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Original article

Impulsive second order control differential equations: Existence and approximate controllability

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ABSTRACT

The primary focus of this manuscript is on the approximate controllability of second-order semilinear control systems with impulses. There have been two sets of necessary requirements discussed. Combining the theories of the sine and cosine families, as well as the compactness of the cosine operator along with the fixed point technique (FPT) yields the first set of results. The following discussion avoids the apply of the compactness and fixed point approach of the cosine function and is shown instead using Gronwall's inequality. The existence and uniqueness of the mild solution are also established. We have also provided a case study of theoretical outcomes that have been confirmed the theory.

1. Introduction

Differential equations are extensively used in a variety of scientific and technological fields, particularly when there is knowledge or speculation about a deterministic relationship between some continuously varying values and their rates of change in space and/or time. This is clear from classical mechanics, which holds that changes in a body's position and speed over time influence how it moves. It is well knowledge that systems with abstract differential equations can be represented by partial differential equations. Pazy (Pazy, 1983) examined a range of solutions to nonlinear and semilinear evolution equations using the semigroups approach. Numerous problems in physics, biological systems, population dynamics, optimal control, ecology, biotechnology, and other fields are solved using differential equations.

In physical sciences, economical theories and population dynamics, impulsive differential equations have become increasingly important. There has been significant progress in impulsive systems, specifically, the systems having fixed instants. These are the very useful type for depicting unexpected changes in large units of the continuous development process and allowing a better comprehension of any physical situation in applied science, the readers can refer (Chadha and Bora, 2018; Jeet and Sukavanam, 2020; Chen and Li, 2010; Sivasankaran et al., 2011; Arora and Sukavanam, 2016; Vijayakumar et al., 2017; Vijayakumar et al., 2021b; Li and Wu, 2018; Zhou et al., 2018). In applied mathematics, control theory is essential since it involves developing and evaluating the control framework. Controllability is used in a different of real-world applications, blood sugar level regulation, including rocket launch challenges for satellites, missiles in defense and economic inflation rate regulation. The author initiated a systematic investigation of controllability in 1963, when Kalman (Kalman, 1963) developed the discussion on controllability for time-invariant and time-varying systems in the state-space form.

A branch of application-oriented mathematics called mathematical control theory focuses on the underlying ideas that underlie the design and analysis of control systems. The two main research areas in control theory have often been complimentary, despite occasionally appearing to move in opposite directions.

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Some of them indicate that the user wishes to modify the object's behaviour in some manner and that a suitable model of the object to be governed is provided. Physical ideas and technological criteria are used, for example, to choose a spacecraft's trajectory that minimises overall voyage duration or fuel consumption. A preprogrammed flight plan is often the outcome of the methods utilized here, which are closely related to other areas of optimization theory and the traditional calculus of variations. The other crucial field of research is founded on the restrictions brought about by uncertainty regarding the model or the environment in which the item performs. The main tactic in this situation is to use feedback to correct for departures from the expected conduct.

The state equation and the restrictions on the control signal are two of the several definitions and criteria for controllability. The great majority of the specifications that are produced for finite dimensional systems in the literature and that can be met. It should be emphasised that there are still a lot of problems around the control theory.In the context of infinite dimensional systems, two key controllability concepts can be separated. There are two types of controllability: exact and approximate controllability. This is intimately related to the existence of non-closed linear subspaces in infinite dimensional spaces. While approximate controllability allows us to guide the system to any small neighbourhood of the final state, exact controllability allows us to lead the system to any end state. To put it another way, approximate controllability makes it possible to point the system in the direction of the state space's dense subspace. Clearly, exact controllability is a more effective concept than approximate controllability in light of this.

Several engineering and scientific problems are nonlinear, and they can be described in infinite-dimensional spaces using ODEs and PDEs. Therefore, nonlinearity is a critical concern in infinite-dimensional spaces; for further information, check (Curtain and Zwart, 1995). The controllability of the system in finite dimensions was studied by several authors. Many scientists studied controllability in finite-dimensional space and generalised their findings to infinite-dimensional control systems. Controllability has been investigated using a variety of methodologies, including fixed-point theorems, which can be seen here (Vijayakumar et al., 2022; Naito, 1987; Mahmudov et al., 2020; Mahmudov et al., 2016; Tomar and Sukavanam, 2011; Sukavanam and Tafesse, 2011; Fu and Mei, 2009; Vijayakumar and Murugesu, 2019; Vijayakumar et al., 2019; Vijayakumar et al., 2021a). Fractional differential systems are mathematical representations of a wide range of applications in science, economics, engineering, and other fields. Several researchers have investigated the existence and controllability of various kinds of integer and fractions order systems can be found in (Shukla et al., 2014; Shukla et al., 2016; Shukla et al., 2015; Mohan Raja et al., 2020; Mohan Raja et al., 2020; Sakthivel et al., 2011; Sakthivel et al., 2012; Kumar and Sukavanam, 2012; Zhou and Jiao, 2010; You et al., 2020).

Instance a consequence of unexpected jumps at crucial points in the advancement approach, many systems in pharmacology, the signal processing, physical in nature, biological in nature, and other fields have manifested impulsive recently. The articles (Li and Wu, 2016; Li et al., 2015) prove the existence of mild solutions as well as stability analysis of impulsive functional differential systems. By using the Lyapunov approach and the Banach contraction theorem, the authors were capable of coming to some significant conclusions. In Bazighifan et al. (2022), Almarri et al. (2022a), Almarri et al. (2022b) authors studied oscillation solutions of the differential equations with delay and impulses. Recently, the authors (Gou and Li, 2021; Mohan Raja et al., 2022) discussed the approximate controllability results for fractional stochastic systems by referring to the fixed point theorems. This paper, we discuss the two sets of sufficient conditions. Combining the theories of the sine and cosine function operators, as well as the compactness of the cosine function with the fixed point theorems (FPT) yields the first set of results. Then, Gronwall's inequality is apply to show the next discussion, which avoids the use of the cosine function's compactness and fixed point methods. The existence and uniqueness of the mild solution are also established. The field of mathematical control theory has benefited significantly from the new and notable results.

The function spaces $\Theta = L_2([0,\jmath];H), Y = L_2\left([0,\jmath];\widehat{H}\right)$, where H and $\widehat{H}$ are Hilbert spaces. Let us consider the second-order impulsive functional differential system as follows:

$$\begin{cases} \frac{dz'(\varphi)}{d\varphi} = Az(\varphi) + Bu(\varphi) + \eta(\varphi, z(\varphi)), \quad \varphi \in J = [0,\jmath], \\ \varphi \neq \varphi_\alpha, \ \alpha = 1,2,\cdots,\beta, \\ z(0) = z_0, \ z'(0) = z_1, \\ \Delta z(\varphi_\alpha) = \chi_\alpha(z(\varphi_\alpha)), \ \Delta z'(\varphi_\alpha) = \widetilde{\chi}_\alpha(z(\varphi_\alpha)), \ \alpha = 1,2,\cdots,\beta, \end{cases} \tag{1.1}$$

where $z(\varphi) \in H$ denotes the state variable; the control function $u(\varphi) \in \widehat{H}$; the linear closed operator $A : D(A) \subseteq H$ into H (refer (Pazy, 1983) for basics), that produces a strongly continuous cosine function $P(\varphi)$, where $D(A)$ is a dense domain of A; the purely non linear function η defined on an into mapping $[0,\jmath] \times H \to H$. $\chi_\alpha, \widetilde{\chi}_\alpha$ maps from H into H, which are the appropriate functions. Assume $\varphi_i < \varphi_{i+1}$ $i = 0,1,\ldots\ldots\beta$ with $\varphi_0 = 0$ and $\varphi_{\beta+1} = \jmath, \Delta z(\varphi_\alpha)$ represent the jump at point of discontinuity φ_α and is denoted as $z(\varphi_\alpha^+) - z(\varphi_\alpha^-) = \Delta z(\varphi_\alpha)$. $z(\varphi_\alpha^+)$ and $z(\varphi_\alpha^-)$ represents the right and left limits of z at φ_α.

The aforementioned (1.1) linear type system is given by

$$\begin{cases} \frac{dz'(\varphi)}{d\varphi} = Az(\varphi) + Bu(\varphi), \quad \varphi \in J = [0,\jmath], \ \varphi \neq \varphi_\alpha, \ \alpha = 1,2,\ldots,\beta, \\ z(0) = z_0, \ z'(0) = z_1, \\ \Delta z(\varphi_\alpha) = \chi_\alpha(z(\varphi_\alpha)), \ \Delta z'(\varphi_\alpha) = \widetilde{\chi}_\alpha(z(\varphi_\alpha)), \ \alpha = 1,2,\ldots,\beta. \end{cases} \tag{1.2}$$

The following is the breakdown of our article's structure: Section 2 presents some more fundamental theories and outcomes related to control theory. The mild solution's outcomes are demonstrated in Section 3. Section 3 explains the apply of the fixed point theorem to determine controllability. Gronwall's inequality is used in Section 5 to provide a new set of necessary conditions. The validation of the theory is demonstrated in Section 6.

2. Preliminaries

This section analyzes the steps required to get the main points of our discussion. Suppose that

$$\mathscr{P}C(J,V) = \{z \text{ maps from } J \text{ into } H, z(\varphi) \text{ is continuous at } \varphi = \varphi_\alpha, \text{ left continuous at } \varphi = \varphi_\alpha, \text{ and } z(\varphi_\alpha^+) \ \exists \ \alpha = 1,2,\cdots,\beta\}.$$

Obviously, the Banach space $\mathscr{P}C(J,V)$ with

$$\|z\|_{\mathscr{P}C} = \sup_{\varphi \in J}\{\|z(\varphi)\| : z \in \mathscr{P}C(J,V)\}.$$

Assume that $\mathscr{P}C^1(J,V)$ consist of all continuously differentiable function z belongs to $\mathscr{P}C(J,V)$, accompanying outcomes:

$$z_R'(\varphi) = \lim_{\xi \to 0+} \frac{z(\varphi+\xi) - z(\varphi^+)}{\xi}, \text{ and } z_L'(\varphi) = \lim_{\xi \to 0-} \frac{z(\varphi+\xi) - z(\varphi^-)}{\xi}.$$

In the above $z_L'(\varphi)$ and $z_R'(\varphi)$ are both continuous on the semi-closed intervals $(0,\jmath]$ and $[0,\jmath)$ respectively. Furthermore, for z belongs to $\mathscr{P}C^1(J,V), z'(0)$, denotes the right derivative at 0, and by $z'(\varphi)$ the left derivative at $0 < \varphi \leqslant \jmath$. In full view of $\mathscr{P}C^1(J,V)$ represents Banach space along with

$\|z\|_{\mathscr{P}C^1} = \max\{\|z\|_{\mathscr{P}C}, \|z'\|_{\mathscr{P}C}\}$.

Definition 2.1 ((*Travis and Webb, 1978*)). A strongly continuous cosine family is defined as the operator $\{P(\varphi)\}_{\varphi\in\mathbb{R}}$ of bounded linear operators mapping Y into Y iff

(a) $P(0) = I$.
(b) $P(\xi+\varphi) + C(\xi-\varphi) = 2P(\xi)P(\varphi)$.
(c) $P(\varphi)w$ is continuous in φ on $\mathbb{R}$ for every $w \in \mathbb{Y}$.

The sine function $\{Q(\varphi), \varphi \in \mathbb{R}\}$ connected with $\{P(\varphi), \varphi \in \mathbb{R}\}$ presented as

$$Q(\varphi) = \int_0^{\varphi} P(\xi)d\xi, \quad \varphi \in \mathbb{R}.$$

The infinitesimal generator of $\{P(\varphi), \varphi \in \mathbb{R}\}$ is $A : \mathbb{Y} \to \mathbb{Y}$ presented as

$$Az = \frac{d^2}{d\varphi^2}P(0)z.$$

The domain of A will now be defined in the following way:

$$dom(A) = \{z \in \mathbb{Y} : \text{the function } P(\varphi)z \text{ is a twice continuously differentiable function of } \varphi\}.$$

The cosine and sine family function introduced above with A satisfy the subsequent characteristics:

Lemma 1. (Travis and Webb, 1978) *Consider A is the infinitesimal generator of $\{P(\varphi) : \varphi \in \mathbb{R}\}$. Then, it satisfies:*

(a) $\exists\, M' \geqslant 1$ and $\omega \geqslant 0$ such that $\|P(\varphi)\| \leqslant M'e^{\omega|t|}$, and then $\|Q(\varphi)\| \leqslant M'e^{\omega|t|}$;
(b) $\exists\, N' \geqslant 1 \ni, \|Q(\xi) - Q(p)\| \leqslant N'\left|\int_{\xi}^{p} e^{\omega|\xi|}d\xi\right|, \forall\, 0 \leqslant \xi \leqslant p < \infty$;
(c) $A\int_{\xi}^{p} Q(u)zdu = [P(p) - P(\xi)]z, \forall\, 0 \leqslant \xi \leqslant p < \infty$.

Because of the uniform boundedness principle and (a), both $P(\varphi)$ and $Q(\varphi)$ are uniformly bounded. Furthermore, $M = M'e^{\omega|b|}$.

Proposition 1. (Travis and Webb, 1978) *Suppose that the strongly continuous cosine family $\{P(\varphi)\}_{\varphi\in\mathbb{R}}$ in $\mathbb{Y}$, then the operator $\widehat{A} : \mathbb{Y} \to \mathbb{Y}$ presented as*

$$\widehat{A}z = \lim_{\varphi\to 0}\frac{(P(2\varphi)z - z)}{2\varphi^2},$$

with domain those $z \in \mathbb{Y}$ for that this limit exists, is the infinitesimal generator of $\{P(\varphi)\}_{\varphi\in\mathbb{R}}$.

Now, we define the mild solution of the system (1.1) according to $u(\cdot)$ belongs to Y as follows:

Definition 2.2. Provided that

$$z(\varphi) = P(\varphi)z_0 + Q(\varphi)z_1 + \int_0^{\varphi} Q(\varphi-\sigma)[Bu(\sigma) + \eta(\sigma, z(\sigma))]d\sigma + \sum_{0<\varphi_\alpha<\varphi} P(\varphi-\varphi_\alpha)\chi_\alpha(z(\varphi_\alpha)) + \sum_{0<\varphi_\alpha<\varphi} Q(\varphi-\varphi_\alpha)\widetilde{\chi}_\alpha(z(\varphi_\alpha)), \; \varphi \in [0, \jmath], \tag{2.1}$$

is satisfied by $z(\cdot) \in \Theta$, next which is called mild solution of the system (1.1).

Definition 2.3. (Sakthivel et al., 2011) The reachable set of (1.1) is presented as $K_\jmath(\eta) = \{z(\jmath) \in H : z(\cdot)$ represents the mild solution of (1.1)$\}$. In case $\eta \equiv 0$, in addition, the system (1.1) reduces to the system (1.2) and is said to be linear system according to the system (1.1). The set $K_\jmath(0)$ denotes a reachable.

Definition 2.4. (Sakthivel et al., 2011) Provided that $\overline{K_\jmath(\eta)} = H$, in addition, the semilinear control system is approximate controllable on J. $\overline{K_\jmath(\eta)}$ denotes the closure of $K_\jmath(\eta)$. If $\overline{K_\jmath(0)} = H$, then linear system is approximate controllable.

The operator $\aleph$ mapping from Θ into Θ is provided as

$[\aleph z](\varphi) = \eta(\varphi, z(\varphi)); t \in (0, \jmath]$.

Assume that ρ mapping from Θ into itself, then

$$[\rho z](\varphi) = \int_0^{\varphi} Q(\varphi-\sigma)z(\sigma)d\sigma.$$

Then, we introduce the operator $\mathscr{L}$ mapping from Θ into H as

$$\mathscr{L}\mu = \int_0^{\jmath} Q(\jmath-\sigma)\mu(\sigma)d\sigma.$$

Assume that $N_0(\mathscr{L})$ as the null space in the manner of $\mathscr{L}$. Then, the closed and the orthogonal space $N_0(\mathscr{L})$ is a subspace of Θ it is denotes $N_0^{\perp}(\mathscr{L})$. Thus, it can then be uniquely defined by $\Theta = N_0(\mathscr{L}) \oplus N_0^{\perp}(\mathscr{L})$. $R(B), \overline{R(B)}$ represents the range of B and closure of $R(B)$.

3. Existence of mild solutions

We assume the following assumptions before moving to the primary outcomes:

Assumption 1. For $\varphi \in \mathbb{R}, \{P(\varphi)\}$ is compact.

Assumption 2. The nonlinear function $\eta(\varphi, z)$, which is fulfills linear growth and Lipschitz condition. Then, $\exists$ a positive constant l fulfilling

$$\|\eta(\varphi, x) - \eta(\varphi, z)\| \leqslant l\|x - z\|, \text{ for all } x, z \in H, 0 \leqslant \varphi \leqslant \jmath.$$

Assumption 3. The function $\chi_\alpha : H \to H$, which is continuous, then $\exists\, q_\alpha, \widetilde{q_\alpha} > 0$ with

$$\|\chi_\alpha(z) - \chi_\alpha(s)\| \leqslant q_\alpha\|z - s\|, \; \|\chi_\alpha(z)\| \leqslant \widetilde{q_\alpha}(1 + \|z\|), \; \alpha = 1, 2, \ldots, \beta,$$

$\forall\, z, s$ belongs to H.

Assumption 4. The continuous function $\widetilde{\chi}_\alpha : H \to H, \exists\, e_\alpha, \widetilde{e_\alpha} > 0$ with

$$\|\widetilde{\chi}_\alpha(z) - \widetilde{\chi}_\alpha(s)\| \leqslant e_\alpha\|z - s\|, \; \|\widetilde{\chi}_\alpha(z)\| \leqslant \widetilde{e_\alpha}(1 + \|z\|), \; \alpha = 1, 2, \ldots, \beta.$$

$\forall\, z, s$ belongs to H.

Theorem 3.1. *If the preceding assumptions (1)-(4) are met, $u(\cdot)$ belongs to Y, according to $u(\cdot)$. Furthermore, the (1.1) system has a unique mild solution in Θ. Here, $\|B\| \leqslant M_B$*

Proof. Assume that $l_\eta = \max_{0\leqslant t\leqslant \jmath}\|\eta(\varphi, 0)\|$.

Determine $\Phi : (L_2([0,\varphi_1];H) \to L_2([0,\varphi_1];H)$ by

$$(\Phi z)(\varphi) = P(\varphi)z_0 + Q(\varphi)z_1 + \int_0^{\varphi} Q(\varphi-\sigma)[Bu(\sigma)+\eta(\sigma,z(\sigma)]d\sigma + \sum_{0<\varphi_\alpha<\varphi} P(\varphi-\varphi_\alpha)\chi_\alpha(z(\varphi_\alpha)) + \sum_{0<\varphi_\alpha<\varphi} Q(\varphi-\varphi_\alpha)\widetilde{\chi}_\alpha(z(\varphi_\alpha)),\ \varphi\in(0,\varphi_1].$$

Now we need to verify that (2.1) represents mild solution on $[0,\varphi 1]$, and that Φ has a fixed point in $L_2([0,\varphi_1];H)$.

Assume that the closed and bounded set $\Omega_R \subset L_2([0,\varphi_1];H)$, where

$$\Omega_R = \left\{ z(\varphi)\in L_2([0,\varphi_1];H) : ||z||_{L_2([0,\varphi_1];H)} \leqslant R, z(0)=z_0,\ z'(0)=z_1 \right\}.$$

For any $z(\cdot)\in\Omega_R$, we have

$$\begin{aligned}
&||(\Phi z)(\varphi)|| \leqslant M||z_0|| + M||z_1|| + MM_B\int_0^{\varphi}||u(\sigma)||d\sigma + M\int_0^{\varphi}||\eta\Big(\sigma,z(\sigma)||d\sigma + M\sum_{0<\varphi_\alpha<\varphi}||\chi_\alpha(z(\varphi_\alpha))|| + M\sum_{0<\varphi_\alpha<\varphi}||\widetilde{\chi}_\alpha(z(\varphi_\alpha))||\\
&\leqslant M||z_0|| + M||z_1|| + MM_B\sqrt{\varphi}||u||_Y + M\int_0^{\varphi}||\eta(\sigma,z(\sigma)) - \eta(\sigma,0)||d\sigma\\
&+ M\int_0^{\varphi}||\eta(\sigma,0)||d\sigma + M\sum_{\alpha=1}^{\beta}||\chi_\alpha(z)|| + M\sum_{\alpha=1}^{\beta}||\widetilde{\chi}_\alpha(z)||\\
&\leqslant M||z_0|| + M||z_1|| + MM_B\sqrt{\varphi}||u||_Y + Ml\int_0^{\varphi}||z(\sigma)||d\sigma + Ml_\eta\int_0^{\varphi}d\sigma\\
&+ M\widetilde{q_\alpha}\sum_{\alpha=1}^{\beta}(1+||z||) + M\widetilde{e_\alpha}\sum_{\alpha=1}^{\beta}(1+||z||) \leqslant M||z_0|| + M||z_1||\\
&+ MM_B\sqrt{\varphi}||u||_Y + MlR\sqrt{\varphi} + Ml_\eta\varphi + M\widetilde{q_\alpha}\beta(1+||z||)\\
&+ M\widetilde{e_\alpha}\beta(1+||z||) \leqslant M||z_0|| + M||z_1|| + MM_B\sqrt{\varphi_1}||u||_Y + MlR\sqrt{\varphi_1}\\
&+ Ml_\eta\varphi_1 + M\widetilde{q_\alpha}\beta(1+R) + M\widetilde{e_\alpha}\beta(1+R).
\end{aligned}$$

Now, let

$$M||z_0|| + M||z_1|| + MM_B\sqrt{\varphi_1}||u||_Y + MlR\sqrt{\varphi_1} + Ml_\eta\varphi_1 + M\widetilde{q_\alpha}\beta(1+R) + M\widetilde{e_\alpha}\beta(1+R) < R.$$

Then

$$M||z_0|| + M||z_1|| + MM_B\sqrt{\varphi_1}||u||_Y + Ml_\eta\varphi_1 + M\widetilde{q_\alpha}\beta + M\widetilde{e_\alpha}\beta < R(1 - Ml\sqrt{\varphi_1} - M\widetilde{q_\alpha}\beta - M\widetilde{e_\alpha}\beta).$$

If the right hand side is positive,

$$\varphi_1 < \left(\frac{1 - M\widetilde{q_\alpha}\beta - M\widetilde{e_\alpha}\beta}{Ml}\right)^2. \tag{3.1}$$

Hence, provided that (3.1) holds the operator $\Phi : \Omega_R \to \Omega_R$.

We now must check that Φ is a contraction on Ω_R. Again for aforementioned reasons, we will consider that z,ξ belongs to Ω_R, consequently we obtain

$$\begin{aligned}
&||(\Phi z)(\varphi) - (\Phi\xi)(\varphi)|| \leqslant M\int_0^{\varphi}||\eta(\sigma,z(\sigma)) - \eta(\sigma,\xi(\sigma))||d\sigma\\
&+ M\sum_{0<\varphi_\alpha<\varphi}||\chi_\alpha(z(\varphi_\alpha)) - \chi_\alpha(\xi(\varphi_\alpha))||\\
&+ M\sum_{0<\varphi_\alpha<\varphi}||\widetilde{\chi}_\alpha(z(\varphi_\alpha)) - \widetilde{\chi}_\alpha(\xi(\varphi_\alpha))||\\
&\leqslant Ml\int_0^{\varphi_1}||z(\sigma) - \xi(\sigma)||d\sigma + Mq_\alpha\sum_{\alpha=1}^{\beta}||z-\xi|| + Me_\alpha\sum_{\alpha=1}^{\beta}||z-\xi||\\
&\leqslant Ml\sqrt{\varphi_1}||z-\xi|| + Mq_\alpha\sum_{\alpha=1}^{\beta}||z-\xi|| + Me_\alpha\sum_{\alpha=1}^{\beta}||z-\xi||.
\end{aligned}$$

As a result, in light of (3.1), Φ is a contraction mapping. Hence, Φ has a unique fixed point in Ω_R, then (2.1) signifies the mild solution on $[0,\varphi_1]$. Likewise, we shall prove (2.1) denotes the mild solution in closed interval $[\varphi_1, t_2],\ \varphi_1 < t_2$. Following in just this way, we can arrive at the conclusion that (2.1) provides a mild solution on $[0,\varphi^*),\ \varphi^* < \infty$. Moreover, we now must check the mild solution's boundedness.

$$\begin{aligned}
||z(\varphi)|| &\leqslant M||z_0|| + M||z_1|| + MM_B\int_0^{\varphi}||u(\sigma)||d\sigma + Ml\int_0^{\varphi}||z(\sigma)||d\sigma + Ml_\eta\varphi\\
&+ M\widetilde{q_\alpha}\beta(1+||z||) + M\widetilde{e_\alpha}\beta(1+||z||)\\
&\leqslant M||z_0|| + M||z_1|| + MM_B\sqrt{\varphi}||u||_Y + Ml_\eta\varphi + Ml\int_0^{\varphi}||z(\sigma)||d\sigma\\
&+ M\widetilde{q_\alpha}\beta + M\widetilde{q_\alpha}\beta||z|| + M\widetilde{e_\alpha}\beta + M\widetilde{e_\alpha}\beta||z||.
\end{aligned}$$

Now that Gronwall's inequality implies that $z(\cdot)$ is bounded, we may conclude that z is properly determined on $[0,\jmath]$.

The notion of uniqueness must be discussed. Then, if $\varphi\in[0,\varphi^*)$, then we suppose that z and ξ are any two solutions of the system (2.1).

$$\begin{aligned}
||(z)(\varphi) - (\xi)(\varphi)|| &\leqslant M\int_0^{\varphi}||\eta(\sigma,z(\sigma)) - \eta(\sigma,\xi(\sigma))||d\sigma\\
&+ M\sum_{0<\varphi_\alpha<\varphi}||\chi_\alpha(z(\varphi_\alpha)) - \chi_\alpha(\xi(\varphi_\alpha))||\\
&+ M\sum_{0<\varphi_\alpha<\varphi}||\widetilde{\chi}_\alpha(z(\varphi_\alpha)) - \widetilde{\chi}_\alpha(\xi(\varphi_\alpha))||\\
&\leqslant Ml\int_0^{\varphi}||z(\sigma) - \xi(\sigma)||d\sigma + Mq_\alpha\beta||z-\xi||\\
&+ Me_\alpha\beta||z-\xi||.
\end{aligned}$$

Utilizing Gronwall's inequality, we have $z(\varphi) = \xi(\varphi)$, for any $\varphi\in[0,\jmath]$. As a result, the mild solution is unique. □

4. Approximate controllability outcomes through FPT

The approximate controllability of the studied system is discussed in this section. Schauder's fixed point theorem is used to produce the primary results. We assume the following when continuing on to the part of an evaluation:

Assumption 5. $\mathscr{L}\mu = \mathscr{L}v$ such that v belongs to $\overline{R(B)}$, for any μ belongs to Θ.

$\exists$ v belongs to $\overline{R(B)}$ with $\mu - v = \theta \in N_0(\mathscr{L})$ for all μ belongs to Θ. Hence, $\Theta = N_0(\mathscr{L}) \oplus \overline{R(B)}$. Then, we can determine E maps from $N_0^{\perp}\mathscr{L}$ into $\overline{R(B)}$ is continuous, linear and presented as $Eu^* = v^*$, which is stands for unique minimum norm element in $\overline{R(B)} \cap \{u^* + N_0(\mathscr{L})\}$, that is,

$$||Eu^*|| = ||v^*|| = \min\left\{||v|| : v\in\{u^* + N_0(\mathscr{L})\} \oplus \overline{R(B)}\right\}.$$

Clearly with the help of assumption (5) $\forall$ u^* belongs to $N_0^{\perp}(\mathscr{L}), \overline{R(B)}$ is the non-void subset of $\{u^* + N_0(\mathscr{L})\}$ and we can express $z\in\Theta$ as $z = \theta + v^*$. uniquely Therefore, E is clearly specified. Then, $||E|| \leqslant \varrho$, where ϱ is a constant.

Lemma 2. *The inequality $||\theta||_{\Theta} \leqslant (1+\varrho)||z||_{\Theta}$, holds, for every $z\in\Theta$ and $\theta\in N_0(\mathscr{L})$.*

Suppose that Θ is the subspace of Θ (refer (Sukavanam and Tafesse, 2011)) $\ni$

$$\Theta = \{\beta\in\Theta : \beta(\varphi) = (\rho\theta)(\varphi),\ \theta\in N_0(\mathscr{L})\},\ t\in[0,\jmath].$$

This is obvious $\beta(\jmath) = 0, \forall\ \beta\in\Theta$.

Now, we introduce the operator η_z mapping from Θ into Θ as

$$\eta_z(\beta) = \rho\theta,\ t\in[0,\jmath].$$

In the above θ is given as the unique decomposition.

$$\aleph(z+\beta) = \theta + v;\ \theta \in N_0(\mathscr{L}),\ v \in \overline{R(B)}. \tag{4.1}$$

Lemma 3. *Based on the condition (3), $\beta_0 \in \Theta$ along with $\eta(\beta_0) = \beta_0$ if $Ml\jmath(1+\varrho) < 1$.*

Proof. Assume that $\Omega_r = \{z \in \Theta : ||z|| \leqslant r,\ r > 0\}$. Our main goal is to demonstrate that $\eta_z : \Omega_r \to \Omega_r$. We will show it by contradiction. Let $\beta \in \Omega_r$, but $\eta_z(\beta) \notin \Omega_r$, that is, $||\eta_z(\beta)|| > r$. From assumption (3) and Lemma 2, we get

$$\begin{aligned} r &< ||\eta_z(\beta)|| = ||\rho\theta|| \leqslant \textstyle\int_0^{\varphi} ||Q(\varphi-\sigma)||\,||\theta(\sigma)||d\sigma \\ &\leqslant M \textstyle\int_0^{\varphi} ||\theta(\sigma)||d\sigma \\ &\leqslant M \textstyle\int_0^{\varphi} (1+\varrho)||\aleph(z+\beta)(\sigma)||d\sigma \\ &\leqslant M(1+\varrho) \textstyle\int_0^{\varphi} ||\eta(\sigma,(z+\beta)(\sigma))||d\sigma \\ &\leqslant M(1+\varrho) \textstyle\int_0^{\varphi} [l||(z+\beta)(\sigma)|| + l_\eta]d\sigma \\ &\leqslant Ml(1+\varrho) \textstyle\int_0^{\varphi} [||z(\sigma) + (\beta)(\sigma)||d\sigma + M\jmath l_\eta(1+\varrho) \\ &\leqslant Ml(1+\varrho)\sqrt{\varphi}||z||_\Theta + Mlr(1+\varrho)t + M\jmath l_\eta(1+\varrho) \\ &\leqslant M(1+\varrho)\left[l\sqrt{\jmath}||z||_\Theta + lr\jmath + l_\eta \jmath\right]. \end{aligned}$$

Applying a limit r tends to ∞ after dividing by r we get

$$Ml\jmath(1+\varrho) \geqslant 1$$

Therefore, by contradiction we deduce that $\eta_z : \Omega_r \to \Omega_r$.

We now validate that ηz is a compact operator. As $P(\varphi)$ is compact, the integral operator ρ is compact, and so η_z is compact. According to Schauder's fixed point theorem, the fixed point of η_z is β_0, that is,

$$\eta_z(\beta_0) = \rho\theta = \beta_0.$$

The proof is now finished. □

Theorem 4.1. *If the conditions (1)-(5) are met, (1.2) system is approximately controllable. In addition, the system (1.1) is also approximate controllable.*

Proof. Assume that the mild solution according to the system (1.2) is denoted by $z(\cdot)$, consequently

$$z(\varphi) = P(\varphi)z_0 + Q(\varphi)z_1 + \rho Bu(\varphi) + \sum_{0<\varphi_\alpha<\varphi} P(\varphi-\varphi_\alpha)\chi_\alpha(z(\varphi_\alpha)) \tag{4.2}$$

$$+ \sum_{0<\varphi_\alpha<\varphi} Q(\varphi-\varphi_\alpha)\widetilde{\chi}_\alpha(z(\varphi_\alpha)),\ \varphi \in (0,\jmath]. \tag{4.3}$$

Our purpose is to show that $s(\varphi) = z(\varphi) + \beta_0(\varphi)$ is the mild solution to the equations below

$$\frac{ds(\varphi)}{d\varphi} = As(\varphi) + (Bu-v)(\varphi) + \eta(\varphi, s(\varphi)),\quad \varphi \in (0,\jmath];$$

$$\varphi \neq \varphi_\alpha, \alpha = 1,2,\ldots,\beta, \tag{4.4}$$

$$s(0) = z_0,\ s'(0) = z_1, \tag{4.5}$$

$$\Delta s(\varphi_\alpha) = \chi_\alpha(z(\varphi_\alpha)),\ \Delta s'(\varphi_\alpha) = \widetilde{\chi}_\alpha(z(\varphi_\alpha)),\ \alpha = 1,2,\ldots,\beta. \tag{4.6}$$

By (4.1), we get

$$\aleph(z+\beta)(\varphi) = \theta(\varphi) + v(\varphi).$$

Using the specification for and Lemma 3 together, one may obtain by acting ρ at $\beta = \beta_0$, which is a fixed point of η_z.

$$\begin{aligned} \rho\aleph(z+\beta_0)(\varphi) &= \rho\theta(\varphi) + \rho v(\varphi), \\ &= \beta_0(\varphi) + \rho v(\varphi). \end{aligned}$$

When we add $z(\cdot)$ to both sides, we obtain

$$z(\varphi) + \rho\aleph(z+\beta_0)(\varphi) = z(\varphi) + \beta_0(\varphi) + \rho v(\varphi).$$

Let $s(\varphi) = z(\varphi) + \beta_0(\varphi)$, then

$$\begin{aligned} & z(\varphi) + \rho\aleph(s)(\varphi) = s(\varphi) + \rho v(\varphi), \\ \Rightarrow\ & s(\varphi) = z(\varphi) + \rho\aleph(s)(\varphi) - \rho v(\varphi). \end{aligned} \tag{4.7}$$

Using Eq. (4.3), we obtain

$$\begin{aligned} s(\varphi) &= P(\varphi)z_0 + Q(\varphi)z_1 + \rho(Bu-v)(\varphi) + \rho\aleph(s)(\varphi) \\ &+ \sum_{0<\varphi_\alpha<\varphi} P(\varphi-\varphi_\alpha)\chi_\alpha(z(\varphi_\alpha)) + \sum_{0<\varphi_\alpha<\varphi} Q(\varphi-\varphi_\alpha)\widetilde{\chi}_\alpha(z(\varphi_\alpha)). \end{aligned}$$

This is really the necessary mild solution (4.4)–(4.6), as well as the control $(Bu-v)$.

In addition, we get $\beta_0(0) = 0 = \beta_0(\jmath)$ and $\beta_0'(0) = 0 = \beta_0'(\jmath)$ so

$$s(0) = z(0) + \beta_0(0) = z_0,\ s'(0) = z'(0) + \beta'(0) = z_1,$$

and

$$s(\jmath) = z(\jmath) + \beta_0(\jmath) = z(\jmath) \in \rho_\jmath(0).$$

Moreover, hence v in $\overline{R(B)}$, there exists υ in Y such that

$||B\upsilon - v|| \leqslant \epsilon$ for all $\epsilon > 0$.

If $w = u - \upsilon$ then consider $z_w(\cdot)$ as mild solution of (1.1). The following can be easily demonstrated:

$$||s(\jmath) - z_w(\jmath)|| = ||z(\jmath) - z_w(\jmath)|| \leqslant \epsilon.$$

It means that $K_\jmath(0) \subseteq K_\jmath(\eta)$. Because $K_\jmath(0)$ is dense belongs to H ((1.2) system is approximate controllable with the help of assumption (5)), $K_\jmath(\eta)$ is dense in H as well. As a result, (1.1) system is approximate controllable. □

5. Approximate controllability outcomes without apply of FPT

We must first establish the following assumption before moving on to the primary topic of this section:

Assumption 6. $\overline{R(B)}$ is a superset of $R(\aleph)$.

Theorem 5.1. *If assumption (2)-(6) are satisfied then (1.1) is approximately controllable provided (1.2) is approximately controllable.*

Proof. Assume that mild solution corresponding to (1.2) system is denoted by $z(\cdot)$, then

$$\begin{aligned} z(\varphi) &= P(\varphi)z_0 + Q(\varphi)z_1 + \rho Bu(\varphi) + \sum_{0<\varphi_\alpha<\varphi} P(\varphi-\varphi_\alpha)\chi_\alpha(z(\varphi_\alpha)) \\ &\quad + \sum_{0<\varphi_\alpha<\varphi} Q(\varphi-\varphi_\alpha)\widetilde{\chi}_\alpha(z(\varphi_\alpha)),\ \varphi \in (0,\jmath]. \end{aligned}$$

Clearly, $\aleph(z)$ belongs to $\overline{R(B)}$ (using assumption (6)). Hence, if $\epsilon > 0$, there exists an element of Y, namely $w(\cdot)$ with

$$||\aleph(z) - Bw||_\Theta \leqslant \epsilon.$$

Let $\xi(\varphi)$ stands for the mild solution according to $(u-w)$ of (1.1). In addition,

$$\begin{aligned} z(\varphi) - \xi(\varphi) = &\textstyle\int_0^{\varphi} Q(\varphi-\sigma)Bw(\sigma)d\sigma - \int_0^{\varphi} Q(\varphi-\sigma)[\aleph\xi](\sigma)d\sigma \\ &+ \sum_{0<\varphi_\alpha<\varphi} P(\varphi-\varphi_\alpha)\chi_\alpha(z(\varphi_\alpha)) - \sum_{0<\varphi_\alpha<\varphi} P(\varphi-\varphi_\alpha)\chi_\alpha(\xi(\varphi_\alpha)) \\ &+ \sum_{0<\varphi_\alpha<\varphi} Q(\varphi-\varphi_\alpha)\widetilde{\chi}_\alpha(z(\varphi_\alpha)) - \sum_{0<\varphi_\alpha<\varphi} Q(\varphi-\varphi_\alpha)\widetilde{\chi}_\alpha(\xi(\varphi_\alpha)) \\ &= \textstyle\int_0^{\varphi} Q(\varphi-\sigma)[Bw - \aleph z](\sigma)d\sigma + \int_0^{\varphi} Q(\varphi-\sigma)[\aleph z - \aleph\xi](\sigma)d\sigma \\ &+ \sum_{0<\varphi_\alpha<\varphi} P(\varphi-\varphi_\alpha)\chi_\alpha(z(\varphi_\alpha)) - \sum_{0<\varphi_\alpha<\varphi} P(\varphi-\varphi_\alpha)\chi_\alpha(\xi(\varphi_\alpha)) \\ &+ \sum_{0<\varphi_\alpha<\varphi} Q(\varphi-\varphi_\alpha)\widetilde{\chi}_\alpha(z(\varphi_\alpha)) - \sum_{0<\varphi_\alpha<\varphi} Q(\varphi-\varphi_\alpha)\widetilde{\chi}_\alpha(\xi(\varphi_\alpha)). \end{aligned}$$

Applying norm, we get

$$\begin{aligned}||z(\varphi)-\xi(\varphi)|| \leqslant\ & M\int_0^{\varphi}||Bw(\sigma)-[\aleph z](\sigma)||_H d\sigma + M\int_0^{\varphi}||[\aleph z](\sigma)\\ &-[\aleph\xi](\sigma)||d\sigma\\ &+M\sum_{0<\varphi_\alpha<\varphi}||\chi_\alpha(z(\varphi_\alpha))-\chi_\alpha(\xi(\varphi_\alpha))||\\ &+M\sum_{0<\varphi_\alpha<\varphi}||\widetilde{\chi}_\alpha(z(\varphi_\alpha))-\widetilde{\chi}_\alpha(\xi(\varphi_\alpha))||\\ &\leqslant M\sqrt{\jmath}||Bw-\aleph z||_\Theta + Ml\int_0^{\varphi}||z(\sigma)-\xi(\sigma)||d\sigma\\ &+Mq_\alpha\beta||z-\xi|| + Me_\alpha\beta||z-\xi||\\ &\leqslant M\sqrt{\jmath}\epsilon + Ml\int_0^{\jmath}||z(\sigma)-\xi(\sigma)||d\sigma + Mq_\alpha\beta||z-\xi||\\ &+Me_\alpha\beta||z-\xi||.\end{aligned}$$

We can do $||z(\jmath)-\xi(\jmath)||_H$ arbitrarily small by choosing an appropriate control w and with the help of Gronwall's inequality. As a result, the (1.1) solution set is dense in (1.2), which is dense in H. □

6. Example

Consider the semilinear impulsive functional heat control system as follows:

$$\begin{cases}\frac{\partial^2 z(\varphi,y)}{\partial\varphi^2}=\frac{\partial^2 z(\varphi,y)}{\partial y^2}+\mu(\varphi,y)+F(\varphi,z(\varphi,y)); \quad 0\leqslant\varphi\leqslant\jmath,\\ \varphi\neq\varphi_\alpha, \alpha=1,2,\ldots,\beta;\\ z(\varphi,0)=z(\varphi,\pi)=0, \quad t>0;\\ z(0,\varphi)=z_0(\varphi), \quad 0\leqslant\varphi\leqslant\pi, \quad \frac{\partial}{\partial\varphi}z(0,\varphi)=z_1(\varphi),\\ \triangle z(\varphi_\alpha,y)=\chi_\alpha(z(\varphi_\alpha,y)),\ \alpha=1,2,..,\beta,\\ \triangle z'(\varphi_\alpha,y)=\widetilde{\chi}_\alpha(z(\varphi_\alpha,y)),\ \alpha=1,2,..,\beta.\end{cases} \tag{6.1}$$

To transform the aforementioned system (6.1) into its abstract version (1.1), let $U=L_2[0,\pi]$. In addition, we introduce the operator $A:D(A)\subset U\to U$ as $Ay=y''$, $y\in D(A)$, where

$$D(A)=\{y \text{ in } U:\ y,\ y' \text{ are absolutely continuous}, \\ y'' \text{ in } U,\ y(0)=y(\pi)=0\}.$$

Choose $h_k(x)=(2/\pi)^{1/2}sin(kr)$, $x\in[0,\pi]$, k belongs to $\mathbb{N}$, so $\{h_k(x)\}$ denote an orthonormal basis of U,A has eigenvalue $\lambda_k=-k^2$ k belongs to $\mathbb{N}$ and the eigenfunction defined by h_k. Therefore, the spectral representation of A is presented by:

$$Az=\sum_{k=1}^{\infty}-k^2\langle y,h_k\rangle h_k,\ \ y\in D(A).$$

Now, we define the $P(\varphi)$ as follows:

$$P(\varphi)z=\sum_{k=1}^{\infty}\cos(kt)\langle y,h_k\rangle h_k, \quad \varphi\in\mathbb{R},$$

along with sine function

$$Q(\varphi)z=\sum_{k=1}^{\infty}\frac{\sin(kt)}{k}\langle y,h_k\rangle h_k, \quad \varphi\in\mathbb{R}.$$

Undoubtedly, $\|P(\varphi)\|\leqslant 1, \forall\ \varphi\in\mathbb{R}$. Therefore, $P(\cdot)$ is uniformly bounded on $\mathbb{R}$.

Define by

$$\widehat{U}=\left\{u|u=\sum_{k=2}^{\infty}u_k h_k,\ with\ \sum_{k=2}^{\infty}u_k^2<\infty\right\}.$$

The norm is represented by $\widehat{U}$, which is an infinite dimensional space.

$$||u||_{\widehat{U}}=\left(\sum_{k=2}^{\infty}u_k^2\right)^{\frac{1}{2}}$$

Determine the linear continuous operator B maps from $\widehat{U}\to U$ by

$$Bu=2u_2e_1+\sum_{k=2}^{\infty}u_k e_k, \quad u=\sum_{k=2}^{\infty}u_k e_k\in\widehat{U}.$$

Choose $z(\varphi)=z(\varphi,\cdot)$, that is, $z(\varphi)(\jmath)=z(\varphi,\jmath)$, φ belongs to J, $\jmath$ belongs to $[0,\pi]$ and $u(\varphi)=\mu(\varphi,\cdot)$, which is mapping from $J\times[0,\pi]$ into $[0,\pi]$ is continuous. Now, we define the function η from $J\times U$ into U as

$$F(\varphi,y)(\varphi)=\eta(\varphi,y(\varphi)), \quad y\in U, \quad \varphi\in[0,\pi]$$

and

$$Bu(\varphi)(\jmath)=\mu(\varphi,y),\ \ \varphi\in J,\ \ 0\leqslant y\leqslant\pi.$$

Furthermore, we provide some suitable demands based on the facts described from above order to show the conclusions of Theorem 4.1 and deduce that (6.1) is approximately controllable.

7. Conclusion

The approximate controllability of second-order semilinear differential systems with impulses was the subject of our paper. Two sets of essential conditions have been developed. Combining the ideas of the sine and cosine families, as well as the compactness of the cosine function, with fixed point approach yielded the first set of findings. The primary point was proven in the second set, that avoided the apply of the fixed point theorem and the compactness of the cosine function. The existence and uniqueness of mild solutions are also demonstrated. We've will provide an application of theoretical outcomes that have been verified.

Future research will discuss on the controllability of the second-order impulsive neutral stochastic control system the usage of the fixed point methods. With appropriate adjustments, the results of this article can be generalized to fractional-order systems.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

Almarri, B., Ali, A.H., Lopes, A.M., Bazighifan, O., 2022a. Nonlinear differential equations with distributed delay: some new oscillatory solutions. Mathematics 10 (6), 995.
Almarri, B., Janaki, S., Ganesan, V., Ali, A.H., Nonlaopon, K., Bazighifan, O., 2022b. Novel oscillation theorems and symmetric properties of nonlinear delay differential equations of fourth-order with a middle term. Symmetry 14 (3), 585.
Arora, U., Sukavanam, N., 2016. Approximate controllability of impulsive semilinear stochastic system with delay in state. Stochastic Anal. Appl. 34 (6), 1111–1123.
Bazighifan, O. et al., 2022. New oscillation solutions of impulsive conformable partial differential equations. AIMS Math. 7 (9), 16328–16348. MR4468417.
Chadha, A., Bora, S.N., 2018. Approximate controllability of impulsive neutral stochastic differential equations driven by Poisson jumps. J. Dyn. Control Syst. 24 (1), 101–128.
Chen, L., Li, G., 2010. Approximate controllability of impulsive differential equations with nonlocal conditions. Int. J. Nonlin. Sci. 10 (4), 438–446.
Curtain, R.F., Zwart, H.J., 1995. An Introduction to Infinite Dimensional Linear Systems Theory. Springer Verlag, New York.
Fu, X., Mei, K., 2009. Approximate controllability of semilinear partial functional differential systems. J. Dyn. Control Syst. 15 (3), 425–443.
Gou, H., Li, Y., 2021. A study on controllability of impulsive fractional evolution equations via resolvent operators. Boundary Value Probl. 2021 (25), 1–22.

Jeet, K., Sukavanam, N., 2020. Approximate controllability of nonlocal and impulsive neutral integro-differential equations using the resolvent operator theory and an approximating technique. Appl. Math. Comput. 364, 124690. 1–15.
Kalman, R.E., 1963. Controllability of linear systems, Contributions to. Diff. Eqs. 1, 190–213.
Kumar, S., Sukavanam, N., 2012. Approximate controllability of fractional order semilinear systems with bounded delay. J. Diff. Eqs. 252 (11), 6163–6174.
Li, X., Wu, J., 2016. Stability of nonlinear differential systems with state-dependent delayed impulses. Automatica J. IFAC 64, 63–69.
Li, X., Wu, J., 2018. Sufficient stability conditions of nonlinear differential systems under impulsive control with state-dependent delay. IEEE Trans. Autom. Control 63 (1), 306–311.
Li, X., Bohner, M., Wang, C.K., 2015. Impulsive differential equations: periodic solutions and applications. Automatica J. IFAC 52, 173–178.
Mahmudov, N.I., Vijayakumar, V., Murugesu, R., 2016. Approximate controllability of second-order evolution differential inclusions in Hilbert spaces. Mediterr. J. Math. 13 (5), 3433–3454.
Mahmudov, N.I., Udhayakumar, R., Vijayakumar, V., 2020. On the approximate controllability of second-order evolution hemivariational inequalities. RM 75 (4), 1–19.
Mohan Raja, M., Vijayakumar, V., Udhayakumar, R., Zhou, Y., 2020. A new approach on the approximate controllability of fractional differential evolution equations of order $1 < r < 2$ in Hilbert spaces. Chaos, Solitons Fractals 141. 1–11. 110310.
Mohan Raja, M., Vijayakumar, V., Udhayakumar, R., 2020. Results on the existence and controllability of fractional integro-differential system of order $1 < r < 2$ via measure of noncompactness. Chaos, Solitons Fractals 139, 110299. 1-10. 110299.
Mohan Raja, M. et al., 2022. On the approximate controllability results for fractional integrodifferential systems of order $1 < r < 2$ with sectorial operators. J. Comput. Appl. Math. 415. Paper No. 114492, 12 pp. MR4441360.
Naito, K., 1987. Controllability of semilinear control systems dominated by the linear part. SIAM J. Control Optim. 25 (3), 715–722.
Pazy, A., 1983. Semigroup of Linear Operators and Application to Partial Differential Equations. Spinger Verlag, New Yok.
Sakthivel, R., Ren, Y., Mahmudov, N.I., 2011. On the approximate controllability of semilinear fractional differential systems. Comput. Math. Appl. 62 (3), 1451–1459.
Sakthivel, R., Mahmudov, N.I., Nieto, J.J., 2012. Controllability for a class of fractional-order neutral evolution control systems. Appl. Math. Comput. 218 (20), 10334–10340.
Shukla, A., Sukavanam, N., Pandey, D.N., 2014. Controllability of semilinear stochastic system with multiple delays in control. IFAC Proc. Vol. 47 (1), 306–312.
Shukla, A., Sukavanam, N., Pandey, D.N., 2015, July. Approximate controllability of semilinear fractional control systems of order $\alpha \in (1, 2]$. In: 2015 Proceedings of the Conference on Control and its Applications. Society for Industrial and Applied Mathematics. pp. 175–180.
Shukla, A., Sukavanam, N., Pandey, D.N., 2016. Complete controllability of semilinear stochastic systems with delay in both state and control. Math. Rep. 18 (2), 247–259.
Sivasankaran, S., Mallika Arjunan, M., Vijayakumar, V., 2011. Existence of global solutions for second order impulsive abstract partial differential equations. Nonlinear Anal.: Theory Methods Appl. 74 (17), 6747–6757.
Sukavanam, N., Tafesse, S., 2011. Approximate controllability of a delayed semilinear control system with growing nonlinear term. Nonlinear Analysis 74 (18), 6868–6875.
Tomar, N.K., Sukavanam, N., 2011. Approximate controllability of non-densely defined semilinear delayed control systems. Nonlinear Stud. 18 (2), 229–234.
Travis, C.C., Webb, G.F., 1978. Cosine families and abstract nonlinear second order differential equations. Acta Math. Acad. Scie. Hungaricae 32 (1–2), 75–96.
Vijayakumar, V., Murugesu, R., 2019. Controllability for a class of second order evolution differential inclusions without compactness. Appl. Anal. 98 (7), 1367–1385.
Vijayakumar, V., Murugesu, R., Poongodi, R., Dhanalakshmi, S., 2017. Controllability of second-order impulsive nonlocal Cauchy problem via measure of noncompactness. Mediterr. J. Math. 14 (1), 1–21.
Vijayakumar, V., Murugesu, R., Tamil Selvan, M., 2019. Controllability for a class of second order functional evolution differential equations without uniqueness. IMA J. Math. Control Infor. 36 (1), 225–246.
Vijayakumar, V., Udhayakumar, R., Dineshkumar, C., 2021a. Approximate controllability of second order nonlocal neutral differential evolution inclusions. IMA J. Math. Control Infor. 38 (1), 192–210.
Vijayakumar, V., Panda, S.K., Nisar, K.S., Baskonus, H.M., 2021b. Results on approximate controllability results for second-order Sobolev-type impulsive neutral differential evolution inclusions with infinite delay. Num. Methods Partial Diff. Eqs. 37 (2), 1200–1221.
Vijayakumar, V., Nisar, K.S., Chalishajar, D., Shukla, A., Malik, M., Alsaadi, A., Aldosary, S.F., 2022. A note on approximate controllability of fractional semilinear integrodifferential control systems via resolvent operators. Fractal Fract. 6 (2), 73.
You, Z., Feckan, M., Wang, J., 2020. Relative controllability of fractional delay differential equations via delayed perturbation of Mittag-Leffler functions. J. Comput. Appl. Math. 378. 1-16. 112939.
Zhou, Y., Jiao, F., 2010. Existence of mild solutions for fractional neutral evolution equations. Comput. Math. Appl. 59 (3), 1063–1077.
Zhou, Y., Suganya, S., Arjunan, M.M., 2018. Existence and controllability for impulsive evolution inclusions without compactness. J. Dyn. Control Syst. 24 (2), 297–311.