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Original article

Explicit and traveling wave solutions of the non-linear couple Drinfeld-Sokolov-Wilson dynamical system arising in shallow water waves

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ABSTRACT

We efficaciously extended the ansatz in the Riccati equation rational expansion technique to a new broad form and is used to attain some new exact wave solutions for the couple non-linear Drinfeld-Sokolov-Wilson (DSW) system. This system arises in water waves, quantum mechanics, fluid mechanics, biology, chemistry etc. Several of the attained solutions which is in the shape of rational solitary wave solutions and periodic wave solutions are entirely new and could be vital for researchers to appreciate. Certain selected solutions are graphically verified by precise values allocated to constant parameters. This technique with the current ansatz can be implemented with a broader range of applicability to handle various other forms of non-linear evolution systems (NLES).

1. Introduction

Of late, it is well observing that several attentions have been focused by researchers on the proliferation of finding the exact solitary wave solutions for coupled non-linear evolution systems (NLES), which play a crucial part in characterizing non-linear problems in mathematical physics and sciences. The coupled non-linear systems have many applications and are often used to exemplify many of the problems in physical sciences, for instance, material science, chemistry, biology, ocean waves, fluid dynamics, optical fiber, plasma physics etc. (Li et al., 2019; Li and Viglialoro, 2018; Abdou and Soliman, 2005; Abdou, 2007a; Abdou, 2007b; Seadawy and Alamri, 2018; Aljahdaly et al., 2019; Arshad et al., 2019; Lu et al., 2019b; Yaro et al., 2019). Attaining numerical and exact solutions for coupled NLES exhibits indispensable task in the investigation of tangible occurrence and has become a vigorous and important task. However, attaining the exact solution of problems involving coupled NLES can aid researchers understand these occurrences well as compare to the numerical solutions (Apeanti et al., 2019a; Seadawy et al., 2018; Nasreen et al., 2019; Ahmed et al., 2019; Cheemaa et al., 2019; Lu et al., 2019a; Apeanti et al., 2019b; Iqbal et al., 2019). Hence, the research of the exact wave solutions of coupled NLES exhibits a significant role in the investigation of these tangible phenomena and numerous controlling techniques already proposed to attain the exact solutions, for example, extended tanh technique, Painlevé technique, extended mapping technique, homogeneous balance technique, homotopy perturbation technique, Exp-function technique, finite difference technique, F-expansion technique, Adomian decomposition technique, various Tanh technique, auxiliary equation technique, rational expansion technique and so on (Agarwal et al., 2020; Agarwal et al., 2017; Wen, 2010; Zenga and Wang, 2009; Selima et al., 2016; Seadawy and El-Rashidy, 2013; Yue et al., 2016; Arshad et al., 2016; Wen, 2011; Wang et al., 2012; Lin et al., 2011; Qin, 2008; Liu and Li, 2016). Several authors (Abdullah and Wang, 2017; Arshad et al., 2017) have investigated the structure and the exact solitary wave solutions for the non-linear Drinfeld–Sokolov–Wilson (DSW) system.

In the meantime the aforementioned method have numerous advantages over the other methods, the easiness and the accessibility of softwares which can smooths the tedious algebraic calculations motivated us to implement the RERE method to ascer-

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tain new exact solutions for the DSW system via new ansatz. For this current paper, the ansatz equation is extended to a new general form in the Riccati equation rational expansion (RERE) technique (Wang et al., 2005) to attain successions of solutions for the non-linear coupled DSW system. Therefore, more or less new and versatile exact traveling wave solutions have been constructed.

The remaining work arrangements are as follows: The Section 2 outlined the RERE technique. In Section 3, this technique is implemented to obtain a new series of solutions for DSW systems. In Section 4, some of the solutions attained are presented graphically to describe the evolutionary shape of the solutions. Section 5 represents the results and discussions. Section 6 points out the conclusion.

2. Summary of the technique

In this aspect, the analytical technique of how the Riccati equation rational technique is used to attain the traveling wave solution of NLES. Most importantly, we check the technique when implementing the NLES defined problems, which has two non-dependent variables, a space (x) dimension and time (t) dimension. Given the subsequent non-linear evolution equation of which we attain traveling wave solutions:

$$P_j(m_j, m_{jt}, m_{jx}, m_{jtt}, m_{jxt}, m_{jxx}, \dots) = 0, \tag{1}$$

where $m(\xi) = m(x,t)$ is an unidentified function, P_j is a polynomial of $m(x,t)$ and its partial derivatives as well as the maximum order derivative and non-linear terms. Consequently, we introduce the significant steps for this technique: Step 1. In joining the cases ξ, ϵ and ζ into a variable $\xi = \beta x - \delta t$, we suggest

$$m_j(x,t) = M_j(\xi), \qquad \xi = \beta x - \delta t, \tag{2}$$

where β and δ are unknowns which will be attained later. Eq. (2) allows us to simplify Eq. (1) to the succeeding ordinary differential equation (ODE):

$$Q_j(M_j, M_j', M_j'', \dots) = 0, \tag{3}$$

where Q_j is the polynomial in $M_j(\xi)$ and its derivatives, whereas $M_j' = \frac{dM}{d\xi}, M_j'' = \frac{d^2M}{d\xi^2}$, etc.

Step 2. We can now suggest that equation (3) has a well founded solution

$$M_j(\xi) = \sigma_0 + \sum_{j=1}^{n} \frac{\sigma_j(\psi(\xi))^k + \lambda_j(\psi(\xi))^{j-1}\psi'(\xi)}{(\alpha\psi(\xi)+1)^j} + \sum_{j=2}^{n}\mu_j(\psi(\xi))^{j-2}\psi'(\xi) + \sum_{j=1}^{n}\gamma_j\left(\frac{\psi'(\xi)}{\psi(\xi)}\right)^j, \tag{4}$$

where $\psi = \psi(\xi)$ fulfill the equation

$$\psi'(\xi) = k_1 + k_2\psi(\xi)^2, \tag{5}$$

with $\alpha, k_1, k_2, \sigma_0, \sigma_j, \lambda_j, \mu_j$ and $\gamma_j (j = 1, 2, \dots, n)$ are unknowns which will be obtained later.

Step 3. The basic systems of a succession of essential solutions, like polynomial exponential, Jacobi, rational solitary waves, and triangular periodic are the diverse special effects of varying waveforms in various non-linear systems, hence, dispersion, dissolution and non-linearity (either in combination or in used) can be balanced. We apply $D[M_j(\xi)] = n_j$ to signify the degree for $M_j(\xi)$ and that of the other terms as:

$$D[M_j^{(\eta)}] = n_j + \eta, \qquad D[M_j^{\eta}(M_l^{(\kappa)})^{\nu}] = n_j\kappa + (\eta + n_l)\nu. \tag{6}$$

Step 4. Now by putting (4) in (3) together with (5) and then by setting all coefficients of $\psi^j(\xi), (j = 1, 2, \dots)$ of the numerator of the obtained system's to zero, the system of non-linear algebraic equations about $\beta, \alpha, \sigma_0, \sigma_j, \lambda_j, \mu_j$ and $\gamma_j (j = 1, 2, \dots, n)$ can be determined.

Step 5. Applying software (ie. *Mathematica* or *Maple*) to find the non-linear algebraic expressions in Step 4, then we obtain the clear solution for the terms $\beta, \alpha, \sigma_0, \sigma_j, \lambda_j, \mu_j$ and $\gamma_j (j = 1, 2, \dots, n)$.

Stage 6. Lastly, (5) has the subsequent validated solutions:

I. For $k_1 = \frac{1}{2}$ and $k_2 = -\frac{1}{2}$,

$$m_j = \sigma_0 + \sum_{j=1}^{n} \frac{\sigma_j(\tanh(\xi) \pm i\,\text{sech}(\xi))^j + \lambda_j(\tanh(\xi) \pm i\,\text{sech}(\xi))^{j-1}(\text{sech}^2(\xi) \mp i\,\text{sech}(\xi)\tanh(\xi))}{(\alpha(\tanh(\xi) \pm i\,\text{sech}(\xi)) + 1)^j} + \sum_{j=2}^{n}\mu_j(\tanh(\xi) \pm i\,\text{sech}(\xi))^{j-2}(\text{sech}^2(\xi) \mp i\,\text{sech}(\xi)\tanh(\xi)) + \sum_{j=1}^{n}\gamma_j\left(\frac{\text{sech}^2(\xi) \mp i\,\text{sech}(\xi)\tanh(\xi)}{\tanh(\xi) \pm i\,\text{sech}(\xi)}\right)^j, \tag{7}$$

$$m_j = \sigma_0 + \sum_{j=1}^{n} \frac{\sigma_j(\coth(\xi) \pm \text{csch}(\xi))^j - \lambda_j(\coth(\xi) \pm \text{csch}(\xi))^{j-1}(\text{csch}^2(\xi) \pm \text{csch}(\xi)\coth(\xi))}{(\alpha(\coth(\xi) \pm i\,\text{csch}(\xi)) + 1)^j} - \sum_{j=2}^{n}\mu_j(\coth(\xi) \pm \text{csch}(\xi))^{j-2}(\text{csch}^2(\xi) \pm \text{csch}(\xi)(\coth(\xi))) + \sum_{j=1}^{n}\gamma_j\left(\frac{-(\text{csch}^2(\xi) \pm \text{csch}(\xi)\coth(\xi))}{\coth(\xi) \pm \text{csch}(\xi)}\right)^j, \tag{8}$$

II. For $k_1 = k_2 = \pm\frac{1}{2}$,

$$m_j = \sigma_0 + \sum_{j=1}^{n} \frac{\sigma_j(\sec(\xi) \pm \tan(\xi))^j + \lambda_j(\sec(\xi) \pm \tan(\xi))^{j-1}(\sec(\xi)\tan(\xi) \pm \sec^2(\xi))}{(\alpha(\sec(\xi) \pm \tan(\xi)) + 1)^j} + \sum_{j=2}^{n}\mu_j(\sec(\xi) \pm \tan(\xi))^{j-2}(\sec(\xi)\tan(\xi) \pm \sec^2(\xi)) + \sum_{j=1}^{n}\gamma_j\left(\frac{\sec(\xi)\tan(\xi) \pm \sec^2(\xi)}{\sec(\xi) \pm \tan(\xi)}\right)^j, \tag{9}$$

$$m_j = \sigma_0 + \sum_{j=1}^{n} \frac{\sigma_j(\csc(\xi) \pm \cot(\xi))^j - \lambda_j(\csc(\xi) \pm \cot(\xi))^{j-1}(\csc(\xi)\cot(\xi) \pm \csc^2(\xi))}{(\alpha(\csc(\xi) \pm \cot(\xi)) + 1)^j} - \sum_{j=2}^{n}\mu_j(\csc(\xi) \pm \cot(\xi))^{j-2}(\csc(\xi)\cot(\xi) \pm \csc^2(\xi)) + \sum_{j=1}^{n}\gamma_j\left(\frac{-(\csc(\xi)\cot(\xi) \pm \csc^2(\xi))}{\csc(\xi) \pm \cot(\xi)}\right)^j, \tag{10}$$

III. For $k_1 = 1$ and $k_2 = -1$,

$$m_j = \sigma_0 + \sum_{j=1}^{n} \frac{\sigma_j\tanh^j(\xi) + \lambda_j\tanh^{j-1}(\xi)\text{sech}^2(\xi)}{(\alpha\tanh(\xi) + 1)^j} + \sum_{j=2}^{n}\mu_j(\tanh(\xi))^{j-2}\text{sech}^2(\xi) + \sum_{j=1}^{n}\gamma_j\left(\frac{\text{sech}^2(\xi)}{\tanh(\xi)}\right)^j, \tag{11}$$

$$m_j = \sigma_0 + \sum_{j=1}^{n} \frac{\sigma_j\coth^j(\xi) - \lambda_j\coth^{j-1}(\xi)\text{csch}^2(\xi)}{(\alpha\coth(\xi) + 1)^j} - \sum_{j=2}^{n}\mu_j(\coth(\xi))^{j-2}\text{csch}^2(\xi) + \sum_{j=1}^{n}\gamma_j\left(\frac{-\text{csch}^2(\xi)}{\coth(\xi)}\right)^j, \tag{12}$$

IV. For $k_1 = k_2 = 1$,

$$m_j = \sigma_0 + \sum_{j=1}^{n} \frac{\sigma_j \tan^j(\xi) + \lambda_j \tan^{j-1}(\xi)\sec^2(\xi)}{(\alpha\tan(\xi)+1)^j} + \sum_{j=2}^{n} \mu_j (\tan(\xi))^{j-2}\sec^2(\xi) + \sum_{j=1}^{n} \gamma_j \left(\frac{\sec^2(\xi)}{\tan(\xi)}\right)^j, \tag{13}$$

V. For $k_1 = k_2 = -1$,

$$m_j = \sigma_0 + \sum_{j=1}^{n} \frac{\sigma_j \cot^j(\xi) - \lambda_j \cot^{j-1}(\xi)\csc^2(\xi)}{(\alpha\cot(\xi)+1)^j} - \sum_{j=2}^{n} \mu_j (\cot(\xi))^{j-2}\csc^2(\xi) + \sum_{j=1}^{n} \gamma_j \left(\frac{-\csc^2(\xi)}{\cot(\xi)}\right)^j, \tag{14}$$

VI. For $k_1 = 0$ and $k_2 \neq 0$,

$$m_j = \sigma_0 + \sum_{j=1}^{n} \frac{\sigma_j (k_2\xi + f_0)^j - \lambda_j k_2 (k_2\xi + f_0)^{j-1}}{\left(\alpha(k_2\xi + f_0) + (k_2\xi + f_0)^2\right)^j} - \sum_{j=2}^{n} \mu_j k_2 (k_2(\xi) + f_0)^{j-2} + \sum_{j=1}^{n} \gamma_j \left(\frac{-k_2}{k_2(\xi) + f_0}\right)^j, \tag{15}$$

where $\xi = \beta x - \delta t, \mathrm{i} = \sqrt{-1}$ and f_0 is constant.

Remark. Notice that the ansatz suggested in this paper is new and different from the one suggested by Wang et al. (2005).

3. Implementation

3.1. The non-linear Drinfeld–Sokolov–Wilson (DSW) system

Here, we take into consideration the couple non-linear DSW system given in the form

$$\begin{aligned} &m_t + (r^2)_x = 0, \\ &r_t - r_{xxx} + 3rm_x + 3mr_x = 0. \end{aligned} \tag{16}$$

The non-linear coupled DSW system have been studied by several researchers (Arshad et al., 2017). By implementing (2) on (16) we attain

$$\begin{aligned} &-\delta M' + \beta (R^2)' = 0, \\ &-\delta R' - \beta^3 R''' + 3\beta R M' + 3\beta M R' = 0. \end{aligned} \tag{17}$$

Now by ignoring the integral constant after integrating the first equation in (17), we get

$$\delta M = \beta R^2. \tag{18}$$

By putting Eq. (18) into the second equation of (17), we get

$$\delta^2 R + \beta^3 \delta R'' - 3\beta^2 R^3 = 0. \tag{19}$$

Now from (19) and by implementing the homogeneous balance principle gives $n = 1$. Therefore,(19) tends to

$$R(\xi) = \sigma_0 + \frac{\sigma_1 \psi(\xi) + \lambda_1 \psi'(\xi)}{\alpha\psi(\xi) + 1} + \gamma_1 \frac{\psi'(\xi)}{\psi(\xi)}, \tag{20}$$

where $\psi(\xi)$ satisfies (5).

Substituting (20) alongside (5), results in a polynomial of $\psi^j(\xi)(j = 0, 1, \ldots)$. By setting the coefficients of the terms $\psi^j(\xi)$ to zero, gives a set of algebraic equation systems. Solving this set of systems for $\sigma_0, \sigma_1, \lambda_1, \gamma_1, \alpha$ and β with the aid of software gives the subsequent values.

Family 1: $k_1 = \frac{1}{2}$ and $k_2 = -\frac{1}{2}$,

$$\sigma_0 = 0, \quad \sigma_1 = 0, \quad \lambda_1 = 0, \quad \gamma_1 = \pm \mathrm{i}\sqrt{\frac{2}{3}}\delta^{2/3}, \quad \alpha = 0, \quad \beta = -\sqrt[3]{\delta},$$

$$\sigma_0 = \pm\frac{\delta^{2/3}}{4\sqrt{3}}, \quad \sigma_1 = \pm\frac{\delta^{2/3}}{\sqrt{3}}, \quad \lambda_1 = 0, \quad \gamma_1 = 0, \quad \alpha = 3, \quad \beta = 2\sqrt[3]{\delta},$$

$$\sigma_0 = \pm\frac{(-1)^{2/3}\delta^{2/3}}{2\sqrt[3]{2}\sqrt{3}}, \quad \sigma_1 = \pm\frac{(-1)^{2/3}\delta^{2/3}}{\sqrt[3]{2}\sqrt{3}}, \quad \lambda_1 = 0, \quad \gamma_1 = 0, \quad \alpha = -3, \quad \beta = -\sqrt[3]{-2}\sqrt[3]{\delta},$$

$$\sigma_0 = 0, \quad \sigma_1 = \pm\frac{(-1)^{2/3}\delta^{2/3}}{\sqrt[3]{2}\sqrt{3}}, \quad \lambda_1 = 0, \quad \gamma_1 = \pm\frac{(-2)^{2/3}\delta^{2/3}}{\sqrt{3}}, \quad \alpha = 0, \quad \beta = -\sqrt[3]{-2}\sqrt[3]{\delta},$$

$$\sigma_0 = \pm\frac{(-1)^{2/3}\delta^{2/3}}{4\sqrt{3}}, \quad \sigma_1 = \pm\frac{(-1)^{2/3}\delta^{2/3}}{\sqrt{3}}, \quad \lambda_1 = 0, \quad \gamma_1 = 0, \quad \alpha = 3, \quad \beta = -2\sqrt[3]{-1}\sqrt[3]{\delta},$$

$$\sigma_0 = 0, \quad \sigma_1 = \pm\frac{(-1)^{2/3}\sqrt[3]{2}\delta^{2/3}}{\sqrt{3}}, \quad \lambda_1 = 0, \quad \gamma_1 = \pm\frac{(-1)^{2/3}\sqrt[3]{2}\delta^{2/3}}{\sqrt{3}}, \quad \alpha = 0, \quad \beta = -\sqrt[3]{-\tfrac{1}{2}}\sqrt[3]{\delta},$$

$$\sigma_0 = 0, \quad \sigma_1 = \pm\frac{\sqrt[3]{2}\delta^{2/3}}{\sqrt{3}}, \quad \lambda_1 = 0, \quad \gamma_1 = \pm\frac{\sqrt[3]{2}\delta^{2/3}}{\sqrt{3}}, \quad \alpha = 0, \quad \beta = \frac{\sqrt[3]{\delta}}{\sqrt[3]{2}},$$

$$\sigma_0 = \pm\frac{\delta^{2/3}}{2\sqrt[3]{2}\sqrt{3}}, \quad \sigma_1 = \pm\frac{\delta^{2/3}}{\sqrt[3]{2}\sqrt{3}}, \quad \lambda_1 = 0, \quad \gamma_1 = 0, \quad \alpha = -3, \quad \beta = \sqrt[3]{2}\sqrt[3]{\delta},$$

$$\sigma_0 = 0, \quad \sigma_1 = \pm\frac{\delta^{2/3}}{\sqrt[3]{2}\sqrt{3}}, \quad \lambda_1 = 0, \quad \gamma_1 = \pm\frac{2^{2/3}\delta^{2/3}}{\sqrt{3}}, \quad \alpha = 0, \quad \beta = \sqrt[3]{2}\sqrt[3]{\delta}.$$

Family 2: $k_1 = k_2 = \pm\frac{1}{2}$,

$$\sigma_0 = 0, \quad \sigma_1 = 0, \quad \lambda_1 = 0, \quad \gamma_1 = \pm\sqrt{\tfrac{2}{3}}\delta^{2/3}, \quad \alpha = 0, \quad \beta = \sqrt[3]{\delta},$$

$$\sigma_0 = 0, \quad \sigma_1 = \pm\frac{\sqrt[6]{-1}\delta^{2/3}}{\sqrt[3]{2}\sqrt{3}}, \quad \lambda_1 = 0, \quad \gamma_1 = 0, \quad \alpha = 0, \quad \beta = \sqrt[3]{-2}\sqrt[3]{\delta},$$

$$\sigma_0 = 0, \quad \sigma_1 = \pm\frac{\sqrt[6]{-1}\delta^{2/3}}{\sqrt[3]{2}\sqrt{3}}, \quad \lambda_1 = 0, \quad \gamma_1 = \pm\frac{\sqrt[6]{-1}2^{2/3}\delta^{2/3}}{\sqrt{3}}, \quad \alpha = 0, \quad \beta = \sqrt[3]{-2}\sqrt[3]{\delta},$$

$$\sigma_0 = 0, \quad \sigma_1 = \pm\frac{\mathrm{i}\sqrt[3]{2}\delta^{2/3}}{\sqrt{3}}, \quad \lambda_1 = 0, \quad \gamma_1 = \pm\frac{\mathrm{i}\sqrt[3]{2}\delta^{2/3}}{\sqrt{3}}, \quad \alpha = 0, \quad \beta = -\frac{\sqrt[3]{\delta}}{\sqrt[3]{2}},$$

$$\sigma_0 = 0, \quad \sigma_1 = \pm\frac{\mathrm{i}\delta^{2/3}}{\sqrt[3]{2}\sqrt{3}}, \quad \lambda_1 = 0, \quad \gamma_1 = \pm\frac{\mathrm{i}2^{2/3}\delta^{2/3}}{\sqrt{3}}, \quad \alpha = 0, \quad \beta = -\sqrt[3]{2}\sqrt[3]{\delta},$$

$$\sigma_0 = \pm\frac{\delta^{2/3}}{2\sqrt{3}\sqrt[3]{4-3\mathrm{i}}}, \quad \sigma_1 = \pm\frac{\sqrt[6]{4-3\mathrm{i}}\delta^{2/3}}{\sqrt{6}}, \quad \lambda_1 = 0, \quad \gamma_1 = 0, \quad \alpha = 3\mathrm{i}, \quad \beta = \sqrt[3]{4-3\mathrm{i}}\sqrt[3]{\delta},$$

$$\sigma_0 = \pm\frac{\delta^{2/3}}{2\sqrt{3}\sqrt[3]{4+3\mathrm{i}}}, \quad \sigma_1 = \pm\frac{\sqrt[6]{4+3i}\delta^{2/3}}{\sqrt{6}}, \quad \lambda_1 = 0, \quad \gamma_1 = 0, \quad \alpha = -3\mathrm{i}, \quad \beta = \sqrt[3]{4+3\mathrm{i}}\sqrt[3]{\delta}.$$

Family 3: $k_1 = 1$ and $k_2 = -1$,

$$\sigma_0 = 0, \quad \sigma_1 = \pm\frac{2\delta^{2/3}}{\sqrt{3}}, \quad \lambda_1 = 0, \quad \gamma_1 = \pm\frac{\delta^{2/3}}{\sqrt{3}}, \quad \alpha = 0, \quad \beta = \frac{\sqrt[3]{\delta}}{2},$$

$$\sigma_0 = \pm\frac{(-1)^{2/3}\delta^{2/3}}{2\sqrt[3]{2}\sqrt{3}}, \quad \sigma_1 = \pm\frac{(-2)^{2/3}\delta^{2/3}}{\sqrt{3}}, \quad \lambda_1 = 0, \quad \gamma_1 = 0, \quad \alpha = 3, \quad \beta = -\sqrt[3]{-2}\sqrt[3]{\delta},$$

$$\sigma_0 = 0, \quad \sigma_1 = \pm\frac{2(-1)^{2/3}\delta^{2/3}}{\sqrt{3}}, \quad \lambda_1 = 0, \quad \gamma_1 = \pm\frac{(-1)^{2/3}\delta^{2/3}}{\sqrt{3}}, \quad \alpha = 0, \quad \beta = -\tfrac{1}{2}\sqrt[3]{-1}\sqrt[3]{\delta},$$

$$\sigma_0=\pm\frac{(-1)^{2/3}\delta^{2/3}}{2^{2/3}\sqrt{3}},\ \ \sigma_1=\pm\frac{(-1)^{2/3}\sqrt[3]{2}\delta^{2/3}}{\sqrt{3}},\ \ \lambda_1=0,\ \ \gamma_1=0,\ \ \alpha=-3,\ \ \beta=-\sqrt[3]{-\tfrac{1}{2}}\sqrt[3]{\delta},$$

$$\sigma_0=\pm\frac{\delta^{2/3}}{2^{2/3}\sqrt{3}},\ \ \sigma_1=\pm\frac{\sqrt[3]{2}\delta^{2/3}}{\sqrt{3}},\ \ \lambda_1=0,\ \ \gamma_1=0,\ \ \alpha=-3,\ \ \beta=\frac{\sqrt[3]{\delta}}{\sqrt[3]{2}},$$

$$\sigma_0=0,\ \ \sigma_1=\pm\frac{\sqrt[3]{2}\delta^{2/3}}{\sqrt{3}},\ \ \lambda_1=0,\ \ \gamma_1=\pm\frac{\sqrt[3]{2}\delta^{2/3}}{\sqrt{3}},\ \ \alpha=0,\ \ \beta=\frac{\sqrt[3]{\delta}}{\sqrt[3]{2}},$$

$$\sigma_0=\pm\frac{\delta^{2/3}}{2\sqrt[3]{2}\sqrt{3}},\quad \sigma_1=\pm\frac{2^{2/3}\delta^{2/3}}{\sqrt{3}},\quad \lambda_1=0,\quad \gamma_1=0,\quad \alpha=3,\quad \beta=\sqrt[3]{2}\sqrt[3]{\delta}.$$

Family 4: $k_1=k_2=1$,

$$\sigma_0=0,\quad \sigma_1=\pm\frac{2i\delta^{2/3}}{\sqrt{3}},\quad \lambda_1=0,\quad \gamma_1=\pm\frac{i\delta^{2/3}}{\sqrt{3}},\quad \alpha=0,\quad \beta=-\frac{\sqrt[3]{\delta}}{2},$$

$$\sigma_0=0,\quad \sigma_1=\pm\frac{2\sqrt[6]{-1}\delta^{2/3}}{\sqrt{3}},\quad \lambda_1=0,\quad \gamma_1=\pm\frac{\sqrt[6]{-1}\delta^{2/3}}{\sqrt{3}},\quad \alpha=0,\quad \beta=\frac{1}{2}\sqrt[3]{-1}\sqrt[3]{\delta},$$

$$\sigma_0=0,\quad \sigma_1=\pm\frac{\sqrt[6]{-1}\sqrt[3]{2}\delta^{2/3}}{\sqrt{3}},\quad \lambda_1=0,\quad \gamma_1=0,\quad \alpha=0,\quad \beta=\sqrt[3]{-\tfrac{1}{2}}\sqrt[3]{\delta},$$

$$\sigma_0=\pm\frac{\delta^{2/3}}{\sqrt{3}\sqrt[3]{8-6i}},\quad \sigma_1=\pm\frac{\sqrt[6]{8-6i}\delta^{2/3}}{\sqrt{3}},\quad \lambda_1=0,\quad \gamma_1=0,\quad \alpha=3i,\quad \beta=\frac{\sqrt[3]{4-3i}\sqrt[3]{\delta}}{2^{2/3}},$$

$$\sigma_0=\pm\frac{(-1)^{2/3}\delta^{2/3}}{\sqrt{3}\sqrt[3]{8-6i}},\quad \sigma_1=\pm\frac{(3-i)\delta^{2/3}}{\sqrt[3]{-8+6i}\sqrt{3}},\quad \lambda_1=0,\quad \gamma_1=0,\quad \alpha=3i,\quad \beta=-\frac{\sqrt[3]{-4+3i}\sqrt[3]{\delta}}{2^{2/3}},$$

$$\sigma_0=\pm\frac{(-1)^{2/3}\delta^{2/3}}{\sqrt{3}\sqrt[3]{8+6i}},\quad \sigma_1=\pm\frac{\sqrt[3]{-3-i}\delta^{2/3}}{\sqrt{3}},\quad \lambda_1=0,\quad \gamma_1=0,\quad \alpha=-3i,\quad \beta=\frac{(-1)^{2/3}\sqrt[3]{4+3i}\sqrt[3]{\delta}}{2^{2/3}}.$$

Family 5: $k_1=k_2=-1$,

$$\sigma_0=0,\quad \sigma_1=\pm\frac{2i\delta^{2/3}}{\sqrt{3}},\quad \lambda_1=0,\quad \gamma_1=\pm\frac{i\delta^{2/3}}{\sqrt{3}},\quad \alpha=0,\quad \beta=-\frac{\sqrt[3]{\delta}}{2},$$

$$\sigma_0=0,\quad \sigma_1=\pm\frac{2\sqrt[6]{-1}\delta^{2/3}}{\sqrt{3}},\quad \lambda_1=0,\quad \gamma_1=\pm\frac{\sqrt[6]{-1}\delta^{2/3}}{\sqrt{3}},\quad \alpha=0,\quad \beta=\frac{1}{2}\sqrt[3]{-1}\sqrt[3]{\delta},$$

$$\sigma_0=0,\ \ \sigma_1=\pm\frac{\sqrt[6]{-1}\sqrt[3]{2}\delta^{2/3}}{\sqrt{3}},\ \ \lambda_1=0,\ \ \gamma_1=0,\ \ \alpha=0,\ \ \beta=\sqrt[3]{-\tfrac{1}{2}}\sqrt[3]{\delta},$$

$$\sigma_0=0,\ \ \sigma_1=\pm\frac{i\sqrt[3]{-2}\delta^{2/3}}{\sqrt{3}},\ \ \lambda_1=0,\ \ \gamma_1=\pm\frac{i\sqrt[3]{-2}\delta^{2/3}}{\sqrt{3}},\ \ \alpha=0,\ \ \beta=-\frac{(-1)^{2/3}\sqrt[3]{\delta}}{\sqrt[3]{2}},$$

$$\sigma_0=\pm\frac{\delta^{2/3}}{\sqrt{3}\sqrt[3]{8-6i}},\quad \sigma_1=\pm\frac{\sqrt[6]{8-6i}\delta^{2/3}}{\sqrt{3}},\quad \lambda_1=0,\quad \gamma_1=0,\quad \alpha=3i,\quad \beta=\frac{\sqrt[3]{4-3i}\sqrt[3]{\delta}}{2^{2/3}},$$

$$\sigma_0=\pm\frac{\delta^{2/3}}{\sqrt{3}\sqrt[3]{8+6i}},\quad \sigma_1=\pm\frac{\sqrt[6]{8+6i}\delta^{2/3}}{\sqrt{3}},\quad \lambda_1=0,\quad \gamma_1=0,\quad \alpha=-3i,\quad \beta=\frac{\sqrt[3]{4+3i}\sqrt[3]{\delta}}{2^{2/3}},$$

$$\sigma_0=\pm\frac{(-1)^{2/3}\delta^{2/3}}{\sqrt{3}\sqrt[3]{8+6i}},\quad \sigma_1=\pm\frac{(-1)^{2/3}\sqrt[6]{8+6i}\delta^{2/3}}{\sqrt{3}},\quad \lambda_1=0,\quad \gamma_1=0,\quad \alpha=-3i,\quad \beta=-\frac{\sqrt[3]{-1}\sqrt[3]{4+3i}\sqrt[3]{\delta}}{2^{2/3}}.$$

Family 6: $k_1=0$ and $k_2\neq 0$,

$$\sigma_0=0,\quad \sigma_1=-k_2\gamma_1,\quad \lambda_1=0,\quad \gamma_1=\gamma_1,\quad \alpha=0,\quad \beta=\beta,$$

$$\sigma_0=\pm\frac{\delta}{\sqrt{3}\beta},\quad \sigma_1=-k_2\gamma_1,\quad \lambda_1=0,\quad \gamma_1=\gamma_1,\quad \alpha=0,\quad \beta=\beta.$$

As indicated in **Family 1**, (16) obtain the following solutions:

$$r_{11}=\pm\frac{i\sqrt{\frac{2}{3}}\delta^{2/3}\left(\mathrm{sech}^2(\xi)\mp i\tanh(\xi)\mathrm{sech}(\xi)\right)}{\tanh(\xi)\pm i\,\mathrm{sech}(\xi)},\tag{21}$$

$$m_{11}=-\frac{1}{\delta^{2/3}}\left(\pm\frac{i\sqrt{\frac{2}{3}}\delta^{2/3}\left(\mathrm{sech}^2(\xi)\mp i\tanh(\xi)\mathrm{sech}(\xi)\right)}{\tanh(\xi)\pm i\,\mathrm{sech}(\xi)}\right)^2,\tag{22}$$

$$r_{12}=-\frac{i\sqrt{\frac{2}{3}}\delta^{2/3}\left(\mathrm{csch}^2(\xi)\pm\coth(\xi)\mathrm{csch}(\xi)\right)}{\coth(\xi)\pm\mathrm{csch}(\xi)},\tag{23}$$

$$m_{12}=-\frac{1}{\delta^{2/3}}\left(-\frac{i\sqrt{\frac{2}{3}}\delta^{2/3}\left(\mathrm{csch}^2(\xi)\pm\coth(\xi)\mathrm{csch}(\xi)\right)}{\coth(\xi)\pm\mathrm{csch}(\xi)}\right)^2,\tag{24}$$

where $\xi=-\sqrt[3]{\delta}x-\delta t$ and δ is constant,

$$r_{13}=\pm\frac{\delta^{2/3}}{4\sqrt{3}}+\frac{\pm\frac{\delta^{2/3}}{\sqrt{3}}(\tanh(\xi)\pm i\,\mathrm{sech}(\xi))}{3(\tanh(\xi)\pm i\,\mathrm{sech}(\xi))+1},\tag{25}$$

$$m_{13}=\frac{2}{\delta^{2/3}}\left(\pm\frac{\delta^{2/3}}{4\sqrt{3}}+\frac{\pm\frac{\delta^{2/3}}{\sqrt{3}}(\tanh(\xi)\pm i\,\mathrm{sech}(\xi))}{3(\tanh(\xi)\pm i\,\mathrm{sech}(\xi))+1}\right)^2,\tag{26}$$

$$r_{14}=\pm\frac{\delta^{2/3}}{4\sqrt{3}}+\frac{\pm\frac{\delta^{2/3}}{\sqrt{3}}(\coth(\xi)\pm\mathrm{csch}(\xi))}{3(\coth(\xi)\pm i\,\mathrm{csch}(\xi))+1},\tag{27}$$

$$m_{14}=\frac{2}{\delta^{2/3}}\left(\pm\frac{\delta^{2/3}}{4\sqrt{3}}+\frac{\pm\frac{\delta^{2/3}}{\sqrt{3}}(\coth(\xi)\pm\mathrm{csch}(\xi))}{3(\coth(\xi)\pm i\,\mathrm{csch}(\xi))+1}\right)^2,\tag{28}$$

where $\xi=2\sqrt[3]{\delta}x-\delta t$ and δ is constant,

$$r_{15}=\pm\frac{(-1)^{2/3}\delta^{2/3}}{2\sqrt[3]{2}\sqrt{3}}+\frac{\pm\frac{(-1)^{2/3}\delta^{2/3}}{\sqrt[3]{2}\sqrt{3}}(\tanh(\xi)\pm i\,\mathrm{sech}(\xi))}{1-3(\tanh(\xi)\pm i\,\mathrm{sech}(\xi))},\tag{29}$$

$$m_{15}=-\frac{\sqrt[3]{-2}}{\delta^{2/3}}\left(\pm\frac{(-1)^{2/3}\delta^{2/3}}{2\sqrt[3]{2}\sqrt{3}}+\frac{\pm\frac{(-1)^{2/3}\delta^{2/3}}{\sqrt[3]{2}\sqrt{3}}(\tanh(\xi)\pm i\,\mathrm{sech}(\xi))}{1-3(\tanh(\xi)\pm i\,\mathrm{sech}(\xi))}\right)^2,\tag{30}$$

$$r_{16}=\pm\frac{(-1)^{2/3}\delta^{2/3}}{2\sqrt[3]{2}\sqrt{3}}+\frac{\pm\frac{(-1)^{2/3}\delta^{2/3}}{\sqrt[3]{2}\sqrt{3}}(\coth(\xi)\pm\mathrm{csch}(\xi))}{1-3(\coth(\xi)\pm i\,\mathrm{csch}(\xi))},\tag{31}$$

$$m_{16}=-\frac{\sqrt[3]{-2}}{\delta^{2/3}}\left(\pm\frac{(-1)^{2/3}\delta^{2/3}}{2\sqrt[3]{2}\sqrt{3}}+\frac{\pm\frac{(-1)^{2/3}\delta^{2/3}}{\sqrt[3]{2}\sqrt{3}}(\coth(\xi)\pm\mathrm{csch}(\xi))}{1-3(\coth(\xi)\pm i\,\mathrm{csch}(\xi))}\right)^2,\tag{32}$$

where $\xi=-\sqrt[3]{-2}\sqrt[3]{\delta}x-\delta t$ and δ is constant,

$$r_{17}=\pm\frac{(-1)^{2/3}\delta^{2/3}}{\sqrt[3]{2}\sqrt{3}}(\tanh(\xi)\pm i\,\mathrm{sech}(\xi))\pm\frac{(-2)^{2/3}\delta^{2/3}\left(\mathrm{sech}^2(\xi)\mp i\tanh(\xi)\mathrm{sech}(\xi)\right)}{\sqrt{3}(\tanh(\xi)\pm i\,\mathrm{sech}(\xi))},\tag{33}$$

$$m_{17}=-\frac{\sqrt[3]{-2}}{\delta^{2/3}}\left(\pm\frac{(-1)^{2/3}\delta^{2/3}}{\sqrt[3]{2}\sqrt{3}}(\tanh(\xi)\pm i\,\mathrm{sech}(\xi))\pm\frac{(-2)^{2/3}\delta^{2/3}\left(\mathrm{sech}^2(\xi)\mp i\tanh(\xi)\mathrm{sech}(\xi)\right)}{\sqrt{3}(\tanh(\xi)\pm i\,\mathrm{sech}(\xi))}\right)^2,\tag{34}$$

$$r_{19} = \pm\frac{(-1)^{2/3}\delta^{2/3}}{4\sqrt{3}} + \frac{\pm\frac{(-1)^{2/3}\delta^{2/3}}{\sqrt{3}}(\tanh(\xi) \pm \mathrm{isech}(\xi))}{3(\tanh(\xi) \pm \mathrm{isech}(\xi)) + 1}, \tag{35}$$

$$m_{19} = -\frac{2\sqrt[3]{-1}}{\delta^{2/3}}\left(\pm\frac{(-1)^{2/3}\delta^{2/3}}{4\sqrt{3}} + \frac{\pm\frac{(-1)^{2/3}\delta^{2/3}}{\sqrt{3}}(\tanh(\xi) \pm \mathrm{isech}(\xi))}{3(\tanh(\xi) \pm \mathrm{isech}(\xi)) + 1}\right)^2, \tag{36}$$

$$r_{110} = \pm\frac{(-1)^{2/3}\delta^{2/3}}{4\sqrt{3}} + \frac{\pm\frac{(-1)^{2/3}\delta^{2/3}}{\sqrt{3}}(\coth(\xi) \pm \mathrm{csch}(\xi))}{3(\coth(\xi) \pm \mathrm{icsch}(\xi)) + 1}, \tag{37}$$

$$m_{110} = -\frac{2\sqrt[3]{-1}}{\delta^{2/3}}\left(\pm\frac{(-1)^{2/3}\delta^{2/3}}{4\sqrt{3}} + \frac{\pm\frac{(-1)^{2/3}\delta^{2/3}}{\sqrt{3}}(\coth(\xi) \pm \mathrm{csch}(\xi))}{3(\coth(\xi) \pm \mathrm{icsch}(\xi)) + 1}\right)^2, \tag{38}$$

where $\xi = -2\sqrt[3]{-1}\sqrt[3]{\delta}x - \delta t$ and δ is constant.

Remark. Not all the solutions of **Family 1** cases listed are attained.

As indicated in **Family 2**, (16) obtain the following solutions:

$$r_{21} = \pm\frac{\sqrt{\frac{2}{3}}\delta^{2/3}(\tan(\xi)\sec(\xi) \pm \sec^2(\xi))}{\sec(\xi) \pm \tan(\xi)}, \tag{39}$$

$$m_{21} = \frac{1}{\delta^{2/3}}\left(\pm\frac{\sqrt{\frac{2}{3}}\delta^{2/3}(\tan(\xi)\sec(\xi) \pm \sec^2(\xi))}{\sec(\xi) \pm \tan(\xi)}\right)^2, \tag{40}$$

$$r_{22} = -\frac{\sqrt{\frac{2}{3}}\delta^{2/3}(\cot(\xi)\csc(\xi) \pm \csc^2(\xi))}{\csc(\xi) \pm \cot(\xi)}, \tag{41}$$

$$m_{22} = \frac{1}{\delta^{2/3}}\left(-\frac{\sqrt{\frac{2}{3}}\delta^{2/3}(\cot(\xi)\csc(\xi) \pm \csc^2(\xi))}{\csc(\xi) \pm \cot(\xi)}\right)^2, \tag{42}$$

where $\xi = \sqrt[3]{\delta}x - \delta t$ and δ is constant,

$$r_{23} = \pm\frac{\sqrt[6]{-1}\delta^{2/3}}{\sqrt[3]{2}\sqrt{3}}(\sec(\xi) \pm \tan(\xi)), \tag{43}$$

$$m_{23} = \frac{\sqrt[3]{-2}}{\delta^{2/3}}\left(\pm\frac{\sqrt[6]{-1}\delta^{2/3}}{\sqrt[3]{2}\sqrt{3}}(\sec(\xi) \pm \tan(\xi))\right)^2, \tag{44}$$

$$r_{24} = \pm\frac{\sqrt[6]{-1}\delta^{2/3}}{\sqrt[3]{2}\sqrt{3}}(\csc(\xi) \pm \cot(\xi)), \tag{45}$$

$$m_{24} = \frac{\sqrt[3]{-2}}{\delta^{2/3}}\left(\pm\frac{\sqrt[6]{-1}\delta^{2/3}}{\sqrt[3]{2}\sqrt{3}}(\csc(\xi) \pm \cot(\xi))\right)^2, \tag{46}$$

where $\xi = \sqrt[3]{-2}\sqrt[3]{\delta}x - \delta t$ and δ is constant,

$$r_{25} = \pm\frac{\sqrt[6]{-1}\delta^{2/3}}{\sqrt[3]{2}\sqrt{3}}(\sec(\xi) \pm \tan(\xi)) \pm \frac{\sqrt[6]{-1}2^{2/3}\delta^{2/3}(\tan(\xi)\sec(\xi) \pm \sec^2(\xi))}{\sqrt{3}(\sec(\xi) \pm \tan(\xi))}, \tag{47}$$

$$m_{25} = \frac{\sqrt[3]{-2}}{\delta^{2/3}}\left(\pm\frac{\sqrt[6]{-1}\delta^{2/3}}{\sqrt[3]{2}\sqrt{3}}(\sec(\xi) \pm \tan(\xi)) \pm \frac{\sqrt[6]{-1}2^{2/3}\delta^{2/3}(\tan(\xi)\sec(\xi) \pm \sec^2(\xi))}{\sqrt{3}(\sec(\xi) \pm \tan(\xi))}\right)^2, \tag{48}$$

$$r_{26} = \pm\frac{\sqrt[6]{-1}\delta^{2/3}}{\sqrt[3]{2}\sqrt{3}}(\csc(\xi) \pm \cot(\xi)) - \frac{\sqrt[6]{-1}2^{2/3}\delta^{2/3}(\cot(\xi)\csc(\xi) \pm \csc^2(\xi))}{\sqrt{3}(\csc(\xi) \pm \cot(\xi))}, \tag{49}$$

$$m_{26} = \frac{\sqrt[3]{-2}}{\delta^{2/3}}\left(\pm\frac{\sqrt[6]{-1}\delta^{2/3}}{\sqrt[3]{2}\sqrt{3}}(\csc(\xi) \pm \cot(\xi)) \pm - \frac{\sqrt[6]{-1}2^{2/3}\delta^{2/3}(\cot(\xi)\csc(\xi) \pm \csc^2(\xi))}{\sqrt{3}(\csc(\xi) \pm \cot(\xi))}\right)^2, \tag{50}$$

where $\xi = \sqrt[3]{-2}\sqrt[3]{\delta}x - \delta t$ and δ is constant,

$$r_{27} = \pm\frac{\mathrm{i}\sqrt[3]{2}\delta^{2/3}}{\sqrt{3}}(\sec(\xi) \pm \tan(\xi)) \pm \frac{\mathrm{i}\sqrt[3]{2}\delta^{2/3}(\tan(\xi)\sec(\xi) \pm \sec^2(\xi))}{\sqrt{3}(\sec(\xi) \pm \tan(\xi))}, \tag{51}$$

$$m_{27} = -\frac{1}{\sqrt[3]{2}\delta^{2/3}}\left(\pm\frac{\mathrm{i}\sqrt[3]{2}\delta^{2/3}}{\sqrt{3}}(\sec(\xi) \pm \tan(\xi)) \pm \frac{\mathrm{i}\sqrt[3]{2}\delta^{2/3}(\tan(\xi)\sec(\xi) \pm \sec^2(\xi))}{\sqrt{3}(\sec(\xi) \pm \tan(\xi))}\right)^2, \tag{52}$$

$$r_{28} = \pm\frac{\mathrm{i}\sqrt[3]{2}\delta^{2/3}}{\sqrt{3}}(\csc(\xi) \pm \cot(\xi)) - \frac{\mathrm{i}\sqrt[3]{2}\delta^{2/3}(\cot(\xi)\csc(\xi) \pm \csc^2(\xi))}{\sqrt{3}(\csc(\xi) \pm \cot(\xi))}, \tag{53}$$

$$m_{28} = -\frac{1}{\sqrt[3]{2}\delta^{2/3}}\left(\pm\frac{\mathrm{i}\sqrt[3]{2}\delta^{2/3}}{\sqrt{3}}(\csc(\xi) \pm \cot(\xi)) - \frac{\mathrm{i}\sqrt[3]{2}\delta^{2/3}(\cot(\xi)\csc(\xi) \pm \csc^2(\xi))}{\sqrt{3}(\csc(\xi) \pm \cot(\xi))}\right)^2, \tag{54}$$

where $\xi = -\frac{\sqrt[3]{\delta}}{\sqrt[3]{2}}x - \delta t$ and δ is constant,

$$r_{29} = \pm\frac{\mathrm{i}\delta^{2/3}}{\sqrt[3]{2}\sqrt{3}}(\sec(\xi) \pm \tan(\xi)) \pm \frac{\mathrm{i}2^{2/3}\delta^{2/3}(\tan(\xi)\sec(\xi) \pm \sec^2(\xi))}{\sqrt{3}(\sec(\xi) \pm \tan(\xi))}, \tag{55}$$

$$m_{29} = -\frac{\sqrt[3]{2}}{\delta^{2/3}}\left(\pm\frac{\mathrm{i}\delta^{2/3}}{\sqrt[3]{2}\sqrt{3}}(\sec(\xi) \pm \tan(\xi)) \pm \frac{\mathrm{i}2^{2/3}\delta^{2/3}(\tan(\xi)\sec(\xi) \pm \sec^2(\xi))}{\sqrt{3}(\sec(\xi) \pm \tan(\xi))}\right)^2, \tag{56}$$

$$r_{210} = \pm\frac{\mathrm{i}\delta^{2/3}}{\sqrt[3]{2}\sqrt{3}}(\csc(\xi) \pm \cot(\xi)) - \frac{\mathrm{i}2^{2/3}\delta^{2/3}(\cot(\xi)\csc(\xi) \pm \csc^2(\xi))}{\sqrt{3}(\csc(\xi) \pm \cot(\xi))}, \tag{57}$$

$$m_{210} = -\frac{\sqrt[3]{2}}{\delta^{2/3}}\left(\pm\frac{\mathrm{i}\delta^{2/3}}{\sqrt[3]{2}\sqrt{3}}(\csc(\xi) \pm \cot(\xi)) - \frac{\mathrm{i}2^{2/3}\delta^{2/3}(\cot(\xi)\csc(\xi) \pm \csc^2(\xi))}{\sqrt{3}(\csc(\xi) \pm \cot(\xi))}\right)^2, \tag{58}$$

where $\xi = -\sqrt[3]{2}\sqrt[3]{\delta}x - \delta t$ and δ is constant,

$$r_{211} = \frac{\pm\frac{\sqrt[6]{4-3\mathrm{i}}\delta^{2/3}}{\sqrt{6}}(\sec(\xi) \pm \tan(\xi))}{1 + 3\mathrm{i}(\sec(\xi) \pm \tan(\xi))} \pm \frac{\delta^{2/3}}{2\sqrt{3}\sqrt[3]{4-3\mathrm{i}}}, \tag{59}$$

$$m_{211} = \frac{\sqrt[3]{4-3\mathrm{i}}}{\delta^{2/3}}\left(\frac{\pm\frac{\sqrt[6]{4-3\mathrm{i}}\delta^{2/3}}{\sqrt{6}}(\sec(\xi) \pm \tan(\xi))}{1 + 3\mathrm{i}(\sec(\xi) \pm \tan(\xi))} \pm \frac{\delta^{2/3}}{2\sqrt{3}\sqrt[3]{4-3\mathrm{i}}}\right)^2, \tag{60}$$

Remark. Not all the solutions of **Family 2** cases listed are attained.

As indicated in **Family 3**, (16) obtain the following solutions:

$$r_{31} = \pm\frac{2\delta^{2/3}}{\sqrt{3}}\tanh(\xi) \pm \frac{\delta^{2/3}\mathrm{csch}(\xi)\mathrm{sech}(\xi)}{\sqrt{3}}, \tag{61}$$

$$m_{31}=\frac{1}{2\delta^{2/3}}\left(\pm\frac{2\delta^{2/3}}{\sqrt{3}}\tanh(\xi)\pm\frac{\delta^{2/3}\text{csch}(\xi)\text{sech}(\xi)}{\sqrt{3}}\right)^2, \tag{62}$$

$$r_{32}=\pm\frac{2\delta^{2/3}}{\sqrt{3}}\coth(\xi)-\frac{\delta^{2/3}\text{csch}(\xi)\text{sech}(\xi)}{\sqrt{3}}, \tag{63}$$

$$m_{32}=\frac{1}{2\delta^{2/3}}\left(\pm\frac{2\delta^{2/3}}{\sqrt{3}}\coth(\xi)-\frac{\delta^{2/3}\text{csch}(\xi)\text{sech}(\xi)}{\sqrt{3}}\right)^2, \tag{64}$$

where $\xi=\frac{\sqrt[3]{\delta}}{2}x-\delta t$ and δ is constant,

$$r_{33}=\pm\frac{(-1)^{2/3}\delta^{2/3}}{2\sqrt[3]{2}\sqrt{3}}+\frac{\pm\frac{(-2)^{2/3}\delta^{2/3}}{\sqrt{3}}\tanh(\xi)}{3\tanh(\xi)+1}, \tag{65}$$

$$m_{33}=-\frac{\sqrt[3]{-2}}{\delta^{2/3}}\left(\pm\frac{(-1)^{2/3}\delta^{2/3}}{2\sqrt[3]{2}\sqrt{3}}+\frac{\pm\frac{(-2)^{2/3}\delta^{2/3}}{\sqrt{3}}\tanh(\xi)}{3\tanh(\xi)+1}\right)^2, \tag{66}$$

$$r_{34}=\pm\frac{(-1)^{2/3}\delta^{2/3}}{2\sqrt[3]{2}\sqrt{3}}+\frac{\left(\pm\frac{(-2)^{2/3}\delta^{2/3}}{\sqrt{3}}\right)\coth(\xi)}{3\coth(\xi)+1}, \tag{67}$$

$$m_{34}=-\frac{\sqrt[3]{-2}}{\delta^{2/3}}\left(\pm\frac{(-1)^{2/3}\delta^{2/3}}{2\sqrt[3]{2}\sqrt{3}}+\frac{\left(\pm\frac{(-2)^{2/3}\delta^{2/3}}{\sqrt{3}}\right)\coth(\xi)}{3\coth(\xi)+1}\right)^2, \tag{68}$$

where $\xi=-\sqrt[3]{-2}\sqrt[3]{\delta}x-\delta t$ and δ is constant,

$$r_{35}=\pm\frac{2(-1)^{2/3}\delta^{2/3}}{\sqrt{3}}\tanh(\xi)\pm\frac{(-1)^{2/3}\delta^{2/3}\text{csch}(\xi)\text{sech}(\xi)}{\sqrt{3}}, \tag{69}$$

$$m_{35}=-\frac{\sqrt[3]{-1}}{2\delta^{2/3}}\left(\pm\frac{2(-1)^{2/3}\delta^{2/3}}{\sqrt{3}}\tanh(\xi)\pm\frac{(-1)^{2/3}\delta^{2/3}\text{csch}(\xi)\text{sech}(\xi)}{\sqrt{3}}\right)^2, \tag{70}$$

$$r_{36}=\pm\frac{2(-1)^{2/3}\delta^{2/3}}{\sqrt{3}}\coth(\xi)-\frac{(-1)^{2/3}\delta^{2/3}\text{csch}(\xi)\text{sech}(\xi)}{\sqrt{3}}, \tag{71}$$

$$m_{36}=-\frac{\sqrt[3]{-1}}{2\delta^{2/3}}\left(\pm\frac{2(-1)^{2/3}\delta^{2/3}}{\sqrt{3}}\coth(\xi)-\frac{(-1)^{2/3}\delta^{2/3}\text{csch}(\xi)\text{sech}(\xi)}{\sqrt{3}}\right)^2, \tag{72}$$

where $\xi=-\frac{1}{2}\sqrt[3]{-1}\sqrt[3]{\delta}x-\delta t$ and δ is constant,

$$r_{37}=\pm\frac{(-1)^{2/3}\delta^{2/3}}{2^{2/3}\sqrt{3}}+\frac{\pm\frac{(-1)^{2/3}\sqrt[3]{2}\delta^{2/3}}{\sqrt{3}}\tanh(\xi)}{1-3\tanh(\xi)}, \tag{73}$$

$$m_{37}=-\frac{\sqrt[3]{-\frac{1}{2}}}{\delta^{2/3}}\left(\pm\frac{(-1)^{2/3}\delta^{2/3}}{2^{2/3}\sqrt{3}}+\frac{\pm\frac{(-1)^{2/3}\sqrt[3]{2}\delta^{2/3}}{\sqrt{3}}\tanh(\xi)}{1-3\tanh(\xi)}\right)^2, \tag{74}$$

$$r_{38}=\pm\frac{(-1)^{2/3}\delta^{2/3}}{2^{2/3}\sqrt{3}}+\frac{\left(\pm\frac{(-1)^{2/3}\sqrt[3]{2}\delta^{2/3}}{\sqrt{3}}\right)\coth(\xi)}{1-3\coth(\xi)}, \tag{75}$$

$$m_{38}=-\frac{\sqrt[3]{-\frac{1}{2}}}{\delta^{2/3}}\left(\pm\frac{(-1)^{2/3}\delta^{2/3}}{2^{2/3}\sqrt{3}}+\frac{\left(\pm\frac{(-1)^{2/3}\sqrt[3]{2}\delta^{2/3}}{\sqrt{3}}\right)\coth(\xi)}{1-3\coth(\xi)}\right)^2, \tag{76}$$

where $\xi=-\sqrt[3]{-\frac{1}{2}}\sqrt[3]{\delta}x-\delta t$ and δ is constant,

$$r_{39}=\pm\frac{i\sqrt[6]{2}\delta^{2/3}\text{csch}(\xi)\text{sech}(\xi)}{\sqrt{3}}, \tag{77}$$

$$m_{39}=-\frac{1}{2^{2/3}\delta^{2/3}}\left(\pm\frac{i\sqrt[6]{2}\delta^{2/3}\text{csch}(\xi)\text{sech}(\xi)}{\sqrt{3}}\right)^2, \tag{78}$$

$$r_{310}=-\frac{i\sqrt[6]{2}\delta^{2/3}\text{csch}(\xi)\text{sech}(\xi)}{\sqrt{3}}, \tag{79}$$

$$m_{310}=-\frac{1}{2^{2/3}\delta^{2/3}}\left(-\frac{i\sqrt[6]{2}\delta^{2/3}\text{csch}(\xi)\text{sech}(\xi)}{\sqrt{3}}\right)^2, \tag{80}$$

where $\xi=-\frac{\sqrt[3]{\delta}}{2^{2/3}}x-\delta t$ and δ is constant.

Remark. Not all the solutions of **Family 3** cases listed are attained.

As indicated in **Family 4**, (16) obtain the following solutions:

$$r_{41}=\pm\frac{2i\delta^{2/3}}{\sqrt{3}}\tan(\xi)\pm\frac{i\delta^{2/3}\csc(\xi)\sec(\xi)}{\sqrt{3}}, \tag{81}$$

$$m_{41}=-\frac{1}{2\delta^{2/3}}\left(\pm\frac{2i\delta^{2/3}}{\sqrt{3}}\tan(\xi)\pm\frac{i\delta^{2/3}\csc(\xi)\sec(\xi)}{\sqrt{3}}\right)^2, \tag{82}$$

where $\xi=-\frac{\sqrt[3]{\delta}}{2}x-\delta t$ and δ is constant,

$$r_{42}=\pm\frac{2\sqrt[6]{-1}\delta^{2/3}}{\sqrt{3}}\tan(\xi)\pm\frac{\sqrt[6]{-1}\delta^{2/3}\csc(\xi)\sec(\xi)}{\sqrt{3}}, \tag{83}$$

$$m_{42}=\frac{\sqrt[3]{-1}}{2\delta^{2/3}}\left(\pm\frac{2\sqrt[6]{-1}\delta^{2/3}}{\sqrt{3}}\tan(\xi)\pm\frac{\sqrt[6]{-1}\delta^{2/3}\csc(\xi)\sec(\xi)}{\sqrt{3}}\right)^2, \tag{84}$$

where $\xi=\frac{1}{2}\sqrt[3]{-1}\sqrt[3]{\delta}x-\delta t$ and δ is constant,

$$r_{43}=\pm\frac{\sqrt[6]{-1}\sqrt[3]{2}\delta^{2/3}}{\sqrt{3}}\tan(\xi), \tag{85}$$

$$m_{43}=\frac{\sqrt[3]{-\frac{1}{2}}}{\delta^{2/3}}\left(\pm\frac{\sqrt[6]{-1}\sqrt[3]{2}\delta^{2/3}}{\sqrt{3}}\tan(\xi)\right)^2, \tag{86}$$

where $\xi=\sqrt[3]{-\frac{1}{2}}\sqrt[3]{\delta}x-\delta t$ and δ is constant,

$$r_{44}=\pm\frac{\sqrt[6]{2}\delta^{2/3}\csc(\xi)\sec(\xi)}{\sqrt{3}}, \tag{87}$$

$$m_{44}=\frac{1}{2^{2/3}\delta^{2/3}}\left(\pm\frac{\sqrt[6]{2}\delta^{2/3}\csc(\xi)\sec(\xi)}{\sqrt{3}}\right)^2, \tag{88}$$

where $\xi=\frac{\sqrt[3]{\delta}}{2^{2/3}}x-\delta t$ and δ is constant,

$$r_{45}=\pm\frac{i\sqrt[6]{-2}\delta^{2/3}\csc(\xi)\sec(\xi)}{\sqrt{3}}, \tag{89}$$

$$m_{45}=-\frac{\sqrt[3]{-1}}{2^{2/3}\delta^{2/3}}\left(\pm\frac{i\sqrt[6]{-2}\delta^{2/3}\csc(\xi)\sec(\xi)}{\sqrt{3}}\right)^2, \tag{90}$$

where $\xi=-\frac{\sqrt[3]{-1}\sqrt[3]{\delta}}{2^{2/3}}x-\delta t$ and δ is constant.

Remark. Not all the solutions of **Family 4** cases listed are attained.

As indicated in **Family 5**, (16) obtain the following solutions:

$$r_{51}=\pm\frac{2i\delta^{2/3}}{\sqrt{3}}\cot(\xi)-\frac{i\delta^{2/3}\csc(\xi)\sec(\xi)}{\sqrt{3}}, \tag{91}$$

$$m_{51}=-\frac{1}{2\delta^{2/3}}\left(\pm\frac{2i\delta^{2/3}}{\sqrt{3}}\cot(\xi)-\frac{i\delta^{2/3}\csc(\xi)\sec(\xi)}{\sqrt{3}}\right)^2, \tag{92}$$

where $\xi = -\frac{\sqrt[3]{\delta}}{2}x - \delta t$ and δ is constant,

$$r_{52} = \pm\frac{2\sqrt[6]{-1}\delta^{2/3}}{\sqrt{3}}\cot(\xi) - \frac{\sqrt[6]{-1}\delta^{2/3}\csc(\xi)\sec(\xi)}{\sqrt{3}}, \tag{93}$$

$$m_{52} = \frac{\sqrt[3]{-1}}{2\delta^{2/3}}\left(\pm\frac{2\sqrt[6]{-1}\delta^{2/3}}{\sqrt{3}}\cot(\xi) - \frac{\sqrt[6]{-1}\delta^{2/3}\csc(\xi)\sec(\xi)}{\sqrt{3}}\right)^2, \tag{94}$$

where $\xi = \frac{1}{2}\sqrt[3]{-1}\sqrt[3]{\delta}x - \delta t$ and δ is constant,

$$r_{53} = \pm\frac{\sqrt[6]{-1}\sqrt[3]{2}\delta^{2/3}}{\sqrt{3}}\cot(\xi), \tag{95}$$

$$m_{53} = \frac{\sqrt[3]{-\frac{1}{2}}}{\delta^{2/3}}\left(\pm\frac{\sqrt[6]{-1}\sqrt[3]{2}\delta^{2/3}}{\sqrt{3}}\cot(\xi)\right)^2, \tag{96}$$

where $\xi = \sqrt[3]{-\frac{1}{2}}\sqrt[3]{\delta}x - \delta t$ and δ is constant,

$$r_{54} = -\frac{\sqrt[6]{2}\delta^{2/3}\csc(\xi)\sec(\xi)}{\sqrt{3}}, \tag{97}$$

$$m_{54} = \frac{1}{2^{2/3}\delta^{2/3}}\left(-\frac{\sqrt[6]{2}\delta^{2/3}\csc(\xi)\sec(\xi)}{\sqrt{3}}\right)^2, \tag{98}$$

where $\xi = \frac{\sqrt[3]{\delta}}{2^{2/3}}x - \delta t$ and δ is constant,

$$r_{55} = -\frac{\mathrm{i}\sqrt[6]{-2}\delta^{2/3}\csc(\xi)\sec(\xi)}{\sqrt{3}}, \tag{99}$$

$$m_{55} = -\frac{\sqrt[3]{-1}}{2^{2/3}\delta^{2/3}}\left(-\frac{\mathrm{i}\sqrt[6]{-2}\delta^{2/3}\csc(\xi)\sec(\xi)}{\sqrt{3}}\right)^2, \tag{100}$$

where $\xi = -\frac{\sqrt[3]{-1}\sqrt[3]{\delta}}{2^{2/3}}x - \delta t$ and δ is constant.

Remark. Not all the solutions of **Family 5** cases listed are attained.

As indicated in **Family 6**, (16) obtain the following solutions:

$$r_{61} = -\frac{2\gamma_1 k_2}{f_0 + k_2\xi}, \tag{101}$$

$$m_{61} = \frac{\beta}{\delta}\left(-\frac{2\gamma_1 k_2}{f_0 + k_2\xi}\right)^2, \tag{102}$$

$$r_{62} = \pm\frac{\delta}{\sqrt{3}\beta} - \frac{2\gamma_1 k_2}{f_0 + k_2\xi}, \tag{103}$$

$$m_{62} = \frac{\beta}{\delta}\left(\pm\frac{\delta}{\sqrt{3}\beta} - \frac{2\gamma_1 k_2}{f_0 + k_2\xi}\right)^2, \tag{104}$$

where $\xi = \beta x - \delta t, \delta, \beta, \gamma_1, f_0$ are constants and $k_2 \neq 0$.

4. Graphical depiction for some of the solutions acquired

In this section, graphically displays for some of the acquired solutions in the preceding section is represented. The numerous acquired solutions are achieved via allocating a constant value to each parameter. The physical nature is presented within the interval $-2 \leqslant x, t \leqslant 2$. The graphs are plotted by taking the values at constants $\delta = 2, \gamma_1 = 1, \kappa_2 = 1.5, \beta = 1.5$ and $f_0 = 1.5$. The nature of the graphs depict how significance the solutions attained for the non-linear coupled DSW system is introduced as system of water waves. Below are some of the plotted figures, thus, Figs. 1–5.

5. Results and discussion

In this paper, we confirm that several of the attained solutions are new by comparing the results acquired with the results of other different works using different techniques. The merits and validity of our technique over other techniques are discussed below:

5.1. Merits

A key advantage of our technique applied over other techniques is, it offers a further flexible and amusing genuine wave solution with several practical parameters. The accurate solution of NLES is of excessive worth for enlightening the essential machinery of cumbersome actual phenomena. In addition to the actual implementation, the approximation solution of NLES aids numerical solvers compare the effectiveness of their solution and contribute to their stability studies.

5.2. Validity

In Arshad et al. (2017) studied the traveling wave solutions of (16) by implementing the modified extended direct algebraic technique. Consequently, they attained various explicit solutions like solitary wave solutions, periodic wave solutions, Jacobi elliptic solutions and Weierstrass elliptic function. It is important to note that most of the results acquired in this paper are comparable to the results attained in Arshad et al. (2017). In addition, the rest

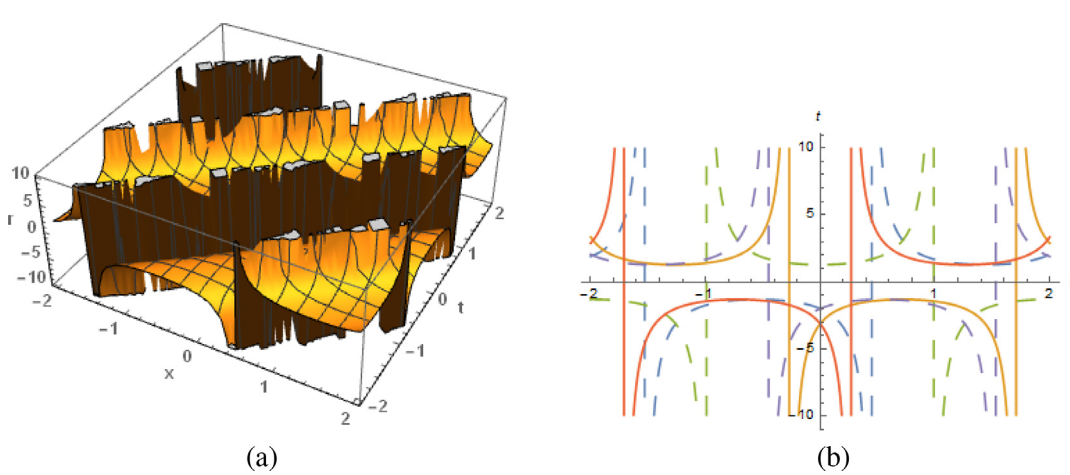

Fig. 1. The results of (39) display from the figures with diverse plotted natures at $\delta = 2$ in the range $-2 \leqslant x, t \leqslant 2$: (a) periodic solitary wave and (b) represent 2-dimensional nature.

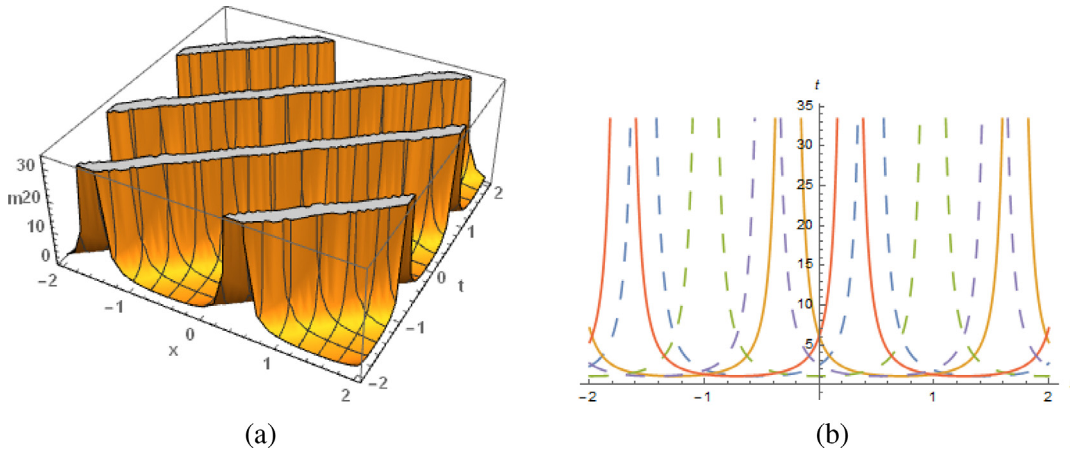

Fig. 2. The results of (40) display from the figures with diverse plotted natures at $\delta = 2$ in the range $-2 \leqslant x, t \leqslant 2$: (a) periodic solitary wave and (b) represent 2–dimensional nature.

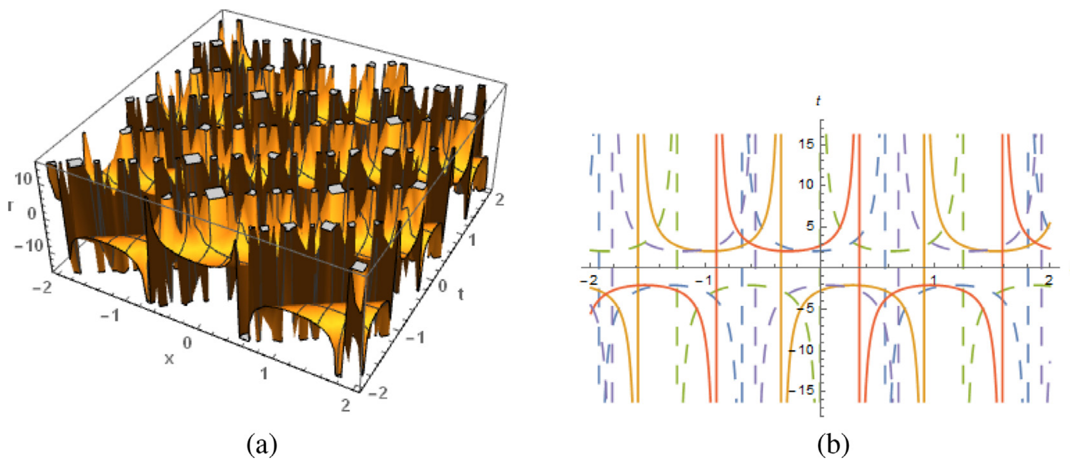

Fig. 3. The results of (87) display from the figures with diverse plotted natures at $\delta = 2$ in the range $-2 \leqslant x, t \leqslant 2$: (a) periodic solitary wave and (b) represent 2–dimensional nature.

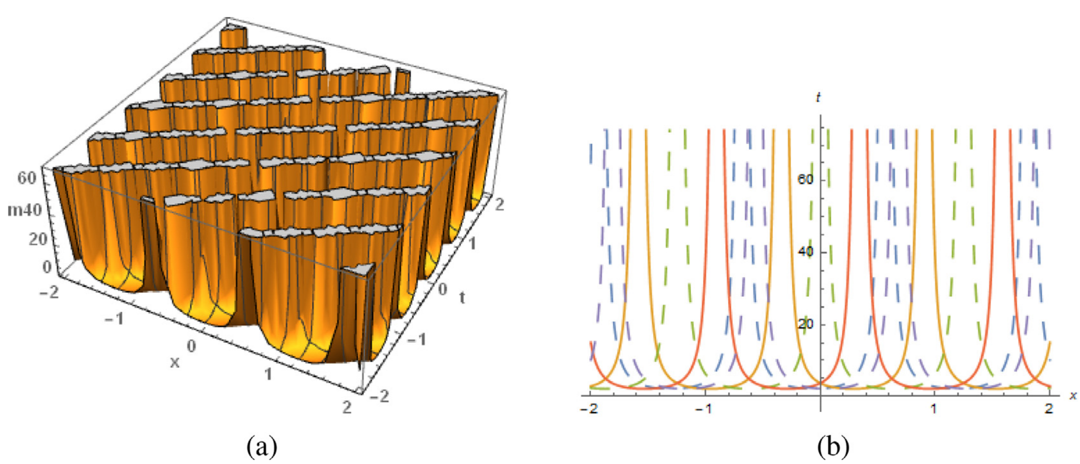

Fig. 4. The results of (88) display from the figures with diverse plotted natures at $\delta = 2$ in the range $-2 \leqslant x, t \leqslant 2$: (a) periodic solitary wave and (b) represent 2–dimensional nature.

of the results are new and valuable for researchers to know more about this technique.

Furthermore, in Khan et al. (2015) K. Khan et al. studied the exact analytical solutions for the (16) by implementing the enhanced $(\frac{G}{G'})$–expansion technique. Therefore, they were able to deduce abundant of traveling wave solutions comprising of hyperbolic solutions. It is important to note that some of the results acquired in this paper are comparable to the results attained in Khan et al. (2015), but the rest of our attained solutions are new and is useful for researchers (Lu et al., 2018a; Lu et al., 2018b;

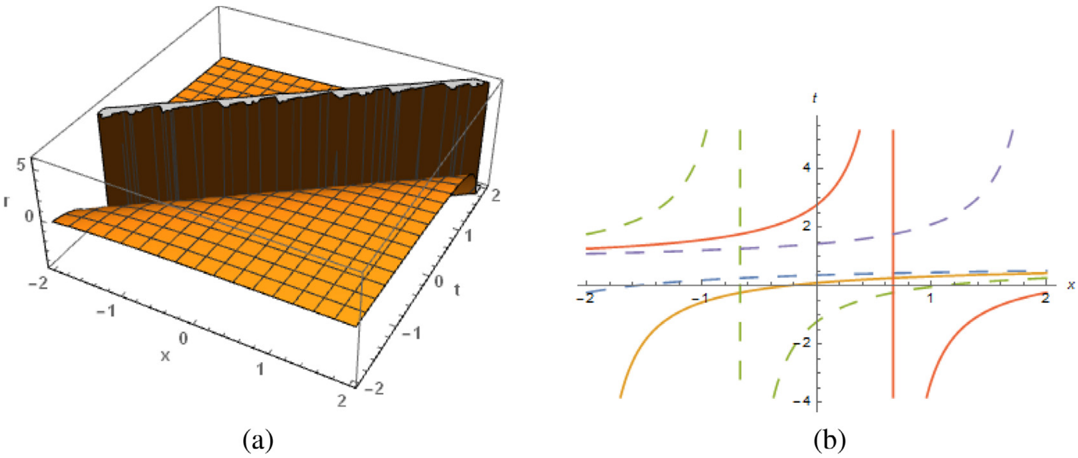

Fig. 5. The results of (103) display from the figures with diverse plotted natures at $\delta = 2, \gamma_1 = 1, k_2 = 1.5, \beta = 1.5$ and $f_0 = 1.5$ in the range $-2 \leqslant x, t \leqslant 2$: (a) solitary wave and (b) represent 2–dimensional nature.

Iqbal et al., 2018; Helal et al., 2014; Ali et al., 2018; Iqbal et al., 2020; Seadawy et al., 2019; Ozkan et al., 2020; Ahmad et al., 2020).

Thus, it is proved that the RERE technique with the current ansatz provides a commanding and essential mathematical tool for finding non-linear evolutionary systems in mathematical physics and sciences. Hence, this new technique is dependable and suggests a diversity of solitary wave solutions that can be applied to numerous other NLES.

6. Conclusion

In this paper, we have efficaciously extended the ansatz in the RERE technique to new and generalized form and attained some new exact traveling wave solutions for (16). Some selected solutions with different physical profiles of (16) are constructed in this paper. Several of the attained solutions which is in the form of rational solitary wave solutions and periodic wave solutions are entirely new and could be vital for researchers to appreciate. The results show that the proposed technique has a broader implementation and is more real than other techniques. Therefore, the technique can be applied to study various NLES that normally seem in scientific real-time applications like mathematics, physics and engineering. In addition, it is essential to investigate the stability of some NLES by using the current formal ansatz applied in this paper which could be useful for researchers.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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