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Asymptotic stability and boundedness criteria for nonlinear retarded Volterra integro-differential equations

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ABSTRACT

In this article, we construct new specific conditions for the asymptotic stability (AS) and boundedness (B) of solutions to nonlinear Volterra integro-differential equations (VIDEs) of first order with a constant retardation. Our analysis is based on the successful construction of suitable Lyapunov–Krasovskii functionals (LKFs). The results of this paper are new, and they improve and complete that can be found in the literature.

1. Introduction

The Volterra integral equations (VIEs) and Volterra integro-differential equations (VIDEs) appeared after their establishment by Vito Volterra, in 1926. Thereafter they have wide applications in sciences and engineering. Namely, these equations appeared in many physical applications such as glass forming process, nanohydrodynamics, heat transfer, diffusion process in general, neutron diffusion and biological species coexisting together with increasing and decreasing rates of generating and wind ripple in the desert (see Wazwaz (2011)). More details about the sources where these equations arise can be found in physics, biology and engineering applications books. In addition, for more details of some of such applications, we referee the readers to the books of Burton (2005) and Wazwaz (2011). By this way we mean that it is worth and deserve to investigate properties of solutions of (VIDEs).

On the other hand, the important techniques used in the literature to search the qualitative behaviors (QBs) of paths of linear and non-linear (VIEs), (IDEs), (VIDEs), and etc., without finding the explicit solutions, are known as the second Lyapunov function(al) method, perturbation theory, fixed point method, the variation of constants formula and so on. In reality, we cannot find the analytical solutions of the equations mentioned, except very particular cases, and some time it become impossible to find the solutions, except numerically. Therefore it is an important need to use the former methods during the investigations.

Particularly, in the last four decades, researchers have produced a vast body of important results on the qualitative properties (QPs) of (VIDEs) by using the methods mentioned. In fact, several qualitative properties (QPs) of solutions; stability (S), boundedness (B), convergence (C), globally existence (GE) of solutions, etc., of different and the same models of linear and nonlinear (VIDEs) have been examined in the literature by many authors. For a comprehensive review and some recent results of (VIEs) and (VIDEs), we refer the reader to see (Atkinson, 1997; Becker, 2009; Brunner, 2004; Burton, 1979; Burton, 1982; Burton, 2005; Costarelli and Spigler, 2014; Furumochi and Matsuoka, 1999; Graef and Tunç, 2015; Graef et al., 2016; Hara et al.,1990; Hritonenko and Yatsenko, 2013; Maleknejad and Najafi, 2011; Miller,1971; Morchalo, 1991; Napoles Valdes, 2001; Raffoul, 2004; Raffoul, 2007; Raffoul, 2013; Staffans, 1988; Tunç, 2016a,b,c; Tunç, 2017a,b,c; Tunc and Ahyan, 2017; Tunç and Mohammed, 2017a,b; Tunç and Mohammed, 2017c; Vanualailai and Nakagiri, 2003; Zhang, 2005; Da Zhang, 1990 and the references therein). In that scientific sources, the authors obtained many interesting and valuable results on the (QPs) of specific (VIDEs). The mentioned authors benefited from the Lyapunov functions (LFs) or (LKFs) and obtained sufficient conditions which imply (S), (B), (C), etc. of solutions.

As renowned from this way, the following scientific works are notable.

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Morchalo (1991) considered the following scalar Volterra integro-differential equation

$$\frac{d}{dt}\left[x(t)-\int_0^t D(t,s)x(s)ds\right]=A(t)x(t)+\int_0^t C(t,s)x(s)ds \qquad (1)$$

with $x(t_0)=x_0$, where $t_0 \geqslant 0$, $x \in \Re$, $A(t)$ is a continuous function for $t \in J$, $J=[0,\infty)$, and $C(t,s)$ and $D(t,s)$ are continuous functions for $0 \leqslant s \leqslant t < \infty$.

The author discussed the (B) of solutions of the (VIDE) (1) by means of a Lyapunov function. The assumptions are constructed in (Morchalo, 1991) are given below.

Assumptions A Morchalo, 1991. Let

$$Z(t,x(t))=x(t)-\int_0^t D(t,s)x(s)ds$$

and

$$a(t,k)=A(t)+k\int_t^\infty |C(u,t)|du+\frac{1}{2}\int_0^t |A(t)D(t,s)+C(t,s)|ds.$$

It is assumed that the following assumption are true.

$(M1)$ *There are positive constants* m and M *such that*

$$x^2 \leqslant mZ^2(t,x) \text{if} |x| \leqslant M, t \in J, J=[0,\infty).$$

$(M2)$ *There is a positive constant* m_1 *such that*

$$|A(t)D(t,s)| \leqslant m_1|C(t,s)| \text{for all} 0 \leqslant s \leqslant t < \infty.$$

$(M3)$ *There is a positive constant b such that the following integral is convergent*;

$$\int_0^t |C(t,s)|Z^2(s,x(s))ds \leqslant b < \infty \text{for all} t \in J, |x| \leqslant M.$$

$(M4)$ *There is a positive constant c such that the following integral is convergent*;

$$\int_0^\infty \left(\int_0^t |C(t,s)|Z^2(s,x(s))ds\right)dt \leqslant c < \infty.$$

$(M5)$ *There are positive constants* a and k *such that*

$$a(t,k) \leqslant -a < 0 \text{for all} t \in J,$$

and

$$\frac{1}{2}mm_1-k \geqslant 0.$$

Theorem A Morchalo, 1991. *Let assumptions* (M1)–(M5) *be hold Then the solution* $x(t)=x(t,t_0,x_0)$ *of* (VIDE) (1) *is* f*-bounded.*

Besides, recently, Tunç (2017) considered the (VIDE) without delay of the form

$$\frac{d}{dt}\left[x(t)-\int_0^t D(t,s)x(s)ds\right]=-A(t)x(t)+\int_0^t C(t,s)x(s)ds+e(t,x) \qquad (2)$$

with $x(t_0)=x_0$, where $t_0 \geqslant 0$, $x \in \Re$, $A(t)$ and $e(t,x)$ are continuous functions for $t \in J$, $J=[0,\infty)$, and $J \times \Re$, respectively, and $C(t,s)$ and $D(t,s)$ are continuous functions for $0 \leqslant s \leqslant t < \infty$. The author investigated the (AS) and (B) of solutions of (VIDE) (2) by defining new suitable Lyapunov functions.

In this paper, instead of (VIDEs) (1) and (2), we are concerned with the (QPs) of solutions of nonlinear first order retarded (VIDEs) equations of the form of

$$\frac{d}{dt}\left[x(t)-\int_{t-\sigma}^t b(t,s)g(x(s))ds\right]=-a(t)x(t)+\int_{t-\sigma}^t c(t,s)g(x(s))ds+p(t,x(t),x(t-\sigma)) \qquad (3)$$

with $x(t_0)=x_0$, where $t-\sigma \geqslant 0$, σ is a positive constant, $x \in \Re$, $a(t)$, $g(x)$ and $p(t,x,x(t-\sigma))$ are continuous functions for $t \in \Re_+$, $\Re_+=[0,\infty)$, on $\Re$, and $\Re_+ \times \Re \times \Re$, respectively, and $b(t,s)$ and $c(t,s)$ are continuous functions for $0 \leqslant s \leqslant t < \infty$.

We assume throughout the paper that when we need x denotes $x(t)$, that is, $x=x(t)$.

For any $t_0 \geqslant 0$ and initial function $\phi \in C([t_0-\sigma,t_0])$, let $x(t)=x(t,t_0,\phi)$ denote the solution of (VIDE) (1) on $[t_0-\sigma,\infty)$ such that $x(t)=\phi(t)$ on $\phi \in C([t_0-\sigma,t_0])$.

Let $C[t_0,t_1]$ and $C[t_0,\infty)$ denote the set of all continuous real-valued functions on $[t_0,t_1]$ and $[t_0,\infty)$, respectively.

2. Stability and boundedness

Definition 2.1. Let $f:\Re_+ \to \Re$ be a continuous and non-negative function. The zero solution of

$$|y(t)-\int_{t-\sigma}^t b(t,s)y(s)ds| \leqslant f(t), y(t_0)=x_0 \qquad (4)$$

for $t \in \Re_+$ is said to be

$(A1)$ $f-$ (S) if for given each $\varepsilon > 0$ and each $t_0 \geqslant 0$, there exists a $\delta=\delta(\varepsilon,t_0)>0$ such that $\phi \in C[0,t_0]$,for all $t \in \Re_+$,$[|\phi| \leqslant \delta$and $f(t) \leqslant \delta] \Rightarrow |y(t,t_0,\phi)| \leqslant \varepsilon$,

$(A2)$ asymptotic $f-$ (S) if it is $f-$ (S) and

$$\lim_{t\to\infty}|y(t,t_0,\phi)|=0$$

for every $|\phi| \leqslant \delta$ and every $f(t) \to 0$ as $t \to \infty$,

$(A3)$ $f-$ bounded if for every bounded function $f:\Re_+ \to \Re$, there exists a bounded solution $y(t,t_0,\phi)$ of (4).

Assumptions A.

Let

$$\omega_1(t,\mu_1)=a(t)-\frac{1}{2}\int_{t-\sigma}^t |a(t)b(t,s)+c(t,s)|ds-\mu_1\int_{t-\sigma}^\infty |c(u+\sigma,t)|du.$$

$(H1)$ There exist positive constants m_1 and m_2 such that

$$|a(t)b(t,s)| \leqslant m_1|c(t,s)| \text{for} 0 \leqslant s \leqslant t < \infty$$

and

$$|g(x)| \leqslant m_2|x| \text{for} x \in \Re.$$

$(H2)$ There exist positive constants μ_1 and k_1 such that

$$\omega_1(t,\mu_1) \geqslant k_1 > 0 \text{for all} t \in \Re_+.$$

Let $p(t,x,x(t-\sigma)) \equiv 0$.

Theorem 2.2. If assumptions (H1)–(H2) are true, *then all solutions of* (VIDE) (3) *are* f-bounded.

Proof. We define a (LKF) $V_1(t) = V_1(t,x)$ by

$$V_1 = \frac{1}{2}[x - \int_{t-\sigma}^{t} b(t,s)g(x(s))ds]^2 + \mu_1 \int_0^t \int_{t-\sigma}^{\infty} |c(u + \sigma, s)|du x^2(s)ds, \quad (5)$$

where the constant $\mu_1 \in \Re$, $\mu_1 > 0$, and it is determined later in the proof. □

We see from (5) that $V_1(t,0) = 0$ and $V_1(t,x) > 0$ when $x \neq 0$. From this reason, it can be followed that (LKF) V_1 is positive definite.

By means of the (LKF) given in (5) and (VIDE) (3), a straightforward calculation implies that

$$\begin{aligned} V_1' &= [x - \int_{t-\sigma}^{t} b(t,s)g(x(s))ds] \\ &\quad \times \frac{d}{dt}[x - \int_{t-\sigma}^{t} b(t,s)g(x(s))ds] \\ &\quad + \mu_1 x^2 \int_{t-\sigma}^{\infty} |c(u+\sigma,t)|du - \mu_1 \int_0^t |c(t,s)|x^2(s)ds \\ &= -[x - \int_{t-\sigma}^{t} b(t,s)g(x(s))ds] \\ &\quad \times [a(t)x - \int_{t-\sigma}^{t} c(t,s)g(x(s))ds] \\ &\quad + \mu_1 x^2 \int_{t-\sigma}^{\infty} |c(u+\sigma,t)|du - \mu_1 \int_0^t |c(t,s)|x^2(s)ds \\ &= -a(t)x^2 + x\int_{t-\sigma}^{t} c(t,s)g(x(s))ds \\ &\quad + a(t)x \int_{t-\sigma}^{t} b(t,s)g(x(s))ds \\ &\quad - \int_{t-\sigma}^{t} b(t,s)g(x(s))ds \times \int_{t-\sigma}^{t} c(t,s)g(x(s))ds \\ &\quad + \mu_1 x^2 \int_{t-\sigma}^{\infty} |c(u+\sigma,t)|du \\ &\quad - \mu_1 \int_0^t |c(t,s)|x^2(s)ds. \end{aligned} \quad (6)$$

We now consider the second and third terms in (6). Indeed, by the hypotheses of Theorem 2.2 and the fact $2|ab| \leqslant a^2 + b^2$, we have

$$\begin{aligned} & x\int_{t-\sigma}^{t} c(t,s)g(x(s))ds + a(t)x\int_{t-\sigma}^{t} b(t,s)g(x(s))ds \\ &= x\int_{t-\sigma}^{t} [a(t)b(t,s) + c(t,s)]g(x(s))ds \\ &\leqslant \frac{1}{2}\int_{t-\sigma}^{t} |a(t)b(t,s) + c(t,s)|x^2(t)ds \\ &\quad + \frac{1}{2}\int_{t-\sigma}^{t} |a(t)b(t,s) + c(t,s)|g^2(x(s)))ds \\ &= \frac{1}{2}x^2\int_{t-\sigma}^{t} |a(t)b(t,s) + c(t,s)|ds \\ &\quad + \frac{1}{2}m_2^2\int_{t-\sigma}^{t} |a(t)b(t,s) + c(t,s)|x^2(s)ds \\ &\leqslant \frac{1}{2}x^2\int_{t-\sigma}^{t} |a(t)b(t,s) + c(t,s)|ds \\ &\quad + \frac{1}{2}m_2^2\int_{t-\sigma}^{t} (|a(t)b(t,s)| + |c(t,s)|)x^2(s)ds \\ &\leqslant \frac{1}{2}x^2\int_{t-\sigma}^{t} |a(t)b(t,s) + c(t,s)|ds \\ &\quad + \frac{1}{2}m_2^2(m_1+1)\int_{t-\sigma}^{t} |c(t,s)|x^2(s)ds. \end{aligned} \quad (7)$$

Then, by (6) and (7), a simple computation shows that

$$\begin{aligned} V_1' &\leqslant -a(t)x^2 + \frac{1}{2}x^2\int_{t-\sigma}^{t} |a(t)b(t,s) + c(t,s)|ds \\ &\quad + \frac{1}{2}m_2^2(m_1+1)\int_{t-\sigma}^{t} |c(t,s)|x^2(s)ds \\ &\quad + \mu_1 x^2 \int_{t-\sigma}^{\infty} |C(u+\sigma,t)|du - \mu_1\int_0^t |c(t,s)|x^2(s)ds \\ &\quad - \int_{t-\sigma}^{t}\int_{t-\sigma}^{t} b(t,s)c(t,u)x(s)x(u)dsdu \\ &= -[a(t) - \frac{1}{2}\int_{t-\sigma}^{t} |a(t)b(t,s) + c(t,s)|ds]x^2 \\ &\quad + [\mu_1 \int_{t-\sigma}^{\infty} |c(u+\sigma,t)|du]x^2 \\ &\quad - [(\mu_1 - \frac{1}{2}m_2^2(m_1+1))\int_{t-\sigma}^{t} |c(t,s)|]x^2(s)ds \\ &\quad - \mu_1\int_0^{t-\sigma} |c(t,s)|x^2(s)ds \\ &\quad - \int_{t-\sigma}^{t}\int_{t-\sigma}^{t} b(t,s)c(t,u)g(x(s))g(x(u))dsdu. \end{aligned} \quad (8)$$

Let

$$\mu_1 = \frac{1}{2}m_2^2(m_1+1)$$

and

$$\int_{t-\sigma}^{t}\int_{t-\sigma}^{t} b(t,s)c(t,u)g(x(s))g(x(u))dsdu \geqslant 0.$$

Then, by the assumption $(H2)$, that is, $\omega_1(t,\mu_1) \geqslant k_1 > 0$, we obtain

$$\begin{aligned} V_1' &\leqslant -[a(t) - \frac{1}{2}\int_{t-\sigma}^{t} |a(t)b(t,s) + c(t,s)|ds]x^2 \\ &\quad + [\mu_1 \int_{t-\sigma}^{\infty} |c(u+\sigma,t)|du]x^2 \leqslant -k_1x^2 \end{aligned}$$

so that $V_1'(t,x(t)) \leqslant 0$.

Integrating the inequality $V_1'(t,x(t)) \leqslant 0$ from $t_0(\geqslant 0)$ to t, we obtain

$V_1(t,x(t)) \leqslant V(t_0,x(t_0))$.

Hence, the proof is complete by observing that

$$\begin{aligned} &\frac{1}{2}[x - \int_{t-\sigma}^{t} b(t,s)x(s)ds]^2 + \mu_1\int_0^t\int_{t-\sigma}^{\infty} |C(u+\sigma,s)|du x^2(s)ds \\ &= V_1(t,x(t)) \leqslant V(t_0,x(t_0)) = K_1 > 0, K_1 \in \Re. \end{aligned}$$

Corollary 2.3. *If the assumptions of Theorem 2.2 hold, then the trivial solution of (VIDE) (3) is (AS). In fact, $V_1 = V_1(t,x(t))$ is positive definite and we find $V_1' = V_1'(t,x(t)) \leqslant -k_1x^2(t)$. This result guarantees that all solutions of (VIDE) (3) are (f-AS) (since V_1 is positive definite and V_1' is negative definite).*

Let $p(t,x,x(t-\sigma)) \neq 0$

and

$$\begin{aligned} \omega(t,\mu_2) &= a(t) - \frac{1}{2}m_2^2(m_1+1)\int_{t-\sigma}^{t} |c(t,s)|ds \\ &\quad - \mu_2\int_{t-\sigma}^{\infty} |c(u+\sigma,t)|du - q(t) - \frac{1}{2}q(t)\int_{t-\sigma}^{t} |b(t,s)|ds. \end{aligned}$$

.

Assumption B. ($H3$) There exist positive constants μ_2 *and* k_2 such that

$\omega_2(t,\mu_2) \geqslant k_2 > 0 for all t \in \Re_+,$

and

$|p(t,x,x(t-\sigma))| \leqslant q(t)|x|,$

where $q(t)$ is a non-negative and continuous function for all $t \in \Re_+$.

Theorem 2.4. *If suppose that assumptions* $(H1)-(H3)$ *are true, then all solutions of (VIDE)* (3) *are* f-bounded.

Proof. In the proof, we benefit from (LKF) $V_2(t) = V_2(t,x(t))$ defined by

$$V_2 = \frac{1}{2}[x - \int_{t-\sigma}^{t} b(t,s)g(x(s))ds]^2 + \mu_2 \int_0^t \int_{t-\sigma}^{\infty} |c(u+\sigma,s)|dux^2(s)ds, \tag{9}$$

where $\mu_2 \in \Re$, $\mu_2 > 0$, we specific it later in the proof. □

By the assumptions of Theorem 2.4, (8) and (VIDE) (3), it can be easily obtained that

$$\begin{aligned}
V_2' \leqslant & -[a(t) - \frac{1}{2}\int_{t-\sigma}^{t} |a(t)b(t,s)+c(t,s)|ds]x^2 \\
& + [\mu_2 \int_{t-\sigma}^{\infty} |c(u+\sigma,t)|du]x^2 \\
& - [(\mu_2 - \frac{1}{2}m_2^2(m_1+1))\int_{t-\sigma}^{t} |c(t,s)|]x^2(s)ds \\
& - \mu_2 \int_0^{t-\sigma} |c(t,s)|x^2(s)ds \\
& + p(t,x,x(t-\sigma))x \\
& - p(t,x,x(t-\sigma))\int_{t-\sigma}^{t} b(t,s)x(s)ds.
\end{aligned}$$

We consider the terms

$$p(t,x,x(t-\sigma))x - p(t,x,x(t-\sigma))\int_{t-\sigma}^{t} b(t,s)x(s)ds.$$

In view of the assumption ($H3$) and the inequality $2|ef| \leqslant |e|^2 + |f|^2$ we have

$$\begin{aligned}
& p(t,x,x(t-\sigma))x - p(t,x,x(t-\sigma))\int_{t-\sigma}^{t} b(t,s)x(s)ds. \\
& \leqslant |p(t,x,x(t-\sigma))||x| \\
& \quad + |p(t,x,x(t-\sigma))|\int_{t-\sigma}^{t} |b(t,s)||x(s)|ds \\
& \leqslant q(t)x^2 + q(t)|x|\int_{t-\sigma}^{t} |b(t,s)||x(s)|ds \\
& \leqslant q(t)x^2 + \frac{1}{2}q(t)\int_{t-\sigma}^{t} |b(t,s)|(x^2 + x^2(s))ds \\
& = [q(t) + \frac{1}{2}q(t)\int_{t-\sigma}^{t} |b(t,s)|ds]x^2 \\
& \quad + \frac{1}{2}q(t)\int_{t-\sigma}^{t} |b(t,s)|x^2(s)ds.
\end{aligned}$$

Then

$$\begin{aligned}
V_2' \leqslant & -\left[a(t) - \frac{1}{2}\int_{t-\sigma}^{t} |a(t)b(t,s)+c(t,s)|ds\right]x^2 \\
& + \left[\mu_2 \int_{t-\sigma}^{\infty} |c(u+\sigma,t)|du + q(t) + \frac{1}{2}q(t)\int_{t-\sigma}^{t} |b(t,s)|ds\right]x^2 \\
& - \left[(\mu_2 - \frac{1}{2}m_2^2(m_1+1))\int_{t-\sigma}^{t} |c(t,s)|\right]x^2(s)ds \\
& + \left[\frac{1}{2}q(t)\int_{t-\sigma}^{t} |b(t,s)|\right]x^2(s)ds - \mu_2 \int_0^{t-\sigma} |c(t,s)|x^2(s)ds \\
\leqslant & -\left[a(t) - \frac{1}{2}m_2^2(m_1+1)\int_{t-\sigma}^{t} |c(t,s)|ds\right]x^2 \\
& + \left[\mu_2 \int_{t-\sigma}^{\infty} |c(u+\sigma,t)|du\right]x^2 \\
& + \left[q(t) + \frac{1}{2}q(t)\int_{t-\sigma}^{t} |b(t,s)|ds\right]x^2 \\
& - \left[(\mu_2 - \frac{1}{2}m_2^2(m_1+1))\int_{t-\sigma}^{t} |c(t,s)|\right]x^2(s)ds \\
& + \left[\frac{1}{2}q(t)\int_{t-\sigma}^{t} |b(t,s)|\right]x^2(s)ds \\
\leqslant & -[\omega_2(t,\mu_2)]x^2 \\
& - \left[\left\{(\mu_2 - \frac{1}{2}m_2^2(m_1+1))\int_{t-\sigma}^{t} |c(t,s)| + q(t)\int_{t-\sigma}^{t} |b(t,s)|\right\}\right]x^2(s)ds.
\end{aligned}$$

Let

$$\mu_2 > m_2^2(m_1+1) > 0$$

such that

$$(\mu_2 - \frac{1}{2}m_2^2(m_1+1))\int_{t-\sigma}^{t} |c(t,s)|x^2(s)ds \geqslant q(t)\int_{t-\sigma}^{t} |b(t,s)|x^2(s)ds$$

Hence, we get

$$V_2' \leqslant -[\omega_2(t,\mu_2)]x^2 \leqslant 0.$$

By the integration of the estimate $V_2'(t) \leqslant 0$ from zero t_0 to t, it follows that

$$\begin{aligned}
V_2 = & \frac{1}{2}[x - \int_{t-\sigma}^{t} b(t,s)g(x(s))ds]^2 \\
& + \mu_2 \int_0^t \int_{t-\sigma}^{\infty} |C(u+\sigma,s)|dux^2(s)ds \\
= & V_2(t,x(t)) \leqslant V_2(t_0,x(t_0)) = K_2 > 0, K_2 \in \Re.
\end{aligned}$$

This result guarantees the f-boundedness of solutions of (VIDE) (3) considered.

3. Conclusion

We consider a specific kind of non-linear functional (VIDEs) of first order with constant retardation. We investigate the (AS) and (B) of solutions that (VIDEs) by the (LKFs) approach. The results of this paper are new and they have a novelty and improve some results can be found in the literature (Morchalo, 1991; Tunç, 2017b).

In addition, the results obtained here complement that results can be found in the literature on (QPs) of solutions of (VIDEs) without or with retardation (see the references of this paper and that in the literature).

Finally, the improvement obtained in the present paper can explained by the following details:

1^0) It is notable that when we compare (VIDE) (3) investigated here with (VIDE) (1), which studied by Morchalo (1991), it follows that (VIDE) (1) is linear, however, (VIDE) (3) is non-linear, and it has also a constant retardation. In addition, if we choose $p(t,x,x(t-\sigma)) = 0$, $g(x) = x$, $b(t,s) = D(t,s)$, $c(t,s) = C(t,s)$, $-a(t) = A(t)$, and take zero $''0''$ instead of $t-\sigma$, then (VIDE)(3) reduces to (VIDE) (1). That is, (VIDE) (3) includes and improves (VIDE) (1) from the linear case to the non-linear case and retarded (VIDEs). The statements just mentioned show the improvement obtained and one of the contribution of this work to the relevant literature.

2^0) When we compare (VIDE) (3) considered in this paper with (VIDE) (2), which studied by Tunç (2017b), it can be seen that (VIDE) (2) is linear provided that $e(t,x) = 0$, and it is without retardation. However, (VIDE) (3) is non-linear, and it has also a constant retardation. In addition, if we choose $p(t,x,x(t-\sigma)) = e(t,x)$, $g(x) = x, a(t) = A(t)$, and take zero $''0''$ instead of $t-\sigma$, then (VIDE) (3) reduces to (VIDE) (2). This information shows clearly the other improvement done by the present paper and display clearly another contribution of the present paper to the literature.

3^0) On the other hand, it is notable that the (LKF) makes necessary to construct a suitable auxiliary functional which gives meaningful result(s) for the problem under investigation. Indeed, there is no general method for constructing such (LKFs) in the literature. Moreover, the problem of Lyapunov- Krasovskii characterization of (S), (AS) and (B) of nonlinear retarded (VIDEs) with non-smooth (LKFs) has remained as an unsolved problem in the related literature by present time, and hence the need continued and is still maintaining for researchers benefiting from that auxiliary functionals.

To arrive the aim of this paper, we construct a new suitable (LKF), and by that auxiliary functional we discuss the (AS) and (B) of solutions of (VIDE) (3).To the best of our information, (AS) and (B) of retarded (VIDEs) of the form of (VIDE) (3) were not discussed by this time in the literature. This paper may be the first attempt in the literature on the topic for that kind of retarded (VIDEs). The results established are also different from that found in the literature (see, Atkinson, 1997; Becker, 2009; Brunner, 2004; Burton, 1979, 1982, 2005; Costarelli and Spigler, 2014; Furumochi and Matsuoka, 1999; Graef and Tunç, 2015; Hara et al.,1990; Hritonenko and Yatsenko, 2013; Maleknejad and Najafi, 2011; Miller,1971; Morchalo, 1991; Napoles Valdes, 2001; Raffoul, 2004, 2007, 2013; Staffans, 1988; Tunç, 2016a,b; Tunç, 2017a,b,c; Vanualailai and Nakagiri, 2003; Zhang, 2005; Da Zhang, 1990 and the references therein). By this way, we would like to mean that the retarded functional (VIDEs) considered and the assumptions established here are different from those currently can be found in the literature and the references of this paper. The information just mentioned indicates the novelty and originality of the present paper.

4^0) Finally, to the best of our knowledge the results of Morchalo (1991) are not true in the general case. We would not like to give here the details of that problems in Morchalo (1991). By present work, we revise, correct and improve the results of Morchalo (1991) for the scalar case, and we obtain the result of Morchalo (1991) under more weaker conditions for the cases of with and without retardation in (VIDE) (3). This case can be seen when we compare assumptions $(M1)-(M5)$ of Morchalo (1991) with that assumptions $(H1)-(H2)$ given above. In addition, we improve of the results of Tunç (2017b) from the situation without retardation to the more general situations with retardation.

In view of all the information mentioned, it can be checked and seen the new and novel properties of the present paper and the improvement obtained in this paper.

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