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The second order price sensitivities for markets in a crisis

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ABSTRACT

Risk management in financial derivative markets requires inevitably the calculation of price sensitivities. The literature contains an abundant amount of research works on these important values. Most of these works consider the well-known Black and Scholes model where the volatility is assumed to be constant. Some works that attempt to deal with markets that are affected by financial crisis have appeared recently. However, none of these papers deal with the calculation of the price sensitivities of the second order. Providing the second order price sensitivities is an important issue in financial risk management because the investor can determine whether or not each source of risk is varying at an increasing rate. This paper treats the computation of the second order prices sensitivities for a market in crisis. The underlying second order price sensitivities are derived explicitly. The obtained formulas are expected to improve on the accuracy of the hedging strategies during a financial crunch.

1. Introduction

Price sensitivities are an integral part of financial risk management nowadays. A number of papers has been devoted to this important issue. Current literature has provided price sensitivities for different volatility models starting with the pioneer Black and Scholes work (Black and Scholes, 1973). Recently, some new ideas have been developed in order to determine whether or not each source of risk is increasing. Thus, within this context the computation of the second order price sensitivities is a pertinent and important issue. The second order price sensitivities provide information about whether or not the underlying risk is changing in a decreasing or increasing rate, this information is not provided by first order price sensitivities. To our best knowledge, the second order price sensitivities have not been introduced for models that account for markets with a crisis in the existing literature. The book of Haug (2007) contains formulas for the second order greeks in the Black and Scholes model. In Capriotti (2015), the author studies a new approach to compute the second order sensitivities for a general model driven by a multidimensional Brownian motion. The new approach is a combination of the Adjoint Algorithmic Differentiation(AAD) and the Likelihood Ratio Method. The work of Landis (2011) analyzes the second order price sensitivities for a general equilibrium model. In Dilloo and Tangman (2017) another numerical approach, namely the non-uniform discretization, is utilized for option pricing. The current paper addresses the second order prices sensitivities of option pricing for a market that is characterized by a financial crisis by providing closed form solutions. The second order price sensitivities studied in this paper are Vanna, Volga and Vega bleed, using the existing denotations from the literature. It should be pointed out that Vega is defined as the change in the price of the option contract with regard to the change in the volatility of the underlying asset. Vanna

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is the change in Delta of an option with respect to the volatility of its underlying asset. Vanna is also the change of the Vega with regards to the price of the underlying asset. Thus, Vanna is a second-order price sensitivity that is useful when an investor is interested in making a Delta or Vega-hedged transaction. This is especially the case when the investors hold a position of complex options or a portfolio of options. The main goal of Vanna is to consider the combined effect of changes in both volatility and the price of the original asset on the option position. Volga represents the variation of Vega with respect to a change in the underlying volatility. Thus, it is the second order derivative of the price of the option with regard to the volatility. This measures are dealt with in the next section for a crisis model.

2. Options pricing and price sensitivities in a crisis period

The Black and Scholes option pricing formula is one of the most applied formulas in modern financial risk management. However, their model, despite its usefulness, has several shortcomings. The constant volatility assumption as well as the continuity of the underlying asset price trajectories is among the main shortcomings of this model. There is an ample quantity of research papers that try to remedy these limitations. For some recent pertinent articles on option pricing and price sensitivities, see Alzghool (2017), El-Khatib and Hatemi-J (2013), El-Khatib and Hatemi-J (2012) or Phaochoo et al. (2016), where an estimation of stochastic volatility, a crisis, a jump-diffusion and a fractional Brownian motion models are studied to address these shortcomings. In this section, the crisis model is presented. Some recent contributions to the modeling of financial assets during crisis are (Dibeh and Harmanani, 2007; El-Khatib and Hatemi-J, 2017; Savit, 1989 and Sornette, 2003). The article written by Dibeh and Harmanani (2007) suggests a stochastic differential equation (SDE) for the valuation of option prices in markets characterized by a crisis. The underlying SDE has the property to generate the volatility as a measure of risk that is dependent on the stock price as well as time. The cited authors make use of the partial differential equation (PDE) for call prices that is derived assuming the risk-neutral situation. Some simulation results are also provided for the European call options. Their findings reveal that the call option prices are systematically less during a crisis compared to those generated by the Black and Scholes approach. The article of El-Khatib and Hatemi-J (2017) provides a closed form solution for option pricing during financial crisis. Savit (1989) proposes that the returns of risky assets might not be generated by a stochastic process that is usually assumed to be the case in the literature. The author suggests that the underlying returns might rather be characterized by a deterministic chaos process in such a way that the prediction errors develop exponentially, which looks like being a stochastic process even though it is deterministic. Sornette (2003) claims that most attempts in the literature try to clarify the market failures based on things that happen during short time before the crisis such as hours, days or sometimes weeks. The author has a totally different view on this issue and he thinks the main causes behind the crisis should be searched months and even years before the underlying crisis. He demonstrates via an empirical approach that the dynamic process for the prices of assets during a crisis is characterized by a converging oscillatory motion. The current paper provides the second order prices sensitivities in a market experiencing a financial crisis by using the pricing formula obtained in El-Khatib and Hatemi-J (2017).

Consider a probability space $(\Omega, \mathcal{F}, P)$ and a Brownian motion process $(W_t)_{t\in[0,T]}$ living in it. Let us denote by $(\mathcal{F}_t)_{t\in[0,T]}$ the natural filtration generated by $(W_t)_{t\in[0,T]}$. The market has an European call option with underlying risky asset S. The return on the asset without risk is denoted by r. For the sake of simplicity, the denotation P is used for the risk-neutral probability. As in El-Khatib and Hatemi-J (2017) the underlying asset price process $S = (S_t)_{t\in[0,T]}$ is assumed to be governed by

$$dS_t = rS_t dt + (\sigma S_t + \alpha e^{rt})dW_t, \tag{2.1}$$

where $t \in [0,T]$ and $S_0 > 0, \sigma$ and α are constant. The solution of (2.1) is

$$S_t = \left(S_0 + \frac{\alpha}{\sigma}\right)\exp\left[\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t\right] - \frac{\alpha}{\sigma}e^{rt}, \qquad t \in [0,T]. \tag{2.2}$$

Notice that when $\alpha = 0, S_t$ is reduced to the log-normal process of the Black–Scholes model. In the next subsection, the option pricing formula as well as the different price sensitivities for the above crash model as derived in El-Khatib and Hatemi-J (2017) is presented.

2.1. Call-Put options prices

The next two propositions from El-Khatib and Hatemi-J (2017) are needed for computing the second order price sensitivities. It is assumed that the price process $(S_t)_{t\in[0,T]}$ under the risk-neutral probability is given by (2.2). Let

$$d_1^\alpha = \frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{S_0 + \frac{\alpha}{\sigma}}{K + \frac{\alpha}{\sigma}e^{rT}}\right) + \left(r + \frac{\sigma^2}{2}\right)T\right), \tag{2.3}$$

and

$$d_2^\alpha = \frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{S_0 + \frac{\alpha}{\sigma}}{K + \frac{\alpha}{\sigma}e^{rT}}\right) + \left(r - \frac{\sigma^2}{2}\right)T\right) = d_1^\alpha - \sigma\sqrt{T}, \tag{2.4}$$

and $\Phi(d) = \int_{-\infty}^{d} \frac{e^{-u^2/2}}{\sqrt{2\pi}}du$. Then, the following proposition can be expressed.

Proposition 1. *The premium of an European call option with underlying asset $S = (S_t)_{t\in[0,T]}$, strike K and maturity T is*

$$\begin{aligned} C(S_T, K) &= E\left[e^{-rT}(S_T - K)^+\right] \\ &= \left(S_0 + \frac{\alpha}{\sigma}\right)\Phi(d_1^\alpha) - \left(Ke^{-rT} + \frac{\alpha}{\sigma}\right)\Phi(d_2^\alpha). \end{aligned} \tag{2.5}$$

Let $(\xi_{t,u}^x)_{u\in[t,T]}$ be the process defined as

$$d\xi_{t,u}^x = r\xi_{t,u}^x du + \sigma\xi_{t,u}^x dW_u, \qquad u \in [t,T], \qquad \xi_{t,t}^x = x.$$

Note that $\xi_t = \xi_{0,t}^1, t \in [0,T]$. The prices of European call and put options at any time t for the crisis model are stated in the next proposition.

Proposition 2. *The prices of an European call and an European put options with underlying asset $S = (S_t)_{t\in[0,T]}$, strike K and maturity T, at time $t \in [0,T]$, are respectively given by*

$$C(t, S_t) = \left(S_t + \frac{\alpha}{\sigma}e^{rt}\right)\Phi(d_{t,1}^\alpha) - \left(Ke^{-r(T-t)} + \frac{\alpha}{\sigma}e^{rt}\right)\Phi(d_{t,2}^\alpha),$$

and

$$P(t, S_t) = \left(S_t + \frac{\alpha}{\sigma}e^{rt}\right)\Phi(d_{t,1}^\alpha) - \left(Ke^{-r(T-t)} + \frac{\alpha}{\sigma}e^{rt}\right)\Phi(d_{t,2}^\alpha) + Ke^{-r(T-t)} - S_t,$$

where

$$d_{t,1}^\alpha = \frac{1}{\sigma\sqrt{T-t}}\left(\ln\left(\frac{S_t + \frac{\alpha}{\sigma}e^{rt}}{K + \frac{\alpha}{\sigma}e^{rT}}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right), \tag{2.6}$$

and

$$d_{t,2}^{\alpha} = \frac{1}{\sigma\sqrt{T-t}}\left(\ln\left(\frac{S_t + \frac{\alpha}{\sigma}e^{rt}}{K + \frac{\alpha}{\sigma}e^{rT}}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)\right).$$

2.2. Price sensitivities

The next proposition gives the price sensitivities for the crisis model.

Proposition 3. *The price sensitivities of an European call option with underlying asset $S = (S_t)_{t\in[0,T]}$, strike K and maturity T, at time $t \in [0,T]$, are respectively given by*

$$\Delta := \frac{\partial C}{\partial S_t} = \Phi(d_{t,1}^{\alpha})$$

$$\Gamma := \frac{\partial^2 C}{\partial S_t^2} = \frac{e^{-(d_{t,1}^{\alpha})^2/2}}{(\sigma S_t + \alpha e^{rt})\sqrt{2\pi(T-t)}} = \frac{e^{-(d_{t,1}^{\alpha})^2/2}}{S_t\sigma\sqrt{2\pi(T-t)}}$$

$$\Theta := \frac{\partial C}{\partial t} = -\frac{S_t\sigma + \alpha e^{rt}}{2\sqrt{2\pi\tau}}e^{-\frac{(d_{t,1}^{\alpha})^2}{2}} - rKe^{-r\tau}\Phi(d_{t,2}^{\alpha}) + \frac{r\alpha}{\sigma}e^{rt}(\Phi(d_{t,1}^{\alpha}) - \Phi(d_{t,2}^{\alpha}))$$

$$= -\frac{S_t\sigma}{2\sqrt{2\pi(T-t)}}e^{-\frac{(d_{t,1}^{\alpha})^2}{2}} - rKe^{-r(T-t)}\Phi(d_{t,2}^{\alpha}) + \frac{r\alpha}{\sigma}e^{rt}(\Phi(d_{t,1}^{\alpha}) - \Phi(d_{t,2}^{\alpha}))$$

$$\rho := (T-t)Ke^{-r(T-t)}\Phi(d_{t,2}^{\alpha}) + \frac{\alpha t}{\sigma}e^{rt}\left(\Phi(d_{t,1}^{\alpha}) - \Phi(d_{t,2}^{\alpha})\right)$$

$$\nu := \frac{\alpha}{\sigma^2}e^{rt}(\Phi(d_{t,2}^{\alpha}) - \Phi(d_{t,1}^{\alpha})) + \frac{e^{-\frac{(d_{t,1}^{\alpha})^2}{2}}}{\sqrt{2\pi}}\left(S_t + \frac{\alpha}{\sigma}e^{rt}\right)\sqrt{T-t}.$$

Moreover, the price sensitivities of an European put option under similar circumstances are as follows.

$$\Delta := \Phi(d_{t,1}^{\alpha}) - 1$$

$$\Gamma := \frac{1}{S_t\sigma\sqrt{2\pi(T-t)}}e^{-(d_{t,1}^{\alpha})^2/2}$$

$$\Theta := -\frac{S_t\sigma + \alpha e^{rt}}{2\sqrt{2\pi\tau}}e^{-\frac{(d_{t,1}^{\alpha})^2}{2}} - rKe^{-r\tau}\Phi(d_{t,2}^{\alpha}) + \frac{r\alpha}{\sigma}e^{rt}(\Phi(d_{t,1}^{\alpha}) - \Phi(d_{t,2}^{\alpha})) + rKe^{-r\tau}$$

$$\rho := (T-t)Ke^{-r(T-t)}\Phi(d_{t,2}^{\alpha}) + \frac{\alpha t}{\sigma}e^{rt}\left(\Phi(d_{t,1}^{\alpha}) - \Phi(d_{t,2}^{\alpha})\right) - \tau Ke^{-r\tau}$$

$$\nu := \frac{\alpha}{\sigma^2}e^{rt}(\Phi(d_{t,2}^{\alpha}) - \Phi(d_{t,1}^{\alpha})) + \frac{e^{-\frac{(d_{t,1}^{\alpha})^2}{2}}}{\sqrt{2\pi}}\left(S_t + \frac{\alpha}{\sigma}e^{rt}\right)\sqrt{T-t}.$$

For the proof of the above proposition see El-Khatib and Hatemi-J (2017).

3. Calculation of the second order price sensitivities in crisis times

In this section, the second order price sensitivities are computed of an European call option for the crisis model. The European put option second order price sensitivities can be obtained similarly. More precisely, the following second order derivatives are calculated:

$$\text{Vanna} := \frac{\partial \text{Vega}}{\partial S} = \frac{\partial^2 C}{\partial S\partial\sigma}, \qquad \text{Volga} := \frac{\partial \text{Vega}}{\partial\sigma} = \frac{\partial^2 C}{\partial\sigma^2} \quad \text{and} \quad \text{Vega bleed} := \frac{\partial^2 C}{\partial T\partial\sigma}.$$

The proposition expressed below provides the second order price sensitivities for the crisis model.

Proposition 4. *The second order sensitivities of an European call option with underlying asset $S = (S_t)_{t\in[0,T]}$, strike K and maturity T, at time $t \in [0,T]$, are respectively given by*

$$\text{Vanna} = \frac{\partial^2 C}{\partial S\partial\sigma} = \frac{\alpha}{\sigma^2}\frac{S_t - Ke^{-r\tau}}{Ke^{-rT} + \frac{\alpha}{\sigma}}\Gamma - \frac{d_{t,2}^{\alpha}e^{-\frac{(d_{t,1}^{\alpha})^2}{2}}}{\sigma\sqrt{2\pi}}. \tag{3.1}$$

$$\text{Volga} = \frac{\partial^2 C}{\partial\sigma^2} = \frac{-2\alpha}{\sigma^3}e^{rt}\left(\Phi(d_{t,2}^{\alpha}) - \Phi(d_{t,1}^{\alpha})\right) + \frac{\alpha}{\sigma^2}e^{rt}\left(\frac{e^{-\frac{(d_{t,2}^{\alpha})^2}{2}}}{\sqrt{2\pi}}\frac{\partial d_{t,2}^{\alpha}}{\partial\sigma} - \frac{e^{-\frac{(d_{t,1}^{\alpha})^2}{2}}}{\sqrt{2\pi}}\frac{\partial d_{t,1}^{\alpha}}{\partial\sigma}\right) + \frac{\sqrt{\tau}}{\sqrt{2\pi}}e^{-\frac{(d_{t,1}^{\alpha})^2}{2}}\left[1 - d_{t,1}^{\alpha}\left(S_t + \frac{\alpha}{\sigma}e^{rt}\right)\frac{\partial d_{t,1}^{\alpha}}{\partial\sigma}\right]. \tag{3.2}$$

$$\text{Vegableed} = \frac{\partial^2 C}{\partial T\partial\sigma} = \frac{\alpha}{\sigma^2}e^{rt}\left(\frac{e^{-\frac{(d_{t,2}^{\alpha})^2}{2}}}{\sqrt{2\pi}}\frac{\partial d_{t,2}^{\alpha}}{\partial\tau} - \frac{e^{-\frac{(d_{t,1}^{\alpha})^2}{2}}}{\sqrt{2\pi}}\frac{\partial d_{t,1}^{\alpha}}{\partial\tau}\right) + \frac{\sqrt{\tau}}{\sqrt{2\pi}}e^{-\frac{(d_{t,1}^{\alpha})^2}{2}}\left(S_t + \frac{\alpha}{\sigma}e^{rt}\right)\left[\frac{1}{2\tau} - d_{t,1}^{\alpha}\frac{\partial d_{t,1}^{\alpha}}{\partial\tau}\right]. \tag{3.3}$$

With

$$\frac{\partial d_{t,2}^{\alpha}}{\partial\sigma} = \frac{-d_{t,1}}{\sigma} + \frac{\alpha}{\sigma^3\sqrt{\tau}}e^{rt}\left[\frac{1}{Ke^{-r\tau} + \frac{\alpha}{\sigma}e^{rt}}\frac{S_t - Ke^{-r\tau}}{S_t + \frac{\alpha}{\sigma}e^{rt}}\right], \tag{3.4}$$

$$\frac{\partial d_{t,2}^{\alpha}}{\partial\sigma} = \frac{\partial d_{t,1}^{\alpha}}{\partial\sigma} - \sqrt{\tau}, \tag{3.5}$$

and

$$\frac{\partial d_{t,2}^{\alpha}}{\partial\tau} = \frac{\partial d_{t,1}^{\alpha}}{\partial\tau} - \frac{\sigma}{2\sqrt{\tau}},$$

where

$$\frac{\partial d_{t,1}^{\alpha}}{\partial\tau} = \frac{1}{2\sigma\tau\sqrt{\tau}}\left(\ln\left(\frac{S_t + \frac{\alpha}{\sigma}e^{rt}}{K + \frac{\alpha}{\sigma}e^{rT}}\right) + \left(r + \frac{\sigma^2}{2}\right)\tau\right) + \frac{1}{2\sigma\tau\sqrt{\tau}}\left(r + \frac{\sigma^2}{2}\right). \tag{3.6}$$

Proof. The first order price sensitivities stated in Proposition 3 are used. Then, Vanna can be computed by differentiating Vega = ν expressed in Proposition 3 with respect to S the underlying asset price, as follows

$$\text{Vanna} = \frac{\partial^2 C}{\partial S\partial\sigma} = \frac{\partial\nu}{\partial S_t} = \frac{\partial\left[\frac{\alpha}{\sigma^2}e^{rt}(\Phi\left(d_{t,2}^{\alpha}\right) - \Phi(d_{t,1}^{\alpha})) + \frac{e^{-\frac{(d_{t,1}^{\alpha})^2}{2}}}{\sqrt{2\pi}}(S_t + \frac{\alpha}{\sigma}e^{rt})\sqrt{T-t}\right]}{\partial S_t}$$

$$= \frac{\alpha}{\sigma^2}e^{rt}\frac{\partial}{\partial S_t}\left(\Phi(d_{t,2}^{\alpha}) - \Phi(d_{t,1}^{\alpha})\right) + \frac{\sqrt{T-t}}{\sqrt{2\pi}}e^{-\frac{(d_{t,1}^{\alpha})^2}{2}}\frac{\partial}{\partial S_t}\left(S_t + \frac{\alpha}{\sigma}e^{rt}\right) - \frac{\sqrt{T-t}}{\sqrt{2\pi}}\left(S_t + \frac{\alpha}{\sigma}e^{rt}\right)\frac{\partial}{\partial S_t}\left(e^{-\frac{(d_{t,1}^{\alpha})^2}{2}}\right)$$

$$= \frac{\alpha}{\sigma^2}e^{rt}\left(\frac{\partial}{\partial S_t}\Phi(d_{t,2}^{\alpha}) - \frac{\partial}{\partial S_t}\Phi(d_{t,1}^{\alpha})\right) + \frac{\sqrt{T-t}e^{-\frac{(d_{t,1}^{\alpha})^2}{2}}}{\sqrt{2\pi}} \times \left[1 - d_{t,1}^{\alpha}\left(S_t + \frac{\alpha}{\sigma}e^{rt}\right)\frac{\partial d_{t,1}^{\alpha}}{\partial S_t}\right].$$

But notice that since $\Delta = \Phi(d_{t,1}^{\alpha})$ then

$$\frac{\partial\Phi(d_{t,1}^{\alpha})}{\partial S_t} = \frac{\partial\Delta}{\partial S_t} = \Gamma \tag{3.7}$$

$$\frac{\partial d^{\alpha}_{t,1}}{\partial S_t} = \frac{\partial\left[\frac{1}{\sigma\sqrt{T-t}}\left(\ln\left(\frac{S_t+\frac{\alpha}{\sigma}e^{rt}}{K+\frac{\alpha}{\sigma}e^{rT}}\right)+(r+\frac{\sigma^2}{2})(T-t)\right)\right]}{\partial S_t} = \frac{1}{\sigma\sqrt{\tau}(S_t+\frac{\alpha}{\sigma}e^{rt})}. \quad (3.8)$$

On the other hand, the partial derivative of $\Phi(d^{\alpha}_{t,2})$ with respect to S_t can be obtained using the chain rule as follows

$$\begin{aligned}\frac{\partial\Phi(d^{\alpha}_{t,2})}{\partial S_t} &= \frac{\partial\Phi(d^{\alpha}_{t,2})}{\partial d^{\alpha}_{t,2}}\frac{\partial d^{\alpha}_{t,2}}{\partial d^{\alpha}_{t,1}}\frac{\partial d^{\alpha}_{t,1}}{\partial S_t} = \frac{\partial\Phi(d^{\alpha}_{t,2})}{\partial d^{\alpha}_{t,2}}\frac{\partial d^{\alpha}_{t,1}}{\partial S_t}\\ &= \frac{e^{-\frac{(d^{\alpha}_{t,1}-\sigma\sqrt{\tau})^2}{2}}}{\sqrt{2\pi}}\frac{\partial d^{\alpha}_{t,1}}{\partial S_t} = \frac{e^{-\frac{(d^{\alpha}_{t,1})^2}{2}-\frac{\sigma^2\tau}{2}+d^{\alpha}_{t,1}\sigma\sqrt{\tau}}}{\sqrt{2\pi}}\frac{\partial d^{\alpha}_{t,1}}{\partial S_t}\\ &= \frac{\partial\Phi(d^{\alpha}_{t,1})}{\partial d^{\alpha}_{t,1}}\frac{\partial d^{\alpha}_{t,1}}{\partial S_t}\frac{S_t+\frac{\alpha}{\sigma}e^{rt}}{Ke^{-r\tau}+\frac{\alpha}{\sigma}e^{rt}} = \frac{\partial\Phi(d^{\alpha}_{t,1})}{\partial S_t}\frac{S_t+\frac{\alpha}{\sigma}e^{rt}}{Ke^{-r\tau}+\frac{\alpha}{\sigma}e^{rt}}\\ &= \Gamma\frac{S_t+\frac{\alpha}{\sigma}e^{rt}}{Ke^{-r\tau}+\frac{\alpha}{\sigma}e^{rt}},\end{aligned}$$

where $\tau := T-t$ is the time to maturity and we have used (3.7) and (3.8). Thus,

$$\begin{aligned}\text{Vanna} &= \frac{\partial^2 C}{\partial S\partial\sigma} = \frac{\partial v}{\partial S_t}\\ &= \frac{\alpha}{\sigma^2}e^{rt}\Gamma\frac{S_t-Ke^{-r\tau}}{Ke^{-r\tau}+\frac{\alpha}{\sigma}e^{rt}} + \frac{\sqrt{\tau}}{\sqrt{2\pi}}e^{-\frac{(d^{\alpha}_{t,1})^2}{2}}\\ &\times\left[1-d^{\alpha}_{t,1}\left(S_t+\frac{\alpha}{\sigma}e^{rt}\right)\frac{1}{\sigma\sqrt{\tau}(S_t+\frac{\alpha}{\sigma}e^{rt})}\right]\\ &= \frac{\alpha}{\sigma^2}e^{rt}\Gamma\frac{S_t-Ke^{-r\tau}}{Ke^{-r\tau}+\frac{\alpha}{\sigma}e^{rt}} + \frac{e^{-\frac{(d^{\alpha}_{t,1})^2}{2}}}{\sigma\sqrt{2\pi}}\left(\sigma\sqrt{\tau}-d^{\alpha}_{t,1}\right),\end{aligned}$$

which gives (3.1). Similarly, Volga can be computed as follows:

$$\begin{aligned}\text{Volga} &= \frac{\partial^2 C}{\partial^2\sigma} = \frac{\partial v}{\partial\sigma} = \frac{\partial\left[\frac{\alpha}{\sigma^2}e^{rt}(\Phi(d^{\alpha}_{t,2})-\Phi(d^{\alpha}_{t,1}))+\frac{e^{-\frac{(d^{\alpha}_{t,1})^2}{2}}}{\sqrt{2\pi}}(S_t+\frac{\alpha}{\sigma}e^{rt})\sqrt{T-t}\right]}{\partial\sigma}\\ &= \frac{-2\alpha}{\sigma^3}e^{rt}\left(\Phi(d^{\alpha}_{t,2})-\Phi(d^{\alpha}_{t,1})\right)+\frac{\alpha}{\sigma^2}e^{rt}\frac{\partial}{\partial\sigma}\left(\Phi(d^{\alpha}_{t,2})-\Phi(d^{\alpha}_{t,1})\right)\\ &+\frac{\sqrt{\tau}}{\sqrt{2\pi}}e^{-\frac{(d^{\alpha}_{t,1})^2}{2}}\left[1-d^{\alpha}_{t,1}\left(S_t+\frac{\alpha}{\sigma}e^{rt}\right)\frac{\partial d^{\alpha}_{t,1}}{\partial\sigma}\right],\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial\sigma}\left(\Phi(d^{\alpha}_{t,2})-\Phi(d^{\alpha}_{t,1})\right) &= \frac{\partial\Phi(d^{\alpha}_{t,2})}{\partial\sigma}-\frac{\partial\Phi(d^{\alpha}_{t,1})}{\partial\sigma}\\ &= \frac{\partial\Phi(d^{\alpha}_{t,2})}{\partial d^{\alpha}_{t,2}}\frac{\partial d^{\alpha}_{t,2}}{\partial\sigma}-\frac{\partial\Phi(d^{\alpha}_{t,1})}{\partial d^{\alpha}_{t,1}}\frac{\partial d^{\alpha}_{t,1}}{\partial\sigma}\\ &= \frac{e^{-\frac{(d^{\alpha}_{t,2})^2}{2}}}{\sqrt{2\pi}}\frac{\partial d^{\alpha}_{t,2}}{\partial\sigma}-\frac{e^{-\frac{(d^{\alpha}_{t,1})^2}{2}}}{\sqrt{2\pi}}\frac{\partial d^{\alpha}_{t,1}}{\partial\sigma},\end{aligned}$$

with

$$\frac{\partial d^{\alpha}_{t,2}}{\partial\sigma} = \frac{-d_{t,1}}{\sigma}+\frac{\alpha}{\sigma^3\sqrt{\tau}}e^{rt}\left[\frac{1}{Ke^{-r\tau}+\frac{\alpha}{\sigma}e^{rt}}\frac{S_t-Ke^{-r\tau}}{S_t+\frac{\alpha}{\sigma}e^{rt}}\right],$$

$$\frac{\partial d^{\alpha}_{t,2}}{\partial\sigma} = \frac{\partial d^{\alpha}_{t,1}}{\partial\sigma}-\sqrt{\tau}.$$

The Vega Bleed is the change of the Vega when there is a time change. The Vega Bleed is calculated as follows:

$$\begin{aligned}\text{Vegableed} &= \frac{\partial^2 C}{\partial\tau\partial\sigma} = \frac{\partial v}{\partial\tau} = \frac{\partial\left[\frac{\alpha}{\sigma^2}e^{rt}(\Phi(d^{\alpha}_{t,2})-\Phi(d^{\alpha}_{t,1}))+\frac{e^{-\frac{(d^{\alpha}_{t,1})^2}{2}}}{\sqrt{2\pi}}(S_t+\frac{\alpha}{\sigma}e^{rt})\sqrt{\tau}\right]}{\partial\tau}\\ &= \frac{\alpha}{\sigma^2}e^{rt}\left(\frac{e^{-\frac{(d^{\alpha}_{t,2})^2}{2}}}{\sqrt{2\pi}}\frac{\partial d^{\alpha}_{t,2}}{\partial\tau}-\frac{e^{-\frac{(d^{\alpha}_{t,1})^2}{2}}}{\sqrt{2\pi}}\frac{\partial d^{\alpha}_{t,1}}{\partial\tau}\right)+\frac{\sqrt{\tau}}{\sqrt{2\pi}}e^{-\frac{(d^{\alpha}_{t,1})^2}{2}}\left(S_t+\frac{\alpha}{\sigma}e^{rt}\right)\\ &\times\left[\frac{1}{2\tau}-d^{\alpha}_{t,1}\frac{\partial d^{\alpha}_{t,1}}{\partial\tau}\right].\end{aligned}$$

Notice that

$$\frac{\partial d^{\alpha}_{t,2}}{\partial\tau} = \frac{\partial d^{\alpha}_{t,1}}{\partial\tau}-\frac{\sigma}{2\sqrt{\tau}}$$

and

$$\frac{\partial d^{\alpha}_{t,1}}{\partial\tau} = \frac{1}{2\sigma\tau\sqrt{\tau}}\left(\ln\left(\frac{S_t+\frac{\alpha}{\sigma}e^{rt}}{K+\frac{\alpha}{\sigma}e^{rT}}\right)+\left(r+\frac{\sigma^2}{2}\right)\tau\right)+\frac{1}{2\sigma\tau\sqrt{\tau}}\left(r+\frac{\sigma^2}{2}\right).$$

which ends the proof. □

Remark 1. The suggested formulas are a generalization of the formulas of the original Black–Scholes model for normal situations. The effect of the crisis is captured by the α parameter. If $\alpha = 0$, then the original model can be obtained.

3.1. An application of Vanna

In this subsection, an application is provided to show how the obtained formulas can improve on the precision of hedging strategies -namely compared to the standard Black and Scholes approach- for offsetting financial risk. For the sake of consistency, the same values for a call option contract on a financial asset that was used by El-Khatib and Hatemi-J (2017) for calculating premiums and first order price sensitivities are considered here to estimate the Vanna. These values are: $\sigma = 0.2, r = 0.05, T = 0.5, S = 1200, K = 1250$ and $\alpha = 0.8$. The Vanna is computed twice using Excel, first based on Eq. (3.1) and second based on the original Black-Shcoles formula, which can be obtained via (3.1) if $\alpha = 0$. Below are the estimated values for the two cases:

- Vanna for the model suggested in this paper is 0.186930317,
- Vanna for the Black and Scholes model is 0.155724683.

One can notice that the Vanna of the crisis model is estimated to be higher than the Vanna of Black and Scholes model. This means the need for hedging is bigger based on the suggested method compared to the standard one.

4. Conclusions

Price sensitivities are regularly used by financial institutions and investors in order to deal with different sources of financial risk. Recently, the literature has put forward formulas for price sensitivities for markets that are characterized by a crisis. This is an important issue because it is exactly during the crisis that the need for successful tools that can neutralize or at least reduce risk is urgent. Another strand of literature has contributed to the introduction of the second order price sensitivities. To the best knowledge, the second order price sensitivities have not been developed for a market with a crisis, a gap that this paper attempt to fill. The formulas that are proposed in this paper are expected to make the hedging against the financial risk more precise. This might be specially the case during the crises.

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