag{2.2}$$

for all $t>0, \alpha\in(0,1)$.

Properties Following properties are satisfied:

$$\begin{aligned}
&1.\ D_t^{\alpha}(ag+bh) = a(D_t^{\alpha}g) + b(D_t^{\alpha}h),\ \text{forall } a,b\in R.\\
&2.\ D_t^{\alpha}(t^p) = p\,t^{p-\alpha},\ \forall\, p\in R.\\
&3.\ D_t^{\alpha}(\lambda) = 0,\ \forall \text{constantfunctions } g(t)=\lambda.\\
&4.\ D_t^{\alpha}(gh) = gD_t^{\alpha}(h) + hD_t^{\alpha}(g).\\
&5.\ D_t^{\alpha}(\tfrac{g}{h}) = \tfrac{gD_t^{\alpha}(h) - hD_t^{\alpha}(g)}{h^2}.\\
&6.\ \text{If } g \text{ isdifferentiablethen, } D_t^{\alpha}(g)(t) = t^{1-\alpha}(\tfrac{dg}{dt}(t)).
\end{aligned} \tag{2.3}$$

Theorem The following property is satisfied by conformable differentiable function $g:(0,\infty)$ and a differentiable function h determined in the range of g

$$D_t^{\alpha}(g\circ h)(t) = t^{1-\alpha}h'(t)g'(h(t)),$$

where $\prime$ denotes the derivative with respect to α.

3. Description of Improved tan$((\zeta)/2)$-Expansion Method

In this section, the improved $\tan(\Psi(\zeta)/2)$-expansion method (Manafian and Lakestani, 2016) is briefly described. The nonlinear partial differential equation is considered in the form

$$\Omega(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}\ldots) = 0, \tag{3.1}$$

where Ω is a polynomial in $u(x,t)$ and its partial derivatives. This technique requires the following steps to be taken.

Step 1: The traveling wave transformation is considered, as

$$u(x,t) = U(\zeta),\quad \zeta = x - \upsilon t, \tag{3.2}$$

where ζ is wave variable. The speed of wave is denoted by the constant υ that is to be evaluated. Eq. (3.1) is transformed into ODE, as

$$Q(U, U', -\upsilon U', U'', \upsilon^2 U, \ldots) = 0. \tag{3.3}$$

Step 2: Eq. (3.3) has solution of the form

$$\begin{aligned} U(\zeta) &= R(\Psi)\\ &= \sum_{i=0}^{k}\alpha_i\left[c+\tan\left(\frac{\Psi(\zeta)}{2}\right)\right]^i + \sum_{i=0}^{k}\beta_i\left[c+\tan\left(\frac{\Psi(\zeta)}{2}\right)\right]^{-i}, \end{aligned} \tag{3.4}$$

where $\alpha_i (0\leqslant i\leqslant k)$ and $\beta_i(0\leqslant i\leqslant k)$ are constants to be found. The function $\Psi = \Psi(\zeta)$ satisfies the relation

$$\Psi'(\zeta) = l\sin(\Psi(\zeta)) + m\cos(\Psi(\zeta)) + n. \tag{3.5}$$

Eq. (3.5) provides the following families of solutions (Akram et al., 2021).

Family 1: If $l^2+m^2-n^2<0$ and $m-n\neq 0$, then

$$\Psi(\zeta) = 2\arctan\left[\frac{l}{m-n} - \frac{\sqrt{n^2-m^2-l^2}}{m-n}\tan\left(\frac{\sqrt{n^2-m^2-l^2}}{2}(\zeta+C)\right)\right]. \tag{3.6}$$

Family 2: If $l^2+m^2-n^2>0$ and $m-n\neq 0$, then

$$\Psi(\zeta) = 2\arctan\left[\frac{l}{m-n} + \frac{\sqrt{m^2+l^2-n^2}}{m-n}\tanh\left(\frac{\sqrt{m^2+l^2-n^2}}{2}(\zeta+C)\right)\right]. \quad (3.7)$$

Family 3: If $l^2+m^2-n^2>0, m\neq 0$ and $n=0$, then

$$\Psi(\zeta) = 2\arctan\left[\frac{l}{m} + \frac{\sqrt{m^2+l^2}}{m}\tanh\left(\frac{\sqrt{m^2+l^2}}{2}(\zeta+C)\right)\right]. \quad (3.8)$$

Family 4: If $l^2+m^2-n^2<0, n\neq 0$ and $m=0$, then

$$\Psi(\zeta) = 2\arctan\left[\frac{-l}{n} + \frac{\sqrt{n^2-l^2}}{n}\tan\left(\frac{\sqrt{n^2-l^2}}{2}(\zeta+C)\right)\right]. \quad (3.9)$$

Family 5: If $l^2+m^2-n^2>0, m-n\neq 0$ and $l=0$, then

$$\Psi(\zeta) = 2\arctan\left[\sqrt{\frac{m+n}{m-n}}\tanh\left(\frac{\sqrt{m^2-n^2}}{2}(\zeta+C)\right)\right]. \quad (3.10)$$

Family 6: If $l=0$ and $n=0$, then

$$\Psi(\zeta) = \arctan\left[\frac{e^{2m(\zeta+C)}-1}{e^{2m(\zeta+C)}+1}, \frac{2e^{m(\zeta+C)}}{e^{2m(\zeta+C)}+1}\right]. \quad (3.11)$$

Family 7: If $m=0$ and $n=0$, then

$$\Psi(\zeta) = \arctan\left[\frac{2e^{l(\zeta+C)}}{e^{2l(\zeta+C)}+1}, \frac{e^{2l(\zeta+C)}-1}{e^{2l(\zeta+C)}+1}\right]. \quad (3.12)$$

Family 8: If $l^2+m^2=n^2$, then

$$\Psi(\zeta) = -2\arctan\left[\frac{(m+n)(l(\zeta+C)+2)}{l^2(\zeta+C)}\right]. \quad (3.13)$$

Family 9: If $l=m=n=il$, then

$$\Psi(\zeta) = 2\arctan\left[e^{il(\zeta+C)}-1\right]. \quad (3.14)$$

Family 10: If $l=n=il$ and $m=-il$, then

$$\Psi(\zeta) = -2\arctan\left[\frac{e^{il(\zeta+C)}}{-1+e^{il(\zeta+C)}}\right]. \quad (3.15)$$

Family 11: If $n=l$, then

$$\Psi(\zeta) = -2\arctan\left[\frac{(l+m)e^{m(\zeta+C)}-1}{(l-m)e^{m(\zeta+C)}-1}\right]. \quad (3.16)$$

Family 12: If $l=n$, then

$$\Psi(\zeta) = 2\arctan\left[\frac{(m+n)e^{m(\zeta+C)}+1}{(m-n)e^{m(\zeta+C)}-1}\right]. \quad (3.17)$$

Family 13: If $n=-l$, then

$$\Psi(\zeta) = 2\arctan\left[\frac{e^{m(\zeta+C)}+m-l}{e^{m(\zeta+C)}-m-l}\right]. \quad (3.18)$$

Family 14: If $m=-n$, then

$$\Psi(\zeta) = -2\arctan\left[\frac{le^{l}(\zeta+C)}{ne^{l(\zeta+C)}-1}\right]. \quad (3.19)$$

Family 15: If $m=0$ and $l=n$, then

$$\Psi(\zeta) = -2\arctan\left[\frac{n(\zeta+C)+2}{n(\zeta+C)}\right]. \quad (3.20)$$

Family 16: If $l=0$ and $m=n$, then

$$\Psi(\zeta) = 2\arctan\left[n(\zeta+C)\right]. \quad (3.21)$$

Family 17: If $l=0$ and $m=-n$, then

$$\Psi(\zeta) = -2\arctan\left[\frac{1}{n(\zeta+C)}\right]. \quad (3.22)$$

Family 18: If $l=0$ and $m=0$, then

$$\Psi(\zeta) = n\zeta + C. \quad (3.23)$$

where $\alpha_i, \beta_i (i=0,1,2,\ldots,k), l, m, n$ are constants to be determined.

Step 3: A positive integer k is determined using homogeneous balance rule. If the constant k is not a positive integer, the following transformation is applied.

(a) If k is a fraction of the form $k=p/q$, then assume

$$U(\zeta) = v^{p/q}(\zeta). \quad (3.24)$$

Substitute Eq. (3.24) in Eq. (3.3), to determine the value of k using homogeneous balance rule.

(b) On the other hand if k is a negative number, then assume

$$U(\zeta) = v^{k}(\zeta). \quad (3.25)$$

Inserting Eq. (3.25) into Eq. (3.3), the value of k is determined by homogeneous balance rule.

Step 4: Eq. (3.4) is substituted into Eq. (3.3) using the value of k obtained in Step 3 and the coefficients of $(\tan(\Psi(\zeta)/2))^i$ and $(\cot(\Psi(\zeta)/2))^i$ are collected. Setting these coefficients to zero, a set of algebraic equations is attained in $\alpha_0, \alpha_i, \beta_i (i=1,2,\ldots,k), l, m, n, \lambda$ and c.

Step 5: Putting the values of $\alpha_0, \alpha_1, \beta_1, \ldots, \alpha_i, \beta_i, \lambda$ and c into Eq. (3.4) the closed form solution of Eq. (3.1) is obtained.

4. Exact solutions of quadratic time-fractional KFG equation

The reduced differential equation form of Eq. (2.1) is acquired by the complex-wave transform

$$\zeta = x - v\frac{t^\alpha}{\alpha}. \quad (4.1)$$

The application of Eq. (4.1) on Eq. (2.1) yields

$$(v^2+\omega_1)q''(\zeta) + \omega_2 q(\zeta) + \omega_3 q^2(\zeta) = 0. \quad (4.2)$$

Using homogeneous balance rule, the integer $m=2$ is obtained. Then, the formal solution by improved $\tan(\Psi(\zeta)/2)$-expansion method for $m=2$ becomes

$$\begin{aligned} u(\zeta) &= \alpha_0 + \alpha_1\left[\tan\left(\tfrac{\Psi(\zeta)}{2}\right)\right] + \alpha_2\left[\tan\left(\tfrac{\Psi(\zeta)}{2}\right)\right]^2 \\ &+\beta_1\left[\tan\left(\tfrac{\Psi(\zeta)}{2}\right)\right]^{-1} + \beta_2\left[\tan\left(\tfrac{\Psi(\zeta)}{2}\right)\right]^{-2}. \end{aligned} \quad (4.3)$$

Eq. (4.3) is inserted into Eq. (4.1) and the coefficients of $\left(\tan(\frac{\Psi(\zeta)}{2})\right)^i, \left(\cot(\frac{\Psi(\zeta)}{2})\right)^i$ are collected. Then, setting each coefficient to zero a system of algebraic equations is formed. As a result, the following sets of traveling wave solutions are obtained:

Set 1:

$$\begin{aligned} &v = \sqrt{-\omega_1 + \tfrac{\omega_2}{l^2+m^2-n^2}}, \ \alpha_0 = -\tfrac{1}{2}\tfrac{(2l^2-m^2+n^2)\omega_2}{(l^2+m^2-n^2)\omega_3}, \ \alpha_1 = 0, \\ &\alpha_2 = 0, \ \beta_1 = -\tfrac{3\omega_2 l(m+n)}{(l^2+m^2-n^2)\omega_3}, \ \beta_2 = -\tfrac{3}{2}\tfrac{(m+n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}, \end{aligned} \quad (4.4)$$

$$U(\zeta) = \alpha_0 + \beta_1\left[\tan\left(\frac{\Psi(\zeta)}{2}\right)\right]^{-1} + \beta_2\left[\tan\left(\frac{\Psi(\zeta)}{2}\right)\right]^{-2}. \quad (4.5)$$

By Eq. (4.5), the **Families 1, 2, 3, 4, 5, 6** and **7** are written respectively, as

$$\begin{aligned} u_1(\zeta) = & -\frac{1}{2}\frac{(2l^2-m^2+n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} \\ & -\frac{3\omega_2 l(m+n)}{(l^2+m^2-n^2)\omega_3}\left[\frac{l}{m-n}-\frac{\sqrt{n^2-m^2-l^2}}{m-n}\tan\left(\frac{\sqrt{n^2-m^2-l^2}}{2}(\zeta+C)\right)\right]^{-1} \\ & -\frac{3}{2}\frac{(m+n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\frac{l}{m-n}-\frac{\sqrt{n^2-m^2-l^2}}{m-n}\tan\left(\frac{\sqrt{n^2-m^2-l^2}}{2}(\zeta+C)\right)\right]^{-2}, \end{aligned} \tag{4.6}$$

$$\begin{aligned} u_2(\zeta) = & -\frac{1}{2}\frac{(2l^2-m^2+n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} \\ & -\frac{3\omega_2 l(m+n)}{(l^2+m^2-n^2)\omega_3}\left[\frac{l}{m-n}+\frac{\sqrt{m^2+l^2-n^2}}{m-n}\tanh\left(\frac{\sqrt{m^2+l^2-n^2}}{2}(\zeta+C)\right)\right]^{-1} \\ & -\frac{3}{2}\frac{(m+n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\frac{l}{m-n}+\frac{\sqrt{m^2+l^2-n^2}}{m-n}\tanh\left(\frac{\sqrt{m^2+l^2-n^2}}{2}(\zeta+C)\right)\right]^{-2}, \end{aligned} \tag{4.7}$$

$$\begin{aligned} u_3(\zeta) = & -\frac{1}{2}\frac{(2l^2-m^2+n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} \\ & -\frac{3\omega_2 l(m+n)}{(l^2+m^2-n^2)\omega_3}\left[\frac{l}{m}+\frac{\sqrt{m^2+l^2}}{m}\tanh\left(\frac{\sqrt{m^2+l^2}}{2}(\zeta+C)\right)\right]^{-1} \\ & -\frac{3}{2}\frac{(m+n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\frac{l}{m}+\frac{\sqrt{m^2+l^2}}{m}\tanh\left(\frac{\sqrt{m^2+l^2}}{2}(\zeta+C)\right)\right]^{-2}, \end{aligned} \tag{4.8}$$

$$\begin{aligned} u_4(\zeta) = & -\frac{1}{2}\frac{(2l^2-m^2+n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} \\ & -\frac{3\omega_2 l(m+n)}{(l^2+m^2-n^2)\omega_3}\left[\frac{-l}{n}+\frac{\sqrt{n^2-l^2}}{n}\tan\left(\frac{\sqrt{n^2-l^2}}{2}(\zeta+C)\right)\right]^{-1} \\ & -\frac{3}{2}\frac{(m+n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\frac{-l}{n}+\frac{\sqrt{n^2-l^2}}{n}\tan\left(\frac{\sqrt{n^2-l^2}}{2}(\zeta+C)\right)\right]^{-2}, \end{aligned} \tag{4.9}$$

$$\begin{aligned} u_5(\zeta) = & -\frac{1}{2}\frac{(2l^2-m^2+n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} \\ & -\frac{3\omega_2 l(m+n)}{(l^2+m^2-n^2)\omega_3}\left[\sqrt{\frac{m+n}{m-n}}\tanh\left(\frac{\sqrt{m^2-n^2}}{2}(\zeta+C)\right)\right]^{-1} \\ & -\frac{3}{2}\frac{(m+n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\sqrt{\frac{m+n}{m-n}}\tanh\left(\frac{\sqrt{m^2-n^2}}{2}(\zeta+C)\right)\right]^{-2}, \end{aligned} \tag{4.10}$$

$$\begin{aligned} u_6(\zeta) = & -\frac{1}{2}\frac{(2l^2-m^2+n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} \\ & -\frac{3\omega_2 l(m+n)}{(l^2+m^2-n^2)\omega_3}\left[\tan\left(\frac{1}{2}\arctan\left(\frac{e^{2m(\zeta+C)}-1}{e^{2m(\zeta+C)}+1},\frac{2e^{m(\zeta+C)}}{e^{2m(\zeta+C)}+1}\right)\right)\right]^{-1} \\ & -\frac{3}{2}\frac{(m+n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\tan\left(\frac{1}{2}\arctan\left(\frac{e^{2m(\zeta+C)}-1}{e^{2m(\zeta+C)}+1},\frac{2e^{m(\zeta+C)}}{e^{2m(\zeta+C)}+1}\right)\right)\right]^{-2}, \end{aligned} \tag{4.11}$$

$$\begin{aligned} u_7(\zeta) = & -\frac{1}{2}\frac{(2l^2-m^2+n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} \\ & -\frac{3\omega_2 l(m+n)}{(l^2+m^2-n^2)\omega_3}\left[\tan\left(\frac{1}{2}\arctan\left(\frac{2e^{l(\zeta+C)}}{e^{2l(\zeta+C)}+1},\frac{e^{2l(\zeta+C)}-1}{e^{2l(\zeta+C)}+1}\right)\right)\right]^{-1} \\ & -\frac{3}{2}\frac{(m+n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\tan\left(\frac{1}{2}\arctan\left(\frac{2e^{l(\zeta+C)}}{e^{2l(\zeta+C)}+1},\frac{e^{2l(\zeta+C)}-1}{e^{2l(\zeta+C)}+1}\right)\right)\right]^{-2}, \end{aligned} \tag{4.12}$$

where $\zeta = x-\left(\sqrt{-\omega_1+\frac{\omega_2}{l^2+m^2-n^2}}\right)\frac{t^\alpha}{\alpha}$.

The above solutions are valid for $\left(-\omega_1+\frac{\omega_2}{l^2+m^2-n^2}\right)>0$.

Using Eq. (4.5), the **Families 9, 10, 11, 12, 13** and **14** can be presented respectively, as

$$\begin{aligned} u_8(\zeta) = & -\frac{1}{2}\frac{(2l^2-m^2+n^2)\omega_2}{(l^2+m^2-n^2)\omega_3}-\frac{3\omega_2 l(m+n)}{(l^2+m^2-n^2)\omega_3}\left[e^{il(\zeta+C)}-1\right]^{-1} \\ & -\frac{3}{2}\frac{(m+n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[e^{il(\zeta+C)}-1\right]^{-2}, \end{aligned} \tag{4.13}$$

$$\begin{aligned} u_9(\zeta) = & -\frac{1}{2}\frac{(2l^2-m^2+n^2)\omega_2}{(l^2+m^2-n^2)\omega_3}-\frac{3\omega_2 l(m+n)}{(l^2+m^2-n^2)\omega_3}\left[-\frac{e^{il(\zeta+C)}}{-1+e^{il(\zeta+C)}}\right]^{-1} \\ & -\frac{3}{2}\frac{(m+n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[-\frac{e^{il(\zeta+C)}}{-1+e^{il(\zeta+C)}}\right]^{-2}, \end{aligned} \tag{4.14}$$

$$\begin{aligned} u_{10}(\zeta) = & -\frac{1}{2}\frac{(2l^2-m^2+n^2)\omega_2}{(l^2+m^2-n^2)\omega_3}-\frac{3\omega_2 l(m+n)}{(l^2+m^2-n^2)\omega_3}\left[-\frac{(l+m)e^{m(\zeta+C)}-1}{(l-m)e^{m(\zeta+C)}-1}\right]^{-1} \\ & -\frac{3}{2}\frac{(m+n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[-\frac{(l+m)e^{m(\zeta+C)}-1}{(l-m)e^{m(\zeta+C)}-1}\right]^{-2}, \end{aligned} \tag{4.15}$$

$$\begin{aligned} u_{11}(\zeta) = & -\frac{1}{2}\frac{(2l^2-m^2+n^2)\omega_2}{(l^2+m^2-n^2)\omega_3}-\frac{3\omega_2 l(m+n)}{(l^2+m^2-n^2)\omega_3}\left[\frac{(m+n)e^{m(\zeta+C)}+1}{(m-n)e^{m(\zeta+C)}-1}\right]^{-1} \\ & -\frac{3}{2}\frac{(m+n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\frac{(m+n)e^{m(\zeta+C)}+1}{(m-n)e^{m(\zeta+C)}-1}\right]^{-2}, \end{aligned} \tag{4.16}$$

$$\begin{aligned} u_{12}(\zeta) = & -\frac{1}{2}\frac{(2l^2-m^2+n^2)\omega_2}{(l^2+m^2-n^2)\omega_3}-\frac{3\omega_2 l(m+n)}{(l^2+m^2-n^2)\omega_3}\left[\frac{e^{m(\zeta+C)}+m-l}{e^{m(\zeta+C)}-m-l}\right]^{-1} \\ & -\frac{3}{2}\frac{(m+n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\frac{e^{m(\zeta+C)}+m-l}{e^{m(\zeta+C)}-m-l}\right]^{-2}, \end{aligned} \tag{4.17}$$

$$\begin{aligned} u_{13}(\zeta) = & -\frac{1}{2}\frac{(2l^2-m^2+n^2)\omega_2}{(l^2+m^2-n^2)\omega_3}-\frac{3\omega_2 l(m+n)}{(l^2+m^2-n^2)\omega_3}\left[-\frac{le^{l}(\zeta+C)}{ne^{l(\zeta+C)}-1}\right]^{-1} \\ & -\frac{3}{2}\frac{(m+n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[-\frac{le^{l}(\zeta+C)}{ne^{l(\zeta+C)}-1}\right]^{-2}, \end{aligned} \tag{4.18}$$

where $\zeta = x-\left(\sqrt{-\omega_1+\frac{\omega_2}{l^2+m^2-n^2}}\right)\frac{t^\alpha}{\alpha}$.

All the above solutions hold true for $\left(-\omega_1+\frac{\omega_2}{l^2+m^2-n^2}\right)>0$.

Using Eq. (4.5), **Family 18** is reported, as

$$\begin{aligned} u_{14}(\zeta) = & -\frac{1}{2}\frac{(2l^2-m^2+n^2)\omega_2}{(l^2+m^2-n^2)\omega_3}-\frac{3\omega_2 l(m+n)}{(l^2+m^2-n^2)\omega_3}\left[\tan\left(\frac{n\zeta+C}{2}\right)\right]^{-1} \\ & -\frac{3}{2}\frac{(m+n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\tan\left(\frac{n\zeta+C}{2}\right)\right]^{-2}, \end{aligned} \tag{4.19}$$

where $\zeta = x-\left(\sqrt{-\omega_1+\frac{\omega_2}{l^2+m^2-n^2}}\right)\frac{t^\alpha}{\alpha}$.

The constraint condition to be satisfied by the above solution is $\left(-\omega_1+\frac{\omega_2}{l^2+m^2-n^2}\right)>0$.

Set 2:

$$\begin{aligned} & \upsilon=\sqrt{-\omega_1+\frac{\omega_2}{l^2+m^2-n^2}},\ \alpha_0=-\frac{1}{2}\frac{(2l^2-m^2+n^2)\omega_2}{(l^2+m^2-n^2)\omega_3},\ \beta_1=0, \\ & \alpha_1=\frac{3\omega_2 l(m-n)}{(l^2+m^2-n^2)\omega_3},\ \alpha_2=-\frac{3}{2}\frac{(m-n)^2\omega_2}{(l^2+m^2-n^2)\omega_3},\ \beta_2=0, \end{aligned} \tag{4.20}$$

$$U(\zeta)=\alpha_0+\alpha_1\left[\tan\left(\frac{\Psi(\zeta)}{2}\right)\right]+\alpha_2\left[\tan\left(\frac{\Psi(\zeta)}{2}\right)\right]^2. \tag{4.21}$$

By Eq. (4.21), the **Families 1, 2, 3, 4, 5, 6** and **7** are respectively given, as

$$\begin{aligned} u_{15}(\zeta) = & -\frac{1}{2}\frac{(2l^2-m^2+n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} \\ & +\frac{3\omega_2 l(m-n)}{(l^2+m^2-n^2)\omega_3}\left[\frac{l}{m-n}-\frac{\sqrt{n^2-m^2-l^2}}{m-n}\tan\left(\frac{\sqrt{n^2-m^2-l^2}}{2}(\zeta+C)\right)\right] \\ & -\frac{3}{2}\frac{(m-n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\frac{l}{m-n}-\frac{\sqrt{n^2-m^2-l^2}}{m-n}\tan\left(\frac{\sqrt{n^2-m^2-l^2}}{2}(\zeta+C)\right)\right]^{2}, \end{aligned} \tag{4.22}$$

$$\begin{aligned} u_{16}(\zeta) = & -\frac{1}{2}\frac{(2l^2-m^2+n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} \\ & +\frac{3\omega_2 l(m-n)}{(l^2+m^2-n^2)\omega_3}\left[\frac{l}{m-n}+\frac{\sqrt{m^2+l^2-n^2}}{m-n}\tanh\left(\frac{\sqrt{m^2+l^2-n^2}}{2}(\zeta+C)\right)\right] \\ & -\frac{3}{2}\frac{(m-n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\frac{l}{m-n}+\frac{\sqrt{m^2+l^2-n^2}}{m-n}\tanh\left(\frac{\sqrt{m^2+l^2-n^2}}{2}(\zeta+C)\right)\right]^{2}, \end{aligned} \tag{4.23}$$

$$\begin{aligned} u_{17}(\zeta) = & -\frac{1}{2}\frac{(2l^2-m^2+n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} \\ & +\frac{3\omega_2 l(m-n)}{(l^2+m^2-n^2)\omega_3}\left[\frac{l}{m}+\frac{\sqrt{m^2+l^2}}{m}\tanh\left(\frac{\sqrt{m^2+l^2}}{2}(\zeta+C)\right)\right] \\ & -\frac{3}{2}\frac{(m-n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\frac{l}{m}+\frac{\sqrt{m^2+l^2}}{m}\tanh\left(\frac{\sqrt{m^2+l^2}}{2}(\zeta+C)\right)\right]^{2}, \end{aligned} \tag{4.24}$$

$$\begin{aligned} u_{18}(\zeta) = & -\frac{1}{2}\frac{(2l^2-m^2+n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} \\ & +\frac{3\omega_2 l(m-n)}{(l^2+m^2-n^2)\omega_3}\left[\frac{-l}{n}+\frac{\sqrt{n^2-l^2}}{n}\tan\left(\frac{\sqrt{n^2-l^2}}{2}(\zeta+C)\right)\right] \\ & -\frac{3}{2}\frac{(m-n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\frac{-l}{n}+\frac{\sqrt{n^2-l^2}}{n}\tan\left(\frac{\sqrt{n^2-l^2}}{2}(\zeta+C)\right)\right]^{2}, \end{aligned} \tag{4.25}$$

$$\begin{aligned} u_{19}(\zeta) = & -\frac{1}{2}\frac{(2l^2-m^2+n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} \\ & +\frac{3\omega_2 l(m-n)}{(l^2+m^2-n^2)\omega_3}\left[\sqrt{\frac{m+n}{m-n}}\tanh\left(\frac{\sqrt{m^2-n^2}}{2}(\zeta+C)\right)\right] \\ & -\frac{3}{2}\frac{(m-n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\sqrt{\frac{m+n}{m-n}}\tanh\left(\frac{\sqrt{m^2-n^2}}{2}(\zeta+C)\right)\right]^2, \end{aligned} \tag{4.26}$$

$$\begin{aligned} u_{20}(\zeta) = & -\frac{1}{2}\frac{(2l^2-m^2+n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} \\ & +\frac{3\omega_2 l(m-n)}{(l^2+m^2-n^2)\omega_3}\left[\tan\left(\frac{1}{2}\arctan\left(\frac{e^{2m(\zeta+C)}-1}{e^{2m(\zeta+C)}+1},\frac{2e^{m(\zeta+C)}}{e^{2m(\zeta+C)}+1}\right)\right)\right] \\ & -\frac{3}{2}\frac{(m-n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\tan\left(\frac{1}{2}\arctan\left(\frac{e^{2m(\zeta+C)}-1}{e^{2m(\zeta+C)}+1},\frac{2e^{m(\zeta+C)}}{e^{2m(\zeta+C)}+1}\right)\right)\right]^2, \end{aligned} \tag{4.27}$$

$$\begin{aligned} u_{21}(\zeta) = & -\frac{1}{2}\frac{(2l^2-m^2+n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} \\ & +\frac{3\omega_2 l(m-n)}{(l^2+m^2-n^2)\omega_3}\left[\tan\left(\frac{1}{2}\arctan\left(\frac{2e^{l(\zeta+C)}}{e^{2l(\zeta+C)}+1},\frac{e^{2l(\zeta+C)}-1}{e^{2l(\zeta+C)}+1}\right)\right)\right] \\ & -\frac{3}{2}\frac{(m-n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\tan\left(\frac{1}{2}\arctan\left(\frac{2e^{l(\zeta+C)}}{e^{2l(\zeta+C)}+1},\frac{e^{2l(\zeta+C)}-1}{e^{2l(\zeta+C)}+1}\right)\right)\right]^2, \end{aligned} \tag{4.28}$$

where $\zeta = x - \left(\sqrt{-\omega_1+\frac{\omega_2}{l^2+m^2-n^2}}\right)\frac{t^\alpha}{\alpha}$.

All the above solutions are valid for $\left(-\omega_1+\frac{\omega_2}{l^2+m^2-n^2}\right)>0$.

By Eq. (4.21), the **Families 9, 10, 11, 12, 13** and **14** can be written respectively, as

$$\begin{aligned} u_{22}(\zeta) = & -\frac{1}{2}\frac{(2l^2-m^2+n^2)\omega_2}{(l^2+m^2-n^2)\omega_3}+\frac{3\omega_2 l(m-n)}{(l^2+m^2-n^2)\omega_3}\left[e^{il(\zeta+C)}-1\right] \\ & -\frac{3}{2}\frac{(m-n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[e^{il(\zeta+C)}-1\right]^2, \end{aligned} \tag{4.29}$$

$$\begin{aligned} u_{23}(\zeta) = & -\frac{1}{2}\frac{(2l^2-m^2+n^2)\omega_2}{(l^2+m^2-n^2)\omega_3}+\frac{3\omega_2 l(m-n)}{(l^2+m^2-n^2)\omega_3}\left[-\frac{e^{il(\zeta+C)}}{-1+e^{il(\zeta+C)}}\right] \\ & -\frac{3}{2}\frac{(m-n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[-\frac{e^{il(\zeta+C)}}{-1+e^{il(\zeta+C)}}\right]^2, \end{aligned} \tag{4.30}$$

$$\begin{aligned} u_{24}(\zeta) = & -\frac{1}{2}\frac{(2l^2-m^2+n^2)\omega_2}{(l^2+m^2-n^2)\omega_3}+\frac{3\omega_2 l(m-n)}{(l^2+m^2-n^2)\omega_3}\left[\frac{(l+m)e^{m(\zeta+C)}-1}{(l-m)e^{m(\zeta+C)}-1}\right] \\ & -\frac{3}{2}\frac{(m-n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\frac{(l+m)e^{m(\zeta+C)}-1}{(l-m)e^{m(\zeta+C)}-1}\right]^2, \end{aligned} \tag{4.31}$$

$$\begin{aligned} u_{25}(\zeta) = & -\frac{1}{2}\frac{(2l^2-m^2+n^2)\omega_2}{(l^2+m^2-n^2)\omega_3}+\frac{3\omega_2 l(m-n)}{(l^2+m^2-n^2)\omega_3}\left[\frac{(m+n)e^{m(\zeta+C)}+1}{(m-n)e^{m(\zeta+C)}-1}\right] \\ & -\frac{3}{2}\frac{(m-n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\frac{(m+n)e^{m(\zeta+C)}+1}{(m-n)e^{m(\zeta+C)}-1}\right]^2, \end{aligned} \tag{4.32}$$

$$\begin{aligned} u_{26}(\zeta) = & -\frac{1}{2}\frac{(2l^2-m^2+n^2)\omega_2}{(l^2+m^2-n^2)\omega_3}+\frac{3\omega_2 l(m-n)}{(l^2+m^2-n^2)\omega_3}\left[\frac{e^{m(\zeta+C)}+m-l}{e^{m(\zeta+C)}-m-l}\right] \\ & -\frac{3}{2}\frac{(m-n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\frac{e^{m(\zeta+C)}+m-l}{e^{m(\zeta+C)}-m-l}\right]^2, \end{aligned} \tag{4.33}$$

$$\begin{aligned} u_{27}(\zeta) = & -\frac{1}{2}\frac{(2l^2-m^2+n^2)\omega_2}{(l^2+m^2-n^2)\omega_3}+\frac{3\omega_2 l(m-n)}{(l^2+m^2-n^2)\omega_3}\left[-\frac{le^{l}(\zeta+C)}{ne^{l(\zeta+C)}-1}\right] \\ & -\frac{3}{2}\frac{(m-n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[-\frac{le^{l}(\zeta+C)}{ne^{l(\zeta+C)}-1}\right]^2, \end{aligned} \tag{4.34}$$

where $\zeta = x - \left(\sqrt{-\omega_1+\frac{\omega_2}{l^2+m^2-n^2}}\right)\frac{t^\alpha}{\alpha}$.

The above solutions are true for $\left(-\omega_1+\frac{\omega_2}{l^2+m^2-n^2}\right)>0$.

Using Eq. (4.21), **Family 18** is given, as

$$\begin{aligned} u_{28}(\zeta) = & -\frac{1}{2}\frac{(2l^2-m^2+n^2)\omega_2}{(l^2+m^2-n^2)\omega_3}+\frac{3\omega_2 l(m-n)}{(l^2+m^2-n^2)\omega_3}\left[\tan\left(\frac{n\zeta+C}{2}\right)\right] \\ & -\frac{3}{2}\frac{(m-n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\tan\left(\frac{n\zeta+C}{2}\right)\right]^2, \end{aligned} \tag{4.35}$$

where $\zeta = x - \left(\sqrt{-\omega_1+\frac{\omega_2}{l^2+m^2-n^2}}\right)\frac{t^\alpha}{\alpha}$.

The above extracted solutions are valid for $\left(-\omega_1+\frac{\omega_2}{l^2+m^2-n^2}\right)>0$.

Set 3:

$$\begin{aligned} & \upsilon=\sqrt{-\omega_1-\frac{\omega_2}{l^2+m^2-n^2}},\ \alpha_0=-\frac{3}{2}\frac{(m^2-n^2)\omega_2}{(l^2+m^2-n^2)\omega_3},\ \alpha_1=0, \\ & \alpha_2=0,\ \beta_1=\frac{3\omega_2 l(m+n)}{(l^2+m^2-n^2)\omega_3},\ \beta_2=\frac{3}{2}\frac{(m+n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}, \end{aligned} \tag{4.36}$$

$$U(\zeta)=\alpha_0+\beta_1\left[\tan\left(\frac{\Psi(\zeta)}{2}\right)\right]^{-1}+\beta_2\left[\tan\left(\frac{\Psi(\zeta)}{2}\right)\right]^{-2}. \tag{4.37}$$

By Eq. (4.37), the **Families 1, 2, 3, 4, 5, 6** and **7** can be presented respectively, as

$$\begin{aligned} u_{29}(\zeta) = & -\frac{3}{2}\frac{(m^2-n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} \\ & +\frac{3\omega_2 l(m+n)}{(l^2+m^2-n^2)\omega_3}\left[\frac{l}{m-n}-\frac{\sqrt{n^2-m^2-l^2}}{m-n}\tan\left(\frac{\sqrt{n^2-m^2-l^2}}{2}(\zeta+C)\right)\right]^{-1} \\ & +\frac{3}{2}\frac{(m+n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\frac{l}{m-n}-\frac{\sqrt{n^2-m^2-l^2}}{m-n}\tan\left(\frac{\sqrt{n^2-m^2-l^2}}{2}(\zeta+C)\right)\right]^{-2}, \end{aligned} \tag{4.38}$$

$$\begin{aligned} u_{30}(\zeta) = & -\frac{3}{2}\frac{(m^2-n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} \\ & +\frac{3\omega_2 l(m+n)}{(l^2+m^2-n^2)\omega_3}\left[\frac{l}{m-n}+\frac{\sqrt{m^2+l^2-n^2}}{m-n}\tanh\left(\frac{\sqrt{m^2+l^2-n^2}}{2}(\zeta+C)\right)\right]^{-1} \\ & +\frac{3}{2}\frac{(m+n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\frac{l}{m-n}+\frac{\sqrt{m^2+l^2-n^2}}{m-n}\tanh\left(\frac{\sqrt{m^2+l^2-n^2}}{2}(\zeta+C)\right)\right]^{-2}, \end{aligned} \tag{4.39}$$

$$\begin{aligned} u_{31}(\zeta) = & -\frac{3}{2}\frac{(m^2-n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} \\ & +\frac{3\omega_2 l(m+n)}{(l^2+m^2-n^2)\omega_3}\left[\frac{l}{m}+\frac{\sqrt{m^2+l^2}}{m}\tanh\left(\frac{\sqrt{m^2+l^2}}{2}(\zeta+C)\right)\right]^{-1} \\ & +\frac{3}{2}\frac{(m+n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\frac{l}{m}+\frac{\sqrt{m^2+l^2}}{m}\tanh\left(\frac{\sqrt{m^2+l^2}}{2}(\zeta+C)\right)\right]^{-2}, \end{aligned} \tag{4.40}$$

$$\begin{aligned} u_{32}(\zeta) = & -\frac{3}{2}\frac{(m^2-n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} \\ & +\frac{3\omega_2 l(m+n)}{(l^2+m^2-n^2)\omega_3}\left[\frac{-l}{n}+\frac{\sqrt{n^2-l^2}}{n}\tan\left(\frac{\sqrt{n^2-l^2}}{2}(\zeta+C)\right)\right]^{-1} \\ & +\frac{3}{2}\frac{(m+n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\frac{-l}{n}+\frac{\sqrt{n^2-l^2}}{n}\tan\left(\frac{\sqrt{n^2-l^2}}{2}(\zeta+C)\right)\right]^{-2}, \end{aligned} \tag{4.41}$$

$$\begin{aligned} u_{33}(\zeta) = & -\frac{3}{2}\frac{(m^2-n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} \\ & +\frac{3\omega_2 l(m+n)}{(l^2+m^2-n^2)\omega_3}\left[\sqrt{\frac{m+n}{m-n}}\tanh\left(\frac{\sqrt{m^2-n^2}}{2}(\zeta+C)\right)\right]^{-1} \\ & +\frac{3}{2}\frac{(m+n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\sqrt{\frac{m+n}{m-n}}\tanh\left(\frac{\sqrt{m^2-n^2}}{2}(\zeta+C)\right)\right]^{-2}, \end{aligned} \tag{4.42}$$

$$\begin{aligned} u_{34}(\zeta) = & -\frac{3}{2}\frac{(m^2-n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} \\ & +\frac{3\omega_2 l(m+n)}{(l^2+m^2-n^2)\omega_3}\left[\tan\left(\frac{1}{2}\arctan\left(\frac{e^{2m(\zeta+C)}-1}{e^{2m(\zeta+C)}+1},\frac{2e^{m(\zeta+C)}}{e^{2m(\zeta+C)}+1}\right)\right)\right]^{-1} \\ & +\frac{3}{2}\frac{(m+n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\tan\left(\frac{1}{2}\arctan\left(\frac{e^{2m(\zeta+C)}-1}{e^{2m(\zeta+C)}+1},\frac{2e^{m(\zeta+C)}}{e^{2m(\zeta+C)}+1}\right)\right)\right]^{-2}, \end{aligned} \tag{4.43}$$

$$\begin{aligned} u_{35}(\zeta) = & -\frac{3}{2}\frac{(m^2-n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} \\ & +\frac{3\omega_2 l(m+n)}{(l^2+m^2-n^2)\omega_3}\left[\tan\left(\frac{1}{2}\arctan\left(\frac{2e^{l(\zeta+C)}}{e^{2l(\zeta+C)}+1},\frac{e^{2l(\zeta+C)}-1}{e^{2l(\zeta+C)}+1}\right)\right)\right]^{-1} \\ & +\frac{3}{2}\frac{(m+n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\tan\left(\frac{1}{2}\arctan\left(\frac{2e^{l(\zeta+C)}}{e^{2l(\zeta+C)}+1},\frac{e^{2l(\zeta+C)}-1}{e^{2l(\zeta+C)}+1}\right)\right)\right]^{-2}, \end{aligned} \tag{4.44}$$

where $\zeta = x - \left(\sqrt{-\omega_1-\frac{\omega_2}{l^2+m^2-n^2}}\right)\frac{t^\alpha}{\alpha}$.

All the above solutions are valid for $\left(-\omega_1-\frac{\omega_2}{l^2+m^2-n^2}\right)>0$.

With the help of Eq. (4.21), the **Families 9, 10, 11, 12, 13** and **14** can be respectively noted, as

$$u_{36}(\zeta) = -\frac{3}{2}\frac{(m^2-n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} + \frac{3\omega_2 l(m+n)}{(l^2+m^2-n^2)\omega_3}\left[e^{il(\zeta+C)}-1\right]^{-1} + \frac{3}{2}\frac{(m+n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[e^{il(\zeta+C)}-1\right]^{-2}, \tag{4.45}$$

$$u_{37}(\zeta) = -\frac{3}{2}\frac{(m^2-n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} + \frac{3\omega_2 l(m+n)}{(l^2+m^2-n^2)\omega_3}\left[-\frac{e^{il(\zeta+C)}}{-1+e^{il(\zeta+C)}}\right]^{-1} + \frac{3}{2}\frac{(m+n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[-\frac{e^{il(\zeta+C)}}{-1+e^{il(\zeta+C)}}\right]^{-2}, \tag{4.46}$$

$$u_{38}(\zeta) = -\frac{3}{2}\frac{(m^2-n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} + \frac{3\omega_2 l(m+n)}{(l^2+m^2-n^2)\omega_3}\left[\frac{(l+m)e^{m(\zeta+C)}-1}{(l-m)e^{m(\zeta+C)}-1}\right]^{-1} + \frac{3}{2}\frac{(m+n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\frac{(l+m)e^{m(\zeta+C)}-1}{(l-m)e^{m(\zeta+C)}-1}\right]^{-2}, \tag{4.47}$$

$$u_{39}(\zeta) = -\frac{3}{2}\frac{(m^2-n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} + \frac{3\omega_2 l(m+n)}{(l^2+m^2-n^2)\omega_3}\left[\frac{(m+n)e^{m(\zeta+C)}+1}{(m-n)e^{m(\zeta+C)}-1}\right]^{-1} + \frac{3}{2}\frac{(m+n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\frac{(m+n)e^{m(\zeta+C)}+1}{(m-n)e^{m(\zeta+C)}-1}\right]^{-2}, \tag{4.48}$$

$$u_{40}(\zeta) = -\frac{3}{2}\frac{(m^2-n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} + \frac{3\omega_2 l(m+n)}{(l^2+m^2-n^2)\omega_3}\left[\frac{e^{m(\zeta+C)}+m-l}{e^{m(\zeta+C)}-m-l}\right]^{-1} + \frac{3}{2}\frac{(m+n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\frac{(e^{m(\zeta+C)}+m-l]}{e^{m(\zeta+C)}-m-l}\right]^{-2}, \tag{4.49}$$

$$u_{41}(\zeta) = -\frac{3}{2}\frac{(m^2-n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} + \frac{3\omega_2 l(m+n)}{(l^2+m^2-n^2)\omega_3}\left[-\frac{le^{l}(\zeta+C)}{ne^{l(\zeta+C)}-1}\right]^{-1} + \frac{3}{2}\frac{(m+n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[-\frac{le^{l}(\zeta+C)}{ne^{l(\zeta+C)}-1}\right]^{-2}, \tag{4.50}$$

where $\zeta = x - \left(\sqrt{-\omega_1 - \frac{\omega_2}{l^2+m^2-n^2}}\right)\frac{t^\alpha}{\alpha}$.

All the above solutions are valid for $\left(-\omega_1 - \frac{\omega_2}{l^2+m^2-n^2}\right) > 0$.

Using Eq. (4.37), **Family 18** is reported, as

$$u_{42}(\zeta) = -\frac{3}{2}\frac{(m^2-n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} + \frac{3\omega_2 l(m+n)}{(l^2+m^2-n^2)\omega_3}\left[\tan\left(\frac{n\zeta+C}{2}\right)\right]^{-1} + \frac{3}{2}\frac{(m+n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\tan\left(\frac{n\zeta+C}{2}\right)\right]^{-2}, \tag{4.51}$$

where $\zeta = x - \left(\sqrt{-\omega_1 - \frac{\omega_2}{l^2+m^2-n^2}}\right)\frac{t^\alpha}{\alpha}$.

The above solution holds true for $\left(-\omega_1 - \frac{\omega_2}{l^2+m^2-n^2}\right) > 0$.

Set 4:

$$\upsilon = \sqrt{-\omega_1 - \frac{\omega_2}{l^2+m^2-n^2}},\quad \alpha_0 = -\frac{3}{2}\frac{(m^2-n^2)\omega_2}{(l^2+m^2-n^2)\omega_3},\quad \beta_1 = 0,$$
$$\alpha_1 = -\frac{3\omega_2 l(m-n)}{(l^2+m^2-n^2)\omega_3},\quad \alpha_2 = \frac{3}{2}\frac{(m-n)^2\omega_2}{(l^2+m^2-n^2)\omega_3},\quad \beta_2 = 0, \tag{4.52}$$

$$U(\zeta) = \alpha_0 + \alpha_1\left[\tan\left(\frac{\Psi(\zeta)}{2}\right)\right] + \alpha_2\left[\tan\left(\frac{\Psi(\zeta)}{2}\right)\right]^2. \tag{4.53}$$

Using Eq. (4.53), the **Families 1, 2, 3, 4, 5, 6** and **7** can be written respectively, as

$$u_{43}(\zeta) = -\frac{3}{2}\frac{(m^2-n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} - \frac{3\omega_2 l(m-n)}{(l^2+m^2-n^2)\omega_3}\left[\frac{l}{m-n} - \frac{\sqrt{n^2-m^2-l^2}}{m-n}\tan\left(\frac{\sqrt{n^2-m^2-l^2}}{2}(\zeta+C)\right)\right] + \frac{3}{2}\frac{(m-n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\frac{l}{m-n} - \frac{\sqrt{n^2-m^2-l^2}}{m-n}\tan\left(\frac{\sqrt{n^2-m^2-l^2}}{2}(\zeta+C)\right)\right]^2, \tag{4.54}$$

$$u_{44}(\zeta) = -\frac{3}{2}\frac{(m^2-n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} - \frac{3\omega_2 l(m-n)}{(l^2+m^2-n^2)\omega_3}\left[\frac{l}{m-n} + \frac{\sqrt{m^2+l^2-n^2}}{m-n}\tanh\left(\frac{\sqrt{m^2+l^2-n^2}}{2}(\zeta+C)\right)\right] + \frac{3}{2}\frac{(m-n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\frac{l}{m-n} + \frac{\sqrt{m^2+l^2-n^2}}{m-n}\tanh\left(\frac{\sqrt{m^2+l^2-n^2}}{2}(\zeta+C)\right)\right]^2, \tag{4.55}$$

$$u_{45}(\zeta) = -\frac{3}{2}\frac{(m^2-n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} - \frac{3\omega_2 l(m-n)}{(l^2+m^2-n^2)\omega_3}\left[\frac{l}{m} + \frac{\sqrt{m^2+l^2}}{m}\tanh\left(\frac{\sqrt{m^2+l^2}}{2}(\zeta+C)\right)\right] + \frac{3}{2}\frac{(m-n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\frac{l}{m} + \frac{\sqrt{m^2+l^2}}{m}\tanh\left(\frac{\sqrt{m^2+l^2}}{2}(\zeta+C)\right)\right]^2, \tag{4.56}$$

$$u_{46}(\zeta) = -\frac{3}{2}\frac{(m^2-n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} - \frac{3\omega_2 l(m-n)}{(l^2+m^2-n^2)\omega_3}\left[\frac{-l}{n} + \frac{\sqrt{n^2-l^2}}{n}\tan\left(\frac{\sqrt{n^2-l^2}}{2}(\zeta+C)\right)\right] + \frac{3}{2}\frac{(m-n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\frac{-l}{n} + \frac{\sqrt{n^2-l^2}}{n}\tan\left(\frac{\sqrt{n^2-l^2}}{2}(\zeta+C)\right)\right]^2, \tag{4.57}$$

$$u_{47}(\zeta) = -\frac{3}{2}\frac{(m^2-n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} - \frac{3\omega_2 l(m-n)}{(l^2+m^2-n^2)\omega_3}\left[\sqrt{\frac{m+n}{m-n}}\tanh\left(\frac{\sqrt{m^2-n^2}}{2}(\zeta+C)\right)\right] + \frac{3}{2}\frac{(m-n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\sqrt{\frac{m+n}{m-n}}\tanh\left(\frac{\sqrt{m^2-n^2}}{2}(\zeta+C)\right)\right]^2, \tag{4.58}$$

$$u_{48}(\zeta) = -\frac{3}{2}\frac{(m^2-n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} - \frac{3\omega_2 l(m-n)}{(l^2+m^2-n^2)\omega_3}\left[\tan\left(\frac{1}{2}\arctan\left(\frac{e^{2m(\zeta+C)}-1}{e^{2m(\zeta+C)}+1}, \frac{2e^{m(\zeta+C)}}{e^{2m(\zeta+C)}+1}\right)\right)\right] + \frac{3}{2}\frac{(m-n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\tan\left(\frac{1}{2}\arctan\left(\frac{e^{2m(\zeta+C)}-1}{e^{2m(\zeta+C)}+1}, \frac{2e^{m(\zeta+C)}}{e^{2m(\zeta+C)}+1}\right)\right)\right]^2, \tag{4.59}$$

$$u_{49}(\zeta) = -\frac{3}{2}\frac{(m^2-n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} - \frac{3\omega_2 l(m-n)}{(l^2+m^2-n^2)\omega_3}\left[\tan\left(\frac{1}{2}\arctan\left(\frac{2e^{l(\zeta+C)}}{e^{2l(\zeta+C)}+1}, \frac{e^{2l(\zeta+C)}-1}{e^{2l(\zeta+C)}+1}\right)\right)\right] + \frac{3}{2}\frac{(m-n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\tan\left(\frac{1}{2}\arctan\left(\frac{2e^{l(\zeta+C)}}{e^{2l(\zeta+C)}+1}, \frac{e^{2l(\zeta+C)}-1}{e^{2l(\zeta+C)}+1}\right)\right)\right]^2, \tag{4.60}$$

where $\zeta = x - \left(\sqrt{-\omega_1 - \frac{\omega_2}{l^2+m^2-n^2}}\right)\frac{t^\alpha}{\alpha}$.

All the preceding solutions are valid for $\left(\sqrt{-\omega_1 - \frac{\omega_2}{l^2+m^2-n^2}}\right) > 0$.

By Eq. (4.53), the **Families 9, 10, 11, 12, 13** and **14** can be respectively reported, as

$$u_{50}(\zeta) = -\frac{3}{2}\frac{(m^2-n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} - \frac{3\omega_2 l(m-n)}{(l^2+m^2-n^2)\omega_3}\left[e^{il(\zeta+C)}-1\right] + \frac{3}{2}\frac{(m-n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[e^{il(\zeta+C)}-1\right]^2, \tag{4.61}$$

$$u_{51}(\zeta) = -\frac{3}{2}\frac{(m^2-n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} - \frac{3\omega_2 l(m-n)}{(l^2+m^2-n^2)\omega_3}\left[-\frac{e^{il(\zeta+C)}}{-1+e^{il(\zeta+C)}}\right] + \frac{3}{2}\frac{(m-n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[-\frac{e^{il(\zeta+C)}}{-1+e^{il(\zeta+C)}}\right]^2, \tag{4.62}$$

$$u_{52}(\zeta) = -\frac{3}{2}\frac{(m^2-n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} - \frac{3\omega_2 l(m-n)}{(l^2+m^2-n^2)\omega_3}\left[\frac{(l+m)e^{m(\zeta+C)}-1}{(l-m)e^{m(\zeta+C)}-1}\right] + \frac{3}{2}\frac{(m-n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\frac{(l+m)e^{m(\zeta+C)}-1}{(l-m)e^{m(\zeta+C)}-1}\right]^2, \tag{4.63}$$

$$u_{53}(\zeta) = -\frac{3}{2}\frac{(m^2-n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} - \frac{3\omega_2 l(m-n)}{(l^2+m^2-n^2)\omega_3}\left[\frac{(m+n)e^{m(\zeta+C)}+1}{(m-n)e^{m(\zeta+C)}-1}\right] + \frac{3}{2}\frac{(m-n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\frac{(m+n)e^{m(\zeta+C)}+1}{(m-n)e^{m(\zeta+C)}-1}\right]^2, \tag{4.64}$$

$$u_{54}(\zeta) = -\frac{3}{2}\frac{(m^2-n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} - \frac{3\omega_2 l(m-n)}{(l^2+m^2-n^2)\omega_3}\left[\frac{e^{m(\zeta+C)}+m-l}{e^{m(\zeta+C)}-m-l}\right] + \frac{3}{2}\frac{(m-n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\frac{e^{m(\zeta+C)}+m-l}{e^{m(\zeta+C)}-m-l}\right]^2, \tag{4.65}$$

$$u_{55}(\zeta) = -\frac{3}{2}\frac{(m^2-n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} - \frac{3\omega_2 l(m-n)}{(l^2+m^2-n^2)\omega_3}\left[-\frac{le^{l}(\zeta+C)}{ne^{l(\zeta+C)}-1}\right] + \frac{3}{2}\frac{(m-n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[-\frac{le^{l}(\zeta+C)}{ne^{l(\zeta+C)}-1}\right]^2, \tag{4.66}$$

where $\zeta = x - \left(\sqrt{-\omega_1 - \frac{\omega_2}{l^2+m^2-n^2}}\right)\frac{t^\alpha}{\alpha}$.

All the above solutions hold true for $\left(-\omega_1 - \frac{\omega_2}{l^2+m^2-n^2}\right) > 0$.

Using Eq. (4.53), **Family 18** is noted, as

$$u_{56}(\zeta) = -\frac{3}{2}\frac{(m^2-n^2)\omega_2}{(l^2+m^2-n^2)\omega_3} - \frac{3\omega_2 l(m-n)}{(l^2+m^2-n^2)\omega_3}\left[\tan\left(\frac{n\zeta+C}{2}\right)\right] + \frac{3}{2}\frac{(m-n)^2\omega_2}{(l^2+m^2-n^2)\omega_3}\left[\tan\left(\frac{n\zeta+C}{2}\right)\right]^2, \quad (4.67)$$

where $\zeta = x - \left(\sqrt{-\omega_1 - \frac{\omega_2}{l^2+m^2-n^2}}\right)\frac{t^\alpha}{\alpha}$.

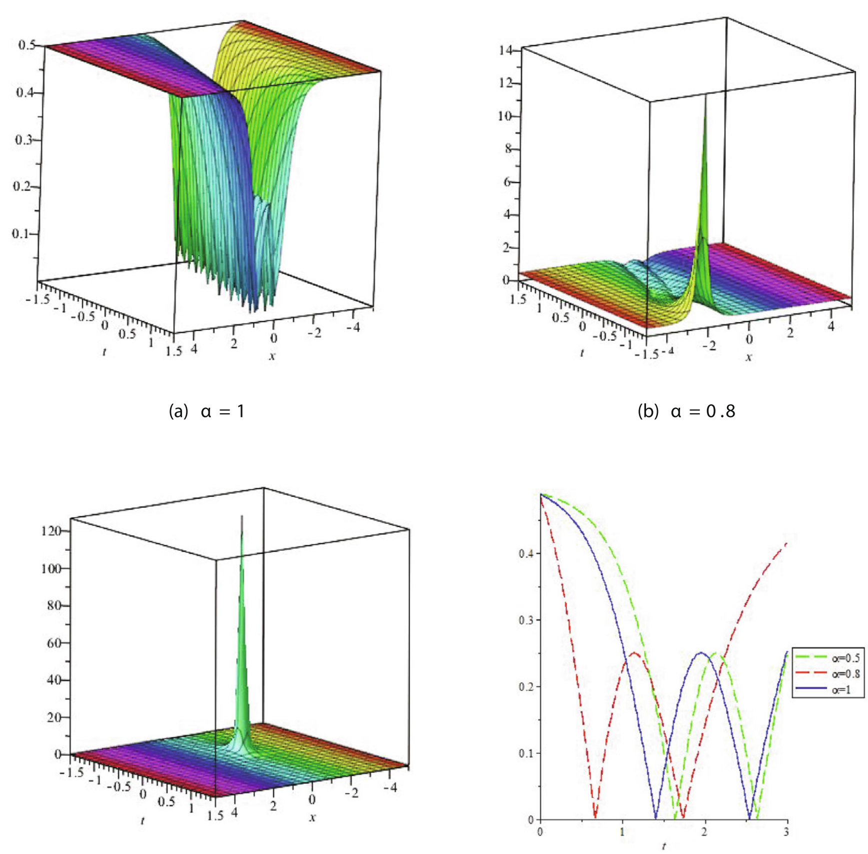
The above extracted solution is valid for $\left(-\omega_1 - \frac{\omega_2}{l^2+m^2-n^2}\right) > 0$.

5. Graphical representations of the obtained solutions

The graphical representation of solutions retrieved by the proposed method is usually as significant as the computation. It helps the reader to grab maximum amount of information of the model. In this section, the behavior of some solutions under fractional effect has been examined using their 3D plots and line plots. The graphical illustration of solutions is given for suitable choices of parameters l, m and n. Figure 1 shows graphical behavior of $u_{19}(x,t)$ that depicts a W-shaped soliton. The corresponding line graph exhibits changing patterns of the solution for two different values of fractional parameter α. Figure 2 interprets singular bright soliton for $u_{39}(x,t)$. Figure 3 depicts singular soliton for $u_{46}(x,t)$ and Figure 4 illustrates a bright soliton for $u_{54}(x,t)$. All these solutions behave differently upon opting different values of α over the interval $(0,1)$. This can be clearly visualized from the line graphs.

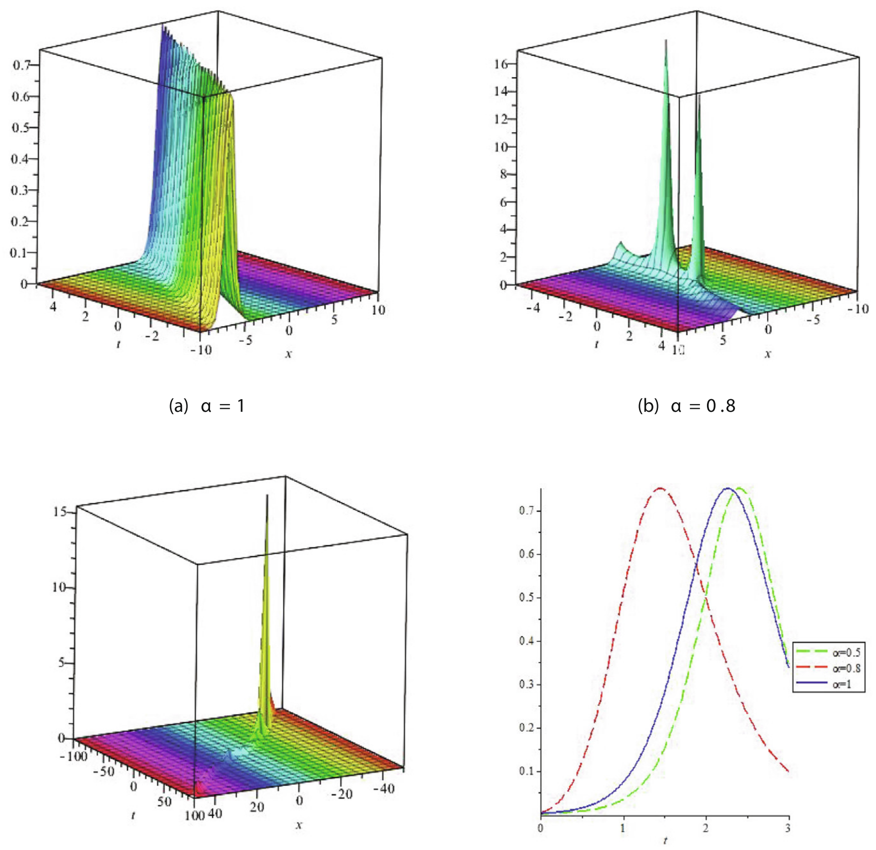
6. Conclusion

This paper addressed the conformable time-fractional Klien-Fock-Gordon equation. The proposed model was examined for quadratic nonlinearity. The conformable derivative was applied on the proposed model for investigating the fractional effects. The conformable derivative opens new horizons for the researchers and scientists to obtain analytical and numerical solutions of problems in science and engineering. The traveling wave transformation along with fractional operator was applied, for transforming the proposed fractional model into an ODE. After obtaining ODE, the improved $\tan(\Psi(\zeta)/2)$-expansion method was employed for the extraction of soliton solutions. New trigonometric, hyperbolic and rational solutions were derived. On comparing our derived results with the results reported in Veeresha et al. (2020) and Alam et al. (2021), it was observed that abundant new traveling wave solutions were obtained. Various bright soliton, W-shaped soliton, singular soliton and periodic wave solutions were reported in this article with fractional effects. The dynamical behavior of the obtained solutions was discussed with the help of 3D plots and 2D line plots by choosing the specific values of arbitrary parameters. It was noticed that the wave profile was changed by taking different values of fractional parameter $\alpha, 0 < \alpha < 1$. It is worth mentioning here that the applied approach is efficient, versatile and powerful to acquire the solitons with real-world applications. The obtained results may be useful for further studies in mathematical physics and engineering.



(a) α = 1 (b) α = 0.8 (c) α = 0.5 (d)

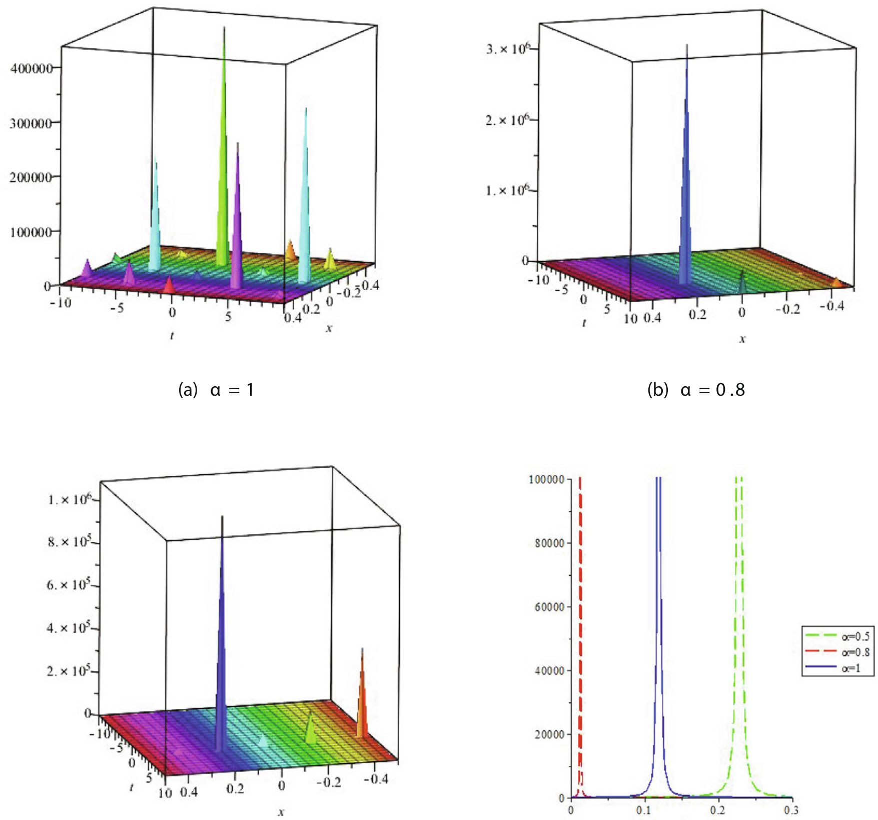
Fig. 1. The Graph for $|u_{19}(x,t)|$ at $l = 0,\ m = 3,\ n = 1,\ \omega_1 = \omega_2 = -1,\ \omega_3 = 2,\ C = 1$.



(a) α = 1 (b) α = 0 .8

(c) α = 0 .5 (d)

Fig. 2. The Graph for $|u_{39}(x,t)|$ at $l=2,\ m=3,\ n=2,\ \omega_1=\omega_2=-1,\ \omega_3=2,\ C=1$.



(a) α = 1 (b) α = 0.8

(c) α = 0.5 (d)

Fig. 3. The Graph for $|u_{46}(x,t)|$ at $l = 1,\ m = 0,\ n = 2,\ \omega_1 = \omega_2 = -1,\ \omega_3 = 2,\ C = 1.$

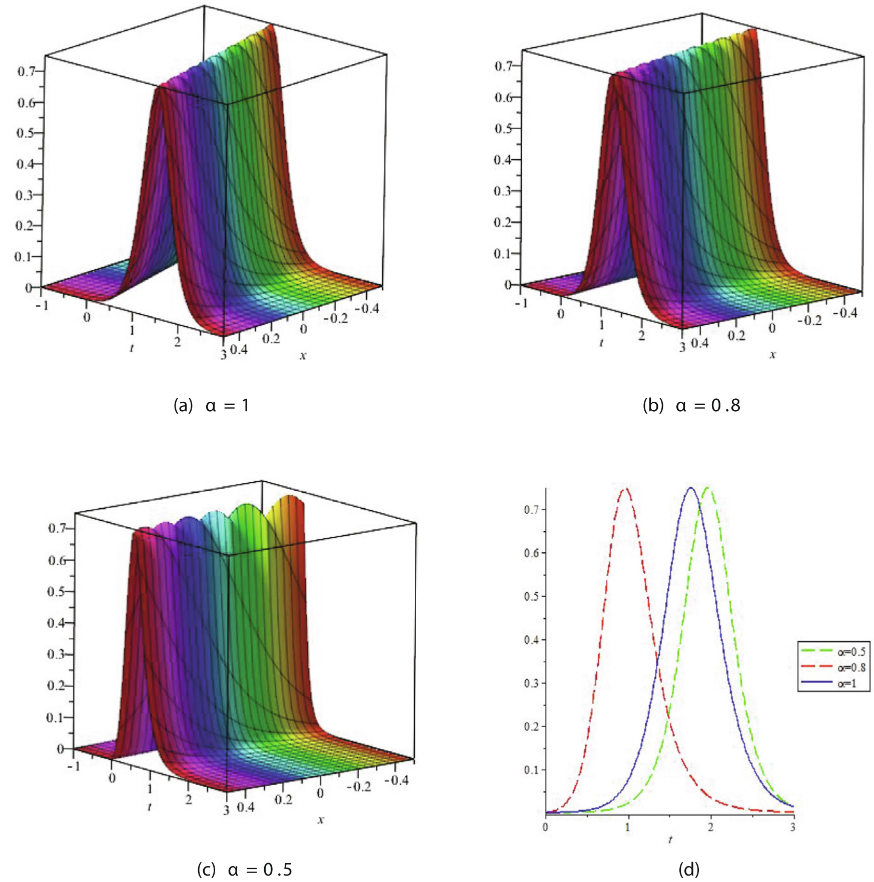


Fig. 4. The graph for $|u_{54}(x,t)|$ at $l = 4,\ m = -5,\ n = -4,\ \omega_1 = \omega_2 = -1,\ \omega_3 = 2,\ C = 1$.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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