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ORIGINAL ARTICLE

Traveling wave solutions of the nonlinear dispersive Klein–Gordon equations

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Abstract In this paper, the traveling wave solutions to the nonlinear dispersive Klein–Gordon equation is obtained. There are five forms of nonlinereity that are studied in this paper. While in each of the cases, the results are in terms of quadratures, the numerical simulation of the solutions are given for completeness.

1. Introduction

Klein–Gordon equation (KGE) appears in the study of relativistic Quantum Mechanics (Aguero et al., 2009; Basak et al., 2009; Bawin and Jaminon, 1984; Biswas et al., 2008; Chen, 2005; Cheng, 2011; Comay, 2004; Lakshmi, 2009; Li et al., 2010; Sassaman and Biswas, 2009, 2010a,b, 2011; Yasuk et al., 2006; Zheng and Lai, 2009). There are various studies on this equation that has been conducted during the past decades. There are soliton solutions, shock wave solutions are also obtained. In addition, the soliton perturbation theory was also studied for this equation. Later, this equation was extended to (1 + 2) dimensions and in addition the evolution and dispersion terms were generalized (Sassaman and Biswas, 2009, 2010a,b, 2011).

The nonlinear dispersive KGE appeared in 2009 (Aguero et al., 2009). In this paper, this nonlinear dispersive KGE will be studied by the aid of traveling wave hypothesis. Traveling wave hypothesis is one of the most fundamental mechanisms of integrating nonlinear evolutions equations. It will be observed that the solutions will be in terms of quadratures. Therefore the numerical simulations will be given for each of these cases. There are five types of nonlinearity that will be taken into consideration.

2. Mathematical analysis

The dimensionless form of the nonlinear dispersive KGE that is going to be studied in this paper is given by (Aguero et al., 2009)

$$q_{tt} - k[3(q_x)^2 q_{xx}] + F(q) = 0 \tag{1}$$

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Here, in (1), the dependent variable q represents the wave profile, while x and t are the independent variables that represents the spatial and temporal variables, respectively. Also, k is a real valued positive constant. The function $F(q)$ represents the nonlinear function. The function $F(q)$ can be written in terms of the potential function $U(q)$ as (Biswas et al., 2008)

$$F(q) = -\frac{\partial U}{\partial q} \tag{2}$$

This potential function $U(q)$ has at least two minima, q_1 and q_3 and a maxima at q_2 such that

$$U(q_1) = U(q_3) \tag{3}$$

2.1. Traveling wave solution

The starting traveling wave hypothesis for the solution to (1) is

$$q(x,t) = g(x - vt) \tag{4}$$

where v is the velocity of the wave while the function g represents the shape of the wave profile. Substituting this hypothesis into (1) gives

$$v^2 g'' - 3k(g')^2 g'' + F(g) = 0 \tag{5}$$

Now multiplying both sides of (2) by g' and integrating yields

$$2v^2(g')^2 - 3k(g')^4 + 4\int g' F(g)\,\mathrm{d}g = 0 \tag{6}$$

which can be solved for g' once the nonlinear functional $F(g)$ is known.

In this paper there are five types of nonlinearity will be considered. They are (Sassaman and Biswas, 2009, 2010a,b, 2011)

$$F(q) = aq - bq^2 \tag{7}$$

$$F(q) = aq - bq^3 \tag{8}$$

$$F(q) = aq - bq^n \tag{9}$$

$$F(q) = aq - bq^n + cq^{2n-1} \tag{10}$$

$$F(q) = aq - bq^{1-n} + cq^{n+1} \tag{11}$$

where in (7)–(11), a, b and c are real valued constants. The exponent n in (9)–(11) are positive integers and their further restrictions will be given in the respective subsections. These five forms will be labeled as Forms I–V, respectively.

3. Analytical and numerical solutions

In this section, the analytical and numerical solutions to (3) will be obtained for the five different forms of nonlinearity that are described in the previous section.

3.1. FORM-I

For the first form, the nonlinear dispersive KGE is given by

$$q_{tt} - k[3(q_x)^2 q_{xx}] + aq - bq^2 = 0 \tag{12}$$

In this case, Eq. (6) reduces to

$$9k(g')^4 - 6v^2(g')^2 - 6ag^2 + 4bg^3 + c_1 = 0 \tag{13}$$

where c_1 is a constant of integration. Solving for g' gives

$$g' = \frac{1}{\sqrt{3k}}\left[v^2 - \sqrt{v^4 - k\left(4bg^3 - 6ag^2 + c_1\right)}\right]^{\frac{1}{2}} \tag{14}$$

Separating variables yield

$$\frac{x - vt}{\sqrt{3k}} + c_2 = \int^g \frac{\mathrm{d}y}{\left[v^2 - \sqrt{v^4 - k\left(4by^3 - 6ay^2 + c_1\right)}\right]^{\frac{1}{2}}} \tag{15}$$

The following figure shows the profile of the traveling wave where the parameter values are $v = k = 1$, $a = b = -1$, while $c_1 = c_2 = 0$ (Fig. 1).

3.2. FORM-II

With the second form, the nonlinear dispersive KGE is given by

$$q_{tt} - k[3(q_x)^2 q_{xx}] + aq - bq^3 = 0 \tag{16}$$

In this case, Eq. (6) reduces to

$$3k(g')^4 - 2v^2(g')^2 - 2ag^2 - bg^4 + c_1 = 0 \tag{17}$$

where c_1 is a constant of integration. Solving for g' gives

$$g' = \frac{1}{\sqrt{3k}}\left[v^2 \pm \sqrt{v^4 - 3k(bg^4 - 2ag^2 + c_1)}\right]^{\frac{1}{2}} \tag{18}$$

By separation of variables,

$$\frac{x - vt}{\sqrt{3k}} + c_2 = \int^g \frac{\mathrm{d}y}{\left[v^2 \pm \sqrt{v^4 - 3k\left(by^4 - 2ay^2 + c_1\right)}\right]^{\frac{1}{2}}} \tag{19}$$

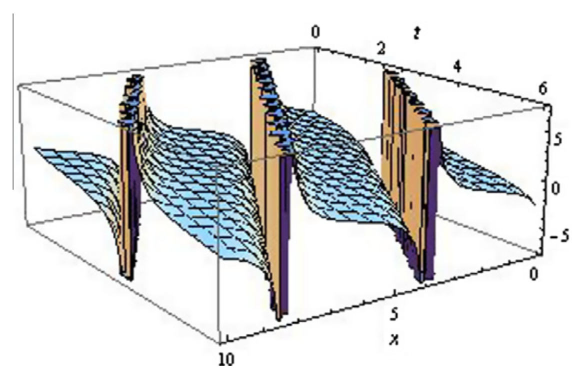

Figure 1 Wave profile for KGE (Form-I).

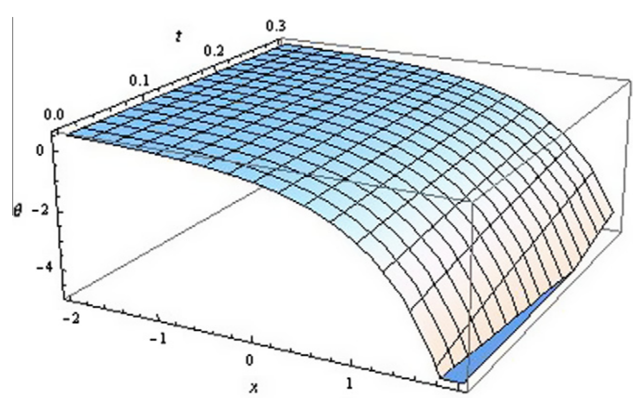

Figure 2 Wave profile for KGE (Form-II).

The following figure shows the profile of the traveling wave where the parameter values are $v = k = a = b = 1$, while $c_1 = c_2 = 0$ (Fig. 2).

3.3. FORM-III

For the third form, the nonlinear dispersive KGE is given by

$$q_{tt} - k[3(q_x)^2 q_{xx}] + aq - bq^n = 0 \tag{20}$$

where n is a positive integer. If $n = 1$ or 2, this case collapses to the first two forms. Here, Eq. (6) reduces to

$$3(n+1)k(g')^4 - 2(n+1)v^2(g')^2 - 2(n+1)ag^2 + 4bg^{n+1} + c_1 = 0 \tag{21}$$

where c_1 is a constant of integration. Solving for g' gives

$$g' = \frac{1}{\sqrt{3(n+1)k}} \times \left[(n+1)v^2 \pm \sqrt{(n+1)^2 v^4 - 3(n+1)k\{4bg^{n+1} - 2(n+1)ag^2 + c_1\}}\right]^{\frac{1}{2}} \tag{22}$$

Once again, separation of variables yield

$$\frac{x - vt}{\sqrt{3(n+1)k}} + c_2 = \int^g \frac{dy}{\left[(n+1)v^2 \pm \sqrt{(n+1)^2 v^4 - 3(n+1)k\{4by^{n+1} - 2(n+1)ay^2 + c_1\}}\right]^{\frac{1}{2}}} \tag{23}$$

The following figure shows the profile of the traveling wave where the parameter values are $v = k = 1$, $a = b = -1$, with $n = 2$ while $c_1 = c_2 = 0$ (Fig. 3).

3.4. FORM-IV

The fourth form of the nonlinear dispersive KGE is given by

$$q_{tt} - k[3(q_x)^2 q_{xx}] + aq - bq^n + cq^{2n-1} = 0 \tag{24}$$

where n is a positive integer. If $n = 1$ or 2, this case collapses to the first two forms. Here, Eq. (6) reduces to

$$3n(n+1)k(g')^4 - 2n(n+1)v^2(g')^2 - 2n(n+1)ag^2 + 4nbg^{n+1} - 2(n+1)cg^{2n} + c_1 = 0 \tag{25}$$

where c_1 is a constant of integration. Solving for g' gives

$$g' = \frac{Q_1}{\sqrt{3n(n+1)k}} \tag{26}$$

where

$$Q_1 = \left[n(n+1)v^2 \pm \left(n^2(n+1)^2 v^4 - 3n(n+1)k\left\{4nby^{n+1} - 2n(n+1)ay^2 - 2(n+1)cy^{2n} + c_1\right\}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} \tag{27}$$

By the aid of separation of variables

$$\frac{x - vt}{\sqrt{3n(n+1)k}} + c_2 = \int^g \frac{dy}{Q_1} \tag{28}$$

The following figure shows the profile of the traveling wave where the parameter values are $v = k = 1$, $a = b = c = 1$, with $n = 2$ while $c_1 = c_2 = 0$ (Fig. 4).

3.5. FORM-V

The fifth and final form of the nonlinear dispersive KGE is given by

$$q_{tt} - k[3(q_x)^2 q_{xx}] + aq - bq^{1-n} + cq^{n+1} = 0 \tag{29}$$

where n is a positive integer. If $n = 1$ or 2, this case collapses to the first two forms. Here, Eq. (6) reduces to

$$3(n^2-4)k(g')^4 - 2(n^2-4)v^2(g')^2 - 2(n^2-4)ag^2 - 4(n+2)bg^{2-n} - 2(n-2)cg^{n+2} + c_1 = 0 \tag{30}$$

where c_1 is a constant of integration. Solving for g' gives

$$g' = \frac{Q_2}{\sqrt{3n(n+1)k}} \tag{31}$$

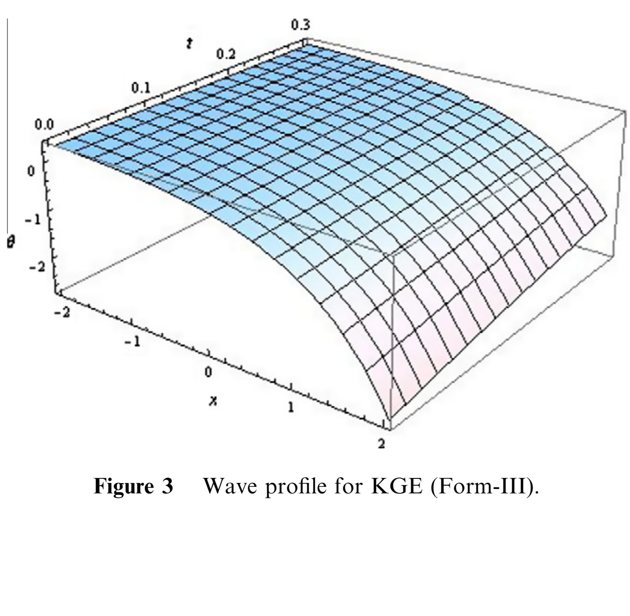

Figure 3 Wave profile for KGE (Form-III).

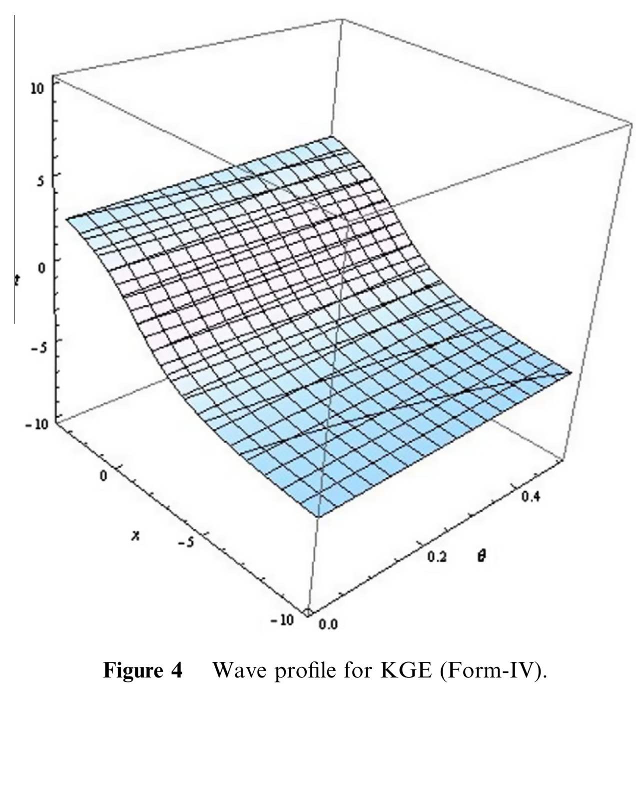

Figure 4 Wave profile for KGE (Form-IV).

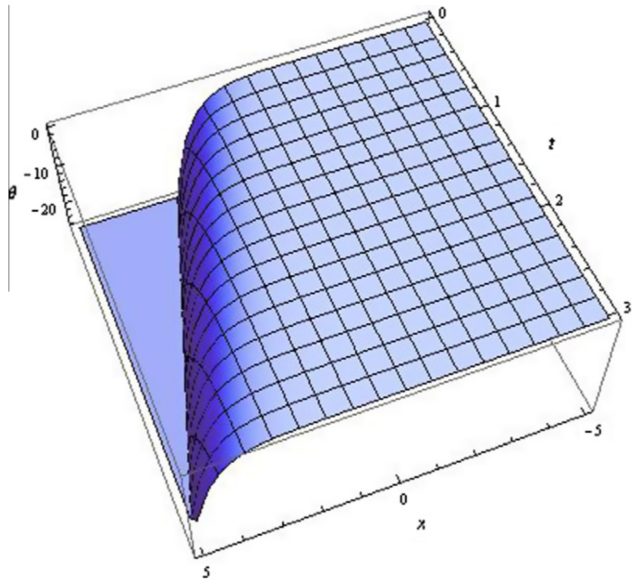

Figure 5 Wave profile for KGE (Form-V).

where

$$Q_2 = \left[(n^2-4)v^2 \pm \left((n^2-4)^2v^4 - 3(n^2-4)k\left\{2(n^2-4)ay^2 + (n+2)by^{2-n} + 4(n-2)cy^{n+2} + c_1\right\}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} \quad (32)$$

Thus, separating variables leads to

$$\frac{x-vt}{\sqrt{3n(n+1)k}} + c_2 = \int^g \frac{dy}{Q_2} \quad (33)$$

The following figure shows the profile of the traveling wave where the parameter values are $v = k = a = b = c = 1$, with $n = 4$ while $c_1 = c_2 = 0$ (Fig. 5).

4. Conclusions

The nonlinear KGE with nonlinear dispersive term is studied in this paper. There are five forms of nonlinearity that are considered. In each of these five cases, the traveling wave solution is obtained. It is observed that the solutions are in terms of quadratures. These integrals cannot be evaluated in terms of known functions. Thus the numerical solutions are given to each of these five cases in order to complete the analysis.

In future, this equation will be further investigated. The conservation laws will be evaluated. The perturbation terms will be taken into account and the adiabatic dynamics of the traveling waves will be evaluated. These results will be reported in future.

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