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Quintic non-polynomial spline methods for third order singularly perturbed boundary value problems

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Abstract In this paper, the non-polynomial spline function is used to find the numerical solution of the third order singularly perturbed boundary value problems of the reaction–diffusion equation type. The convergence analysis is discussed and the method is shown to have fourth order convergence. To validate the applicability of the method, two model examples have been solved for different values of the perturbation parameter and mesh sizes. The numerical results have been tabulated and also presented in graphs. It can be observed from the results that the present method approximates the exact solution very well.

1. Introduction

Any differential equation whose solution changes rapidly in some parts of the interval or domain is known as singular perturbation problem. These problems arise very frequently in diversified fields of applied mathematics and engineering, for instance in fluid mechanics hydrodynamics, quantum mechanics, chemical-reactor theory, aerodynamics, plasma dynamics, rarefied-gas dynamics, oceanography, meteorology, modeling of semiconductor devices, diffraction theory and reaction–diffusion processes and many other allied areas. The numerical solution of perturbed differential equation of the form of self-adjoint second order two point boundary value problems has been presented using the methods such as optimal quadratic and cubic spline collocation on non-uniform partitions (Christara and Ng, 2006), a fourth order adaptive collocation approach and patching approach (Khuri and Sayfy, 2012, 2014). Parametric Quintic and non-polynomial Quintic spline solutions have been presented for third-order boundary value problems and for the system of third order boundary value problems respectively (Khan and Sultana, 2012a,b).

A singular perturbation problem is said to be reaction diffusion type problem, if the order of differential equation is reduced by two (Phaneendral et al. 2012). Basically, the problem of ineffectiveness for solving singularly perturbed problems has been associated with the perturbation parameter. Accordingly, more efficient and simpler numerical methods are required to solve singularly perturbed two-point boundary value problems. In recent years, a large number of methods have been established to provide accurate results (Temsah, 2008; Rashidinia et al., 2007; Jalilian et al., 2015; Reza and Rashidinia, 2009; Ghazala, 2012, 2015; Ghazala and Imran, 2014; Sonali and Hradyesh, 2015). Those shows that a

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considerable amount of work has been done for the development of numerical methods to boundary value problems using various splines, yet there is lack of accuracy and convergence because the treatment of singular perturbation problems is not trivial distribution and the solution profile depends on perturbation parameter and mesh size h, (Doolan et al., 1980). It is necessary to develop efficient and accurate numerical methods for third order singularly perturbed problems. So,the purpose of this study is to develop a new spline method for the solution of third order singularly perturbed boundary value problem which is more accurate than the existing methods. This method depends on a non-polynomial spline function which has a trigonometric part and a polynomial part.

2. Description of the method

Consider the third order self adjoint singularly perturbed boundary value problem of the form:

$$Ly(x) \equiv -\varepsilon y'''(x) + u(x)y = f(x), \quad 0 \leqslant x \leqslant 1 \tag{1}$$

with boundary conditions

$$y(0) = \alpha_1, \quad y(1) = \beta_1, \quad y'(0) = \gamma \tag{2}$$

where $u(x) \geqslant u > 0$ and $\alpha_1, \beta_1, \gamma, u$ are constants and ε is a small positive parameter ($0 < \varepsilon << 1$), and $f(x)$ are sufficiently smooth functions. We consider a uniform mesh Δ with nodal points x_i on $[0,1]$ such that:

$$\Delta : 0 = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = 1, x_i = x_0 + ih,$$
$$i = 0, 1, \ldots, n; \quad \text{where} \quad h = \frac{1}{n}$$

The spline function we propose has a form $T_5 = span\{1, x, x^2, x^3, cos(kx), sin(kx)\}$ where k is the frequency of trigonometric part of the splines function which can be real or imaginary and will be used to raise the accuracy of the method.

For each segment $[x_i, x_{i+1}]$, $\mathrm{i} = 0, 1, \ldots, n-1$, the non-polynomial $S_\Delta(x)$ has the following form:

$$\begin{aligned} S_\Delta(x) &= a_i \cos k(x - x_i) + b_i \sin k(x - x_i) \\ &\quad + c_i(x - x_i)^3 + d_i(x - x_i)^2 + e_i(x - x_i) + f_i, \\ &\quad i = 1, 2, \ldots n - 1 \end{aligned} \tag{3}$$

where, a_i, b_i, c_i, d_i and e_i are constants.

Let $y(x)$ be the exact solution of Eq. (1) with boundary conditions Eq. (2) and S_i be an approximation to $y_i = y(x)$ obtained by the spline function $S_\Delta(x)$ passing through the points (x_i, S_Δ) and $(x_{i+1}, S_{\Delta+1})$. Following the technique of (Srivastava and Kumar, 2011):

$$\begin{matrix} S_\Delta(x_i) = y_i & S_\Delta(x_{i+1}) = y_{i+1} & \\ S''_\Delta(x_i) = D_i & S''_\Delta(x_{i+1}) = D_{i+1} & \text{for } i = 0, 1, \ldots, n-1 \\ S'''_\Delta(x_{i+1}) = T_{i+1} & S'''_\Delta(x_i) = T_i & \\ S^{(4)}_\Delta(x_i) = F_i & S^{(4)}_\Delta(x_{i+1}) = F_{i+1} & \end{matrix} \tag{4}$$

The coefficients in Eq. (3) using Eq. (4) are determined as:

$a_i = \frac{F_i}{k^4}$, $b_i = \frac{F_{i+1} - F_i \cos(\theta)}{k^4 \sin(\theta)}$, $c_i = \frac{D_{i+1} - D_i}{6h} + \frac{F_{i+1} - F_i}{6hk^2}$, $d_i = \frac{1}{2}(D_i + \frac{F_i}{k^2})$
$e_i = \frac{y_{i+1} - y_i}{h} + \frac{F_i - F_{i+1}}{hk^4} - \frac{h}{6}(D_{i+1} + 2D_i) - \frac{h}{6k^2}(F_{i+1} + 2F_i)$
$f_i = y_i - \frac{F_i}{k^4}$; for $\mathrm{i} = 0, 1, \ldots, \mathrm{n} - 1$ and $\theta = kh$.

Applying the first and second derivative continuity at knots, that is, $S^{(m)}_{\Delta-1}(x_i) = S^{(m)}_\Delta(x_i)$, for $m = 1, 3$, the following relations are derived:

$$\begin{aligned} D_{i-1} + 4D_i + D_{i+1} &= \frac{6}{h^2}(y_{i+1} - 2y_i + y_{i-1}) \\ &\quad - 6h^2(\lambda_1 F_{i-1} + 2\rho_1 F_i + \lambda_1 F_{i+1}) \end{aligned} \tag{5}$$

where $\lambda_1 = \frac{3h}{6k^2} - \frac{2h}{3k^2} + \frac{1}{k^2 \sin\theta}$ and $\rho_1 = \frac{6 \cot\theta}{hk^3}$

$$D_{i-1} - 2D_i + D_{i+1} = \lambda_2 F_{i+1} + 2\rho_2 F_i + \lambda_2 F_{i-1} \tag{6}$$

where $\lambda_2 = \frac{h}{k^3} - \frac{1}{k^2}$, $\rho_2 = \frac{1}{k^2} - \frac{hcot\theta}{k^3}$

Subtracting Eq. (6) from Eq. (5), we obtain:

$$\begin{aligned} D_i &= \frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1}) \\ &\quad - h^2\left(\left(\lambda_1 + \frac{\lambda_2}{6}\right)F_{i+1} + 2\left(\rho_1 + \frac{\rho_2}{6}\right)F_i + \left(\lambda_1 + \frac{\lambda_2}{6}\right)F_{i-1}\right) \end{aligned} \tag{7}$$

Using the continuity of third derivative and Eq. (7), we obtain the following relation:

$$\begin{aligned} T_i &= \frac{1}{h^3}(y_{i+2} - 3y_{i+1} + 3y_i - y_{i-1}) - h(pF_{i+2} + (p_0 - p + \alpha)F_{i+1} \\ &\quad + (p - p_0 + \beta)F_i - pF_{i-1}) \end{aligned} \tag{8}$$

where: $p = \lambda_1 + \frac{\lambda_2}{6}$, $p_0 = 2(\rho_1 + \frac{\rho_2}{6})$, $\beta = \frac{1}{\theta^2}(1 - \theta cot\theta)$ and $\alpha = \frac{1}{\theta^2}(\theta csc\theta - 1)$

We define the operator $\wedge$ by $\wedge w \equiv p(w_{i+2} + w_{i-2}) + sw_i + q(w_{i+1} + w_{i-1})$ for any function w evaluated at the mesh points. On applying this operator for T in Eq. (8), (Kumar and Srivastava, 2009), we have:

$$\wedge T_i \equiv pT_{i+2} + qT_{i+1} + sT_i + qT_{i-1} + pT_{i-2} \tag{9}$$

Substituting Eq. (8) into Eq. (9), we derive the following useful relation:

$$\wedge T_i \equiv \frac{1}{h^3}\left((\alpha + \beta)(y_{i+2} - y_{i-2}) + (2\alpha - 4\beta)(y_{i+1} + y_{i-1})\right) \tag{10}$$

Using Eqs. (9) and (10) we obtain:

$$\begin{aligned} &\frac{1}{h^3}\left((\alpha + \beta)(y_{i+2} - y_{i-2}) + (2\alpha - 4\beta)(y_{i+1} + y_{i-1})\right) \\ &\quad = pT_{i+2} + qT_{i+1} + sT_i + qT_{i-1} + pT_{i-2} \end{aligned} \tag{11}$$

where: $s = 2(\frac{1}{6}(\alpha + \beta) + \lambda_1 - 2\rho_1)$ and $q = 2(\frac{1}{6}(2\alpha + \beta) - (\lambda_1 + \rho_1))$ for $i = 2, 3 \ldots, n-2$

Rearranging Eq. (1) and evaluating at 'xi' we have:

$$y'''_i = \frac{uy_i}{\varepsilon} - \frac{f_i}{\varepsilon} \tag{12}$$

Substituting Eq. (12) into Eq. (11) and simplifying we obtain:

$$
\begin{aligned}
&((\alpha+\beta)\varepsilon + uph^3)y_{i-2} + ((2\alpha-4\beta)\varepsilon + quh^3)y_{i-1} + suh^3 y_i \\
&\quad - ((2\alpha-4\beta)\varepsilon + uqh^3)y_{i+1} - ((\alpha+\beta)\varepsilon + uph^3)y_{i+2} \\
&\qquad = -h^3(p(f_{i+2}+f_{i-2}) + sf_i + q(f_{i-1}+f_{i+1})), \\
&\qquad \text{for } i = 2\ldots n-2
\end{aligned} \tag{13}
$$

End conditions:

The system in Eq. (13) gives $(n-3)$ linear algebraic equations in the $(n-1)$ unknowns y_i, for $i = 1, 2, \ldots n-1$, So we need two more equations, at each ends. Following the procedure given by Reza and Rashidina (2009), the required end condition can be written as:

$$\sum_{l=0}^{3} b_l y_l + c_1 h y_0' + h^3 \sum_{l=0}^{3} d_l y_l''' + t_1 = 0, \quad i = 1 \tag{14}$$

$$\sum_{l=n-3}^{n} m_l y_l + h^3 \sum_{l=n-4}^{n} k_l y_l''' + t_{n-1} = 0, \quad i = n-1 \tag{15}$$

where b_l, c_1, d_l, m_l and k_l are arbitrary parameters which are to be calculated using method of undetermined coefficients.

For $l = 0, 1, \ldots n$. Eqs. (14) and (15) can be written as:

$$
\begin{aligned}
&c_1 h y_0' + b_0 y_0 + b_1 y_1 + b_2 y_2 + b_3 y_3 + h^3[d_0 y_0''' + d_1 y_1''' + d_2 y_2''' \\
&\quad + d_3 y_3'''] + t_1 = 0, \quad \text{for } i = 1
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
&m_{n-3}y_{n-3} + m_{n-2}y_{n-2} + m_{n-1}y_{n-1} + m_n y_n + h^3(k_{n-4}y_{n-4}''' \\
&\quad + k_{n-3}y_{n-3}''' + k_{n-2}y_{n-2}''' + k_{n-1}y_{n-1}''' + k_n y_n''') + t_{n-1} \\
&\quad = 0, \quad \textit{for } i = n-1
\end{aligned} \tag{17}
$$

Expanding each terms of Eq. (16) by Taylor's series about x_0. then, substituting and collecting coefficients of the same order we obtain:

$$
\begin{aligned}
&(b_0+b_1+b_2+b_3)y_0 + (b_1+2b_2+3b_3+c_1)hy_0' \\
&+(b_1+4b_2+9b_3)h^2 y_0'' \\
&+\left(\tfrac{b_1+8b_2+27b_3}{6} + d_0+d_1+d_2+d_3\right)h^3 y_0''' \\
&+\left(\tfrac{b_1+16b_2+81b_3}{24} + d_1+2d_2+3d_3\right)h^4 y_0^{(4)} \\
&+\left(\tfrac{b_1+32b_2+243b_3}{120} + \tfrac{d_1+4d_2+9d_3}{2}\right)h^5 y_0^{(5)} \\
&+\left(\tfrac{b_1+64b_2+729b_3}{720} + \tfrac{d_1+8d_2+27d_3}{6}\right)h^6 y_0^{(6)} \\
&+\left(\tfrac{b_1+128b_2+2187b_3}{5040} + \tfrac{d_1+16d_2+81d_3}{24}\right)h^7 y_0^{(7)} \\
&+\left(\tfrac{b_1+256b_2+6561b_3}{40320} + \tfrac{d_1+32d_2+243d_3}{120}\right)h^8 y_0^{(8)} + O(h^9) = 0
\end{aligned} \tag{18}
$$

Expanding Eq. (13) in Taylor's series about x_i, we obtain the following local truncation error:

$$
\begin{aligned}
t_i &= 4h(2\alpha-\beta)y_i' + \frac{1}{3}h^3(6p+6q+3s-10\alpha-4\beta)y_i^{(3)} \\
&\quad + \frac{1}{30}h^5(-30(4p+q)+17\alpha+14\beta)y_i^{(5)} \\
&\quad + \frac{1}{1260}h^7(-105(16p+q)+65\alpha+62\beta)y_i^{(7)} + O(h^8)
\end{aligned} \tag{19}
$$

Similarly, we can expand each term for Eq. (17), substituting and collecting coefficients of the same order.

Using Eq. (19) and eliminating the coefficients of various powers of h for different choices of parameters α, β, p, q and s, where $\alpha + \beta = \frac{1}{2}$,

And truncating the terms in Eq. (19) that contains h^7 and above, for arbitrary α and β provided that $\alpha + \beta = \frac{1}{2}$, the value of p, q and s are evaluated from:

$$
\begin{cases}
2p + 2q + s = 1 \\
120p + 30q = 7.5 \\
1680p + 105q = 31.5
\end{cases}; \quad \text{we obtain} : (\alpha, \beta, p, q, s) = \left(\frac{1}{6}, \frac{1}{3}, \frac{1}{240}, \frac{7}{30}, \frac{21}{40}\right)
$$

After equating each coefficient of orders of Eq. (18) with *0* we obtain the values of parameters

$$
\begin{aligned}
&(b_0, b_1, b_2, b_3, c_1, d_0, d_1, d_2, d_3) \\
&\quad = (320, -360, 0, 40, 240, -16, -90, -12, -2)
\end{aligned} \tag{20}
$$

In a similar manner we obtain parameters of the other end condition as:

$$
\begin{aligned}
&(m_{n-3}, m_{n-2}, m_{n-1}, m_n, k_{n-4}, k_{n-3}, k_{n-2}, k_{n-1}, k_n) \\
&\quad = (-120, 360, -360, 120, 0, 0, -60, -60, 0)
\end{aligned} \tag{21}
$$

Using Eqs. (20) and (21) in Eq. (16) and (17) respectively, with the help of Eq. (12) we obtain the two end conditions as:

$$
\begin{aligned}
&-(360\varepsilon + 90uh^3)y_1 - 12uh^3 y_2 + (40\varepsilon - 2uh^3)y_3 \\
&\quad = -240\varepsilon h\gamma - (320\varepsilon - 16uh^3)\alpha_1 \\
&\qquad - h^3(16f_0 + 90f_1 + 12f_2 + 2f_3), \quad \text{for } i = 1
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
&-120\varepsilon y_{n-3} + (360\varepsilon - 60uh^3)y_{n-2} - (360\varepsilon + 60uh^3)y_{n-1} \\
&\quad = -120\varepsilon\beta_1 - 60h^3(f_{n-1} + f_{n-2}), \quad \text{for, } i = n-1
\end{aligned} \tag{23}
$$

Hence, Eqs. (13), (22) and (23) gives penta - diagonal system for $i = 1, 2, \ldots, n-1$ and can easily be solved using Gauss-Elimination method.

3. Convergence analysis of the method

Consider the system of Eqs. (11), (22) and (23) in the matrix form as:

$$Ay + h^3 DF = C \tag{24}$$

where;

$$
A = \begin{bmatrix}
-360\varepsilon - 90uh^3 & -12uh^3 & 40\varepsilon - 2uh^3 & \ldots & \ldots & \ldots \\
-\varepsilon + quh^3 & suh^3 & \varepsilon + quh^3 & -0.5\varepsilon + uph^3 & \ldots & \ldots \\
0.5\varepsilon + uph^3 & -\varepsilon + quh^3 & suh^3 & \varepsilon + quh^3 & -0.5\varepsilon + uph^3 & \ldots \\
\ldots & \ddots & \ddots & \ddots & \ddots & \ldots \\
\ldots & \ddots & \ddots & \ddots & \ddots & \ldots \\
\ldots & \ddots & 0.5\varepsilon + uph^3 & -\varepsilon + quh^3 & suh^3 & \varepsilon + quh^3 \\
\ldots & \ldots & \cdots & -120\varepsilon & 360\varepsilon - 60uh^3 & -360\varepsilon - 60uh^3
\end{bmatrix}
$$

$$
D = \begin{bmatrix}
90 & -12 & 2 & \ldots & \ldots & \ldots \\
q & s & q & p & \ldots & \ldots \\
p & q & s & q & p & \ldots \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\ldots & \ldots & p & q & s & q \\
\ldots & \ldots & \cdots & \ldots & 60 & 60
\end{bmatrix}
$$

$$C=(c_1,c_2,\ldots c_{n-2},c_{n-1})^T,\ c_1=-240\varepsilon h\gamma-(320\varepsilon-16uh^3)-16h^3f_0$$
$$c_2=-ph^3f_0+(\varepsilon(\alpha+\beta)-uph^3)\alpha_1$$
$$c_i=0\ i=3,4\ldots,n-3\ \text{and}$$
$$c_{n-2}=(\varepsilon(\alpha+\beta)-uph^3)\beta_1-h^3pf_n$$
$$c_{n-1}=-120\varepsilon\beta_1$$

Also, $y=(y_1,y_2,\ldots y_{n-2},y_{n-1})^T$ and $F=(f_1,f_2,\ldots f_{n-2},f_{n-1})^T$

The exact solution is defined as $\bar{y}=(y(x_1), y(x_2),\ldots,y(x_{n-1}))$, Eq. (24) is written as

$$A\bar{y}-h^3DF=T+C \tag{25}$$

where $T=[t_1,t_2,\ldots,t_{n-1}]^T$ with truncation error:

$$t_1=\tfrac{2161}{140}\varepsilon h^7y^{(7)}(\varphi),\quad x_0<(\varphi)<x_3$$
$$t_i=-\tfrac{7}{288}\varepsilon h^7y^{(7)}(\varphi),\quad x_{i-2}<(\varphi)<x_{i+1}\quad \text{for } i=2,3,\ldots,n-1$$
$$t_{n-1}=\tfrac{1}{2}\varepsilon h^7y^{(7)}(\varphi),\quad x_{n-3}<(\varphi)<x_{n+1} \tag{26}$$

Moreover, $A(\bar{y}-y)=AE=T$ (27)

$$E=\bar{y}-y=(e_1,e_2,\ldots,e_{n-1})^T \tag{28}$$

To determine the error bounds the row sums, $s_1,s_2,\ldots,s_{n-1}$ of the matrix A are calculated

$$s_1=\sum_{j=1}^{n-1}a_{1j}=-320\varepsilon-104uh^3$$

$$s_2=\sum_{j=1}^{n-1}a_{2j}=-0.5\varepsilon-\frac{1314}{1805}uh^3$$

$$s_i=\sum_{j=1}^{n-1}a_{ij}=-\frac{41}{56}uh^3\ \text{for } i=3,\ldots,n-3$$

$$s_{n-2}=\sum_{j=1}^{n-1}a_{n-2j}=0.5\varepsilon-\frac{1314}{1805}uh^3$$

$$s_{n-1}=\sum_{j=1}^{n-1}a_{n-1j}=-120\varepsilon-120uh^3$$

Since $0<\varepsilon 1$, we can choose h sufficiently small so that the matrix A is irreducible and monotone. Using, Ghazala (2012) and Mohanty and Jha (2005), it follows that A^{-1} exists and its elements are non-negative. Hence, from Eq. (27), we get $E=A^{-1}T$

Also from the theory of matrices $A^{-1}A=I_{(n-1)\times(n-1)}$

Since each row of sum of matrix is $I_{(n-1)\times(n-1)}=1$ and $A^{-1}A=1$.

That is $a_{11}^{-1}(a_{11}+a_{12}+\ldots a_{1,n-1})+a_{12}^{-1}(a_{21}+a_{21}+\ \ldots a_{2,n-1})+\ldots+a_{1,n-1}^{-1}(a_{n-1,1}+a_{n-1,2}+\ldots a_{n-1,n-1})=1$

$\Rightarrow a_{11}^{-1}(s_1)+a_{12}^{-1}(s_2)+\ldots a_{n-1}^{-1}(s_{n-1})=1$

This is written in a compact form

$$\sum_{i=1}^{n-1}a_{k,i}^{-1}s_i=1,\ for\ k=1,2,\ldots,n-1. \tag{29}$$

Defining $S_j=\min s_i$, from Eq. (29)

$$S_j(a_{k,1}^{-1}+a_{k,2}^{-1}+\ldots+a_{k,n-1}^{-1})\leqslant 1$$

It follows that

$$\sum_{i=1}^{n-1}a_{k,i}^{-1}\leqslant\frac{1}{s_j}=\frac{1}{h^3B_{io}} \tag{30}$$

where $B_{io}=\left(\frac{1}{h^3}\right)s_j>0,1\leqslant io\leqslant n-1$

From Eq. (27), the error terms can be written as

$$e_j=\sum_{i=1}^{n-1}a_{j,i}^{-1}T_i,\quad j=1,2,\ldots,n-1 \tag{31}$$

Using Eqs. (26) and (30)

$$|e_k|\leqslant max\left\|\tfrac{1}{h^3B_{io}}\right\|\left\|\tfrac{7}{288}\varepsilon h^7y^{(7)}(\varphi)\right\|_{,x_{i-2}<(\varphi)<x_{i+2}}\leqslant\tfrac{1}{h^3B_i}\left(\tfrac{7\varepsilon h^7}{288}\right)max\left\|y^{(7)}(\varphi)\right\|_{,x_{i-2}<(\varphi)<x_{i+2}}$$
$$|e_k|\leqslant(\mu\varepsilon h^4)max\left\|y^{(7)}(\varphi)\right\|_{,x_{i-2}<(\varphi)<x_{i+2}}\ \text{for j}=1,2,\ldots,\text{n}-1$$

where μ is a constant and independent of h, and hence it follows that $\|E\|=O(h^4)$.

This result can be summarized in the following theorem.

Table 1 Maximum absolute errors for Example 1.

ε	N = 10	N = 20	N = 40
Our Method			
1/16	6.8572e−6	1.1698e−7	1.8578e−9
1/32	2.9156e−6	4.9916e−8	7.9252e−10
1/64	1.2223e−6	2.0005e−8	3.2111e−10
Sonali and Hradyesh (2015)			
1/16	4.7e−4	1.1e−4	2.6e−5
1/32	1.9e−4	4.7e−5	1.2e−5
1/64	8.0e−5	1.9e−5	4.8e−6
Ghazala (2012)			
1/16	2.9e−3	1.2e−4	6.4e−6
1/32	9.2e−4	3.8e−5	2.1e−6
1/64	1.4e−4	6.8e−6	4.6e−7

Table 2 Maximum absolute errors for Example 2

ε	N = 10	N = 20	N = 40
Our Method			
1/16	3.1247e−7	4.9269e−9	7.4543e−11
1/32	1.3421e−7	2.1095e−9	3.1741e−11
1/64	5.6587e−8	8.4937e−10	1.2904e−11
Ghazala and Imran (2014)			
1/16	5.70e−7	5.97e−8	4.14e−9
1/32	2.50e−7	2.52e−8	1.75e−9
1/64	1.00e−7	9.90e−9	6.8e−10
Sonali and Hradyesh (2015)			
1/16	2.4e−4	6.1×10^{-5}	1.5e−5
1/32	1.0e−5	2.6×10^{-5}	6.4e−6
1/64	4.0e−5	1.0×10^{-6}	2.5e−6
Ghazala (2012)			
1/16	1.3e−2	1.1e−3	7.8e−5
1/32	3.2e−3	2.7e−4	1.8e−5
1/64	3.4e−4	2.2e−5	1.1e−6

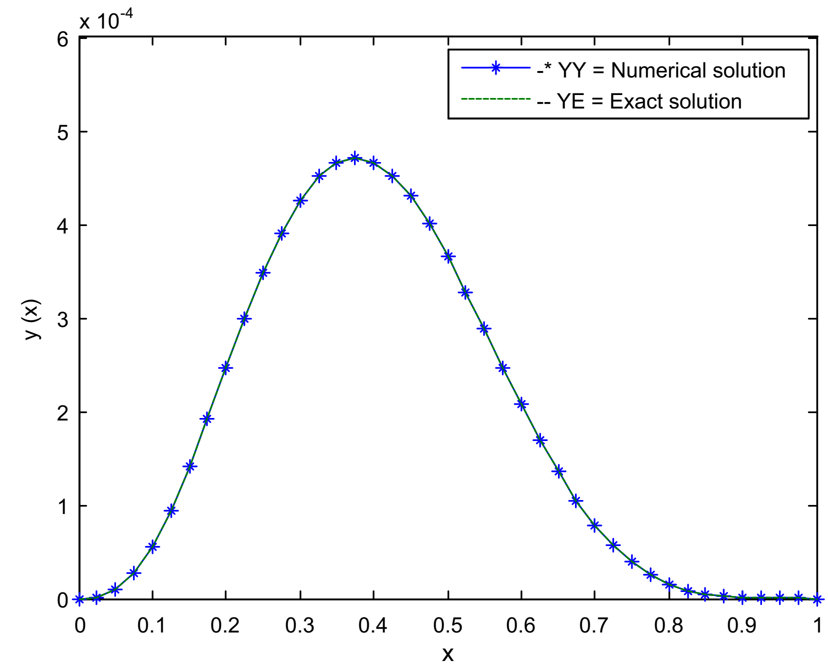

Figure. 1 The graph of exact and numerical solution of Example 1 for $N = 40$ and $\varepsilon = \frac{1}{64}$.

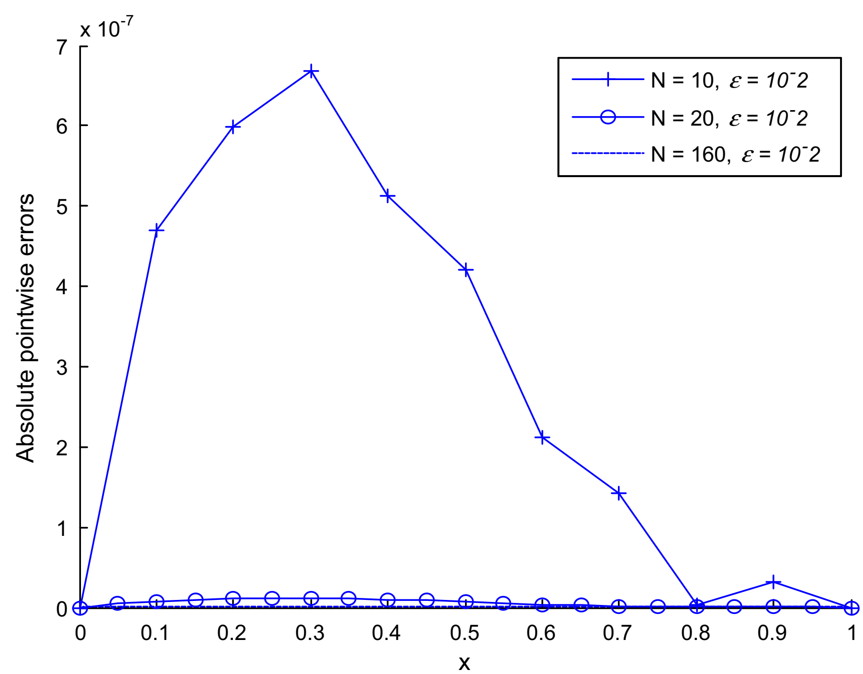

Figure. 2 Absolute errors decrease as the Number of mesh N increases for Example 1.

Theorem. The method given by Eqs. (13), (22) and (23) for solving the boundary value problem of Eqs. (1) and (2) for sufficiently small h gives a fourth order convergent solution.

4. Numerical examples and results

To demonstrate the validity of the methods, two model singularly perturbed problems have been considered. These examples have been chosen because they have been widely discussed in the literature and their exact solutions were available for comparison.

Example 4.1. Consider the following singularly perturbed problem.

$$-\varepsilon y'''(x) + y(x) = 6\varepsilon(1-x)^5 x^3 - 6\varepsilon^2\Big(6(1-x)^5 - 90(1-x)^4 x + 180(1-x)^3 x^2 - 60(1-x)^2 x^3\Big)$$

Table 3 Maximum absolute errors for the give two Examples at different ε and N.

ε	$N = 10$	$N = 20$	$N = 40$	$N = 80$	$N = 160$
Example 1					
10^{-1}	1.1975e−05	2.0219e−07	3.1957e−09	4.7607e−11	5.7583e−13
10^{-2}	6.6766e−07	1.1287e−08	1.7827e−10	2.7109e−12	3.6632e−14
10^{-3}	3.2989e−08	5.2434e−10	8.5508e−12	1.3163e−13	1.9575e−15
10^{-4}	1.2010e−09	2.7305e−11	3.8892e−13	6.3114e−15	9.6848e−17
10^{-5}	1.6891e−11	8.9636e−13	2.1008e−14	2.8475e−16	4.5782e−18
10^{-6}	1.7571e−13	1.1945e−14	6.2565e−16	1.5583e−17	2.1979e−19
10^{-7}	1.7642e−15	1.2335e−16	7.9310e−18	4.2141e−19	1.1354e−20
Example 2					
10^{-1}	5.4458e−07	8.5081e−09	1.2746e−10	3.8280e−12	2.5562e−12
10^{-2}	3.1090e−08	4.8206e−10	7.1755e−12	1.0009e−13	4.7587e−14
10^{-3}	1.6607e−09	2.3244e−11	3.5263e−13	5.2109e−15	4.0246e−16
10^{-4}	6.0523e−11	1.2982e−12	1.6745e−14	2.5603e−16	7.2777e−18
10^{-5}	9.5650e−13	4.2374e−14	9.5845e−16	1.1907e−17	1.3447e−19
10^{-6}	1.1282e−14	6.8915e−16	2.8422e−17	6.9320e−19	8.6027e−21
10^{-7}	1.1478e−16	7.7799e−18	4.5896e−19	1.8729e−20	4.9879e−22

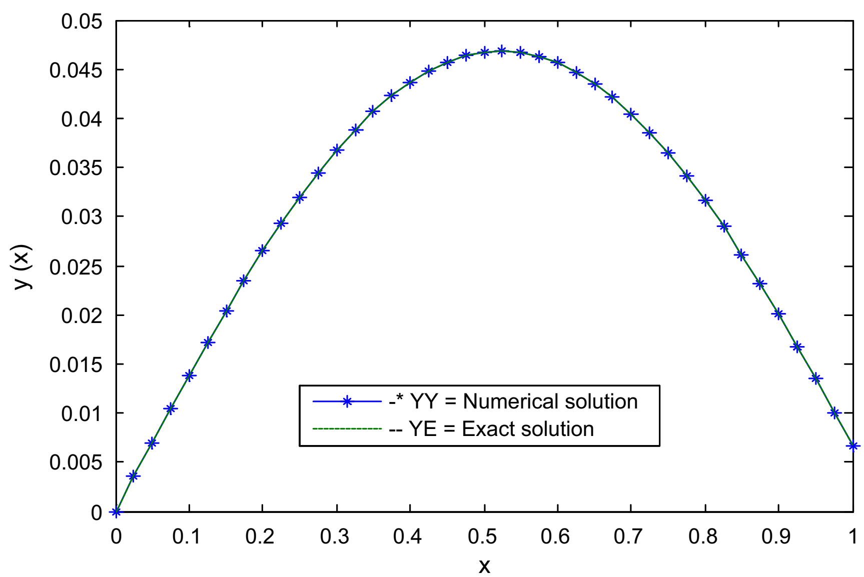

Figure. 3 The graph of exact and numerical solution of Example 2 for $N = 40$ and $\varepsilon = \frac{1}{64}$.

$y(0) = 0, \quad y(1) = 0, \quad y'(0) = 0$ for $0 \leqslant x \leqslant 1$

The analytic solution is $y(x) = 6x^3\varepsilon(1-x)^5$

The numerical solutions in terms of maximum absolute errors are given in Tables 1 and 3 with its graph in Figs. 1 and 2 as follows:

Example 4.2. Consider the following singularly perturbed problem.

$$-\varepsilon y'''(x) + y(x) = 81\varepsilon^2 \cos 3x + 3\varepsilon sin3x, \quad \text{for}, 0 \leqslant x \leqslant 1$$

$y(0) = 0, \quad y(1) = 3\varepsilon sin3, \quad y'(0) = 9\varepsilon$. The analytical solution is $y(x) = 3\varepsilon sin3x$.

The numerical results in terms of maximum absolute errors are tabulated in Tables 2 and 3 with its graph given in Fig. 3 and 4.

5. Discussion and conclusion

In this paper, the quintic non-polynomial spline method has been presented for solving third order singularly perturbed boundary value problems of the reaction–diffusion equation type. First, the given domain is discretized and the derivative of the given differential equation is replaced by the spline approximations. Then, the system is transformed to penta-diagonal system, which can easily be solved using any appropriate methods for solving the systems of linear equations. To validate the applicability of the proposed method, two model examples have been considered and solved for different values of perturbation parameter and different mesh sizes. The order of convergence has been established for the method and is convergent to order four. As it can be observed from the numerical results presented in Tables 1–3 and graphs (Figs 1

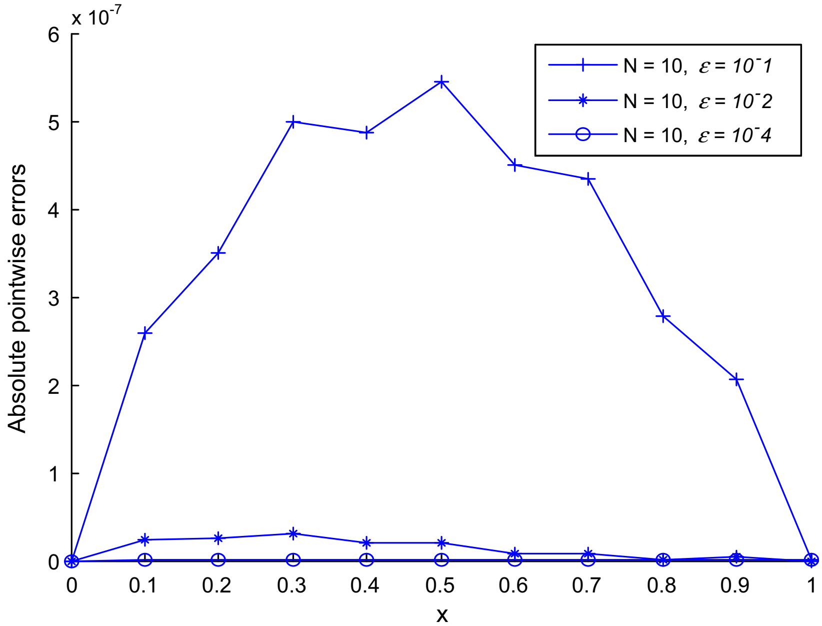

Figure. 4 Absolute errors decrease as perturbation parameter ε decreases for Example 2.

and 3), the present method approximates the exact solution very well. Moreover, the method has been analyzed by taking sufficiently small size of h and perturbation parameter where other existing numerical methods reported in the literature may fail.

The results obtained by the present method have been compared with the numerical results obtained by (Ghazala, 2012; Ghazala and Imran, 2014; and Sonali and Hradyesh, 2015) and is observed to be more accurate than the methods proposed by the aforementioned scholars. Furthermore, the absolute errors decrease rapidly as N increases and the perturbation parameter decreases (Figs. 2 and 4). This method can be extended to higher order Quintic non-polynomial spline methods for solving similar problems.

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