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# Closed-form solution of optimal control problem of a fractional order system

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ABSTRACT

A formulation and an analytical solution technique of fractional optimal control problem (FOCP) is presented in this paper. The performance index (PI) considered is a conformable fractional integral (CFI) function and is a function of both the state and the control variables. Dynamic behaviour of the system is described by conformable fractional differential equation (FDE). The necessary conditions of optimality and the general transversality condition in terms of Hamiltonian are obtained using variational approach. Both the fixed and free final end point conditions have been considered. An analytical solution technique is presented for solving the conformable FOCPs. To validate the formulation and solution scheme numerical examples are presented.



## 1. Introduction

Fractional calculus has been attracted by many researchers from last few decades in different fields because systems described using FDEs give more accurate behaviour (Abuasad et al., 2019; Albzeirat et al., 2018; Dzielinski et al., 2010; Katsikadelis, 2015; Phaochoo et al., 2016; Tarasov, 2016). It should be mentioned here that impedance of a real capacitor is $\frac{1}{Cs^{\alpha}}, (0 < \alpha < 1)$, where $C$ is a constant nearly equal to the conventional capacitance (Westeriund and Ekstam, 1994). This leads to a fractional order model and it contains most of the properties of the real capacitor. The capacitor remembers voltages it has been subjected to earlier. As fractional derivative represents nonlocal property or memory, FDE represents better description of the dynamic system. Many applications of fractional derivative could be found in (Bagley and Torvik, 1984; Diethelm, 2013; Ding et al., 2012; He and Li, 2016; Heymans, 2004; Jenson and Jeffreys, 1977; Luo et al., 2018; Mandelbrot, 1982; Rangaig and Convicto, 2018; Suribabu and Chiranjeevi, 2016) and references therein. Therefore, this subject is developing extensively.

Several definitions of a fractional derivative (FD) exist in the literature. For example, Riemann-Liouville (RL), Caputo, Riesz, Riesz-Caputo, Grünwald-Letnikov (GL) and Weyl FDs (Oldham and Spanier, 1974; Podlubny, 1999). All these FDs are defined via fractional integrals, so they represent nonlocal properties. As the FD is not a local property and it takes into account the effect of history, perhaps for this reason system description using FDE is more accurate. However, some discrepancies arise with these FDs because these derivatives in general do not follow chain rule, product rule, quotient rule, etc. Due to these discrepancies mathematical analysis becomes difficult. To overcome some of these difficulties, Kolwankar and Gangal (1996) define a type of local fractional derivative (LFD) using Riemann-Liouville formulation of fractional derivative.

It has been observed in the literature (Adda and Cresson, 2001; Babakhani and Daftardar-Gejji, 2002; Chen et al., 2010; Xiaojun and Baleanu, 2013), several authors have extended the properties of LFDs and its applications. The advantage of LFD is that it is limit based rather than defined via fractional integral. Recently, Khalil et al. (2014) introduce another kind of limit based FD called conformable fractional derivative (CFD). It is interesting to note that CFD follows most of the properties of integer derivative. Therefore, it is advantageous to use CFD from mathematical analysis point of view. The detailed analysis of CFDs and various solution methods of conformable FDEs are presented in (Abdeljawad, 2015; Abu Hammad and Khalil, 2014a,b,c,d; Atangana, 2015; Atangana et al., 2015; Batarfi et al., 2015; Bayour and Torres, 2017; Benkhettou et al., 2016; Khalil and Abu-Shaab, 2015; Lazo and Torres, 2017; Salahshour et al., 2015; Ünal et al., 2015, 2017;

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Ünal and Gökdoğan, 2017; Gökdoğan et al., 2016; Yang et al., 2016). Moreover, applications of CFD in different fields are found in (Avci et al., 2016,2017a,2017b; Çenesiz and Kurt, 2015, Çenesiz et al., 2017; Chung, 2015; Eslami and Rezazadeh, 2016; Kurt et al., 2015).

In Katugampola (2014), CFD is generalized and author introduces a new LFD which is also following most of the classical properties of integer derivative. Anderson and Ulness (2015) follow Katugampola's derivative and address different applications in quantum mechanics. Zhao and Luo (2017) introduce a new LFD, called "general conformable fractional derivative" (GCFD), and also address the physical interpretation of GCFD. It is demonstrated that the CFD is a special case of GCFD. Also, the CFD has clear physical interpretations (Zhao and Luo, 2017). Therefore, CFD deals better with the dynamics of complex systems than other FDs in the literature. In the present work, we explore the application of CFDs and CFIs in FOCPs.

The problem of finding state and control variables which minimizes a given PI subjected to system constraints is referred as optimal control problem (OCP) (Lewis and Syrmos, 1995; Naidu, 2003). So far most of the FOCPs have been reported in terms of Riemann-Liouville ( Agrawal, 2004; Agrawal and Baleanu, 2007; Baleanu et al., 2009; Biswas and Sen, 2010, 2011b, 2014; Dzielinski and Czyronis, 2012,2013; Heydari et al., 2016; Keshavarz et al., 2016; Lotfi et al., 2011), Caputo (Agrawal, 2008; Alizadeh and Effati, 2018; Almeida and Torres, 2015; Biswas and Sen, 2011a; Chiranjeevi and Biswas, 2017; Dehghan et al., 2016; Ezz-Eldien et al., 2017; Javad Sabouri et al., 2017; Lotfi et al., 2013; Nemati and Yousefi, 2016) FDs and few works with other FDs like Riesz (Agrawal, 2007), Riesz-Caputo (Almeida, 2012; Frederico and Torres, 2010; Jarad et al., 2012) and combined Caputo (Odzijewicz et al., 2012). In this regard, Agrawal (2004), Agrawal and Baleanu (2007), Agrawal (2008) and Baleanu et al. (2009) present formulation and different solution schemes for FOCPs. Biswas and Sen (2010, 2011a) propose FOCP formulation and solution scheme using pseudo-state-space approach. Variational iteration method is used in Alizadeh and Effati (2018) for the solution of FOCPs. Almeida and Torres (2015) propose a discrete method for the solution of FOCPs. Biswas and Sen (2011b, 2014) investigate the formulation and solution of free and fixed final end point conditions of FOCPs. Solution of very large class of FOCPs by means of rational approximation is presented in Tricaud and Chen (2010). Fractional optimal control problem solution method based on Legendre wavelets is presented in Heydari et al. (2016). Javad Sabouri et al. (2017) obtain an approximate solution of FOCP using neural networks. An FOCP of discrete-time system with fixed final time have been investigated in (Dzielinski and Czyronis, 2012, 2013; Chiranjeevi and Biswas, 2017). Formulation of both continuous-time and discrete-time FOCPs with control constraints have been presented in Chiranjeevi and Biswas (2018). Dehghan et al. (2016) investigate the solution of FOCPs using the modified Jacobi polynomials. Numerical solution of FOCPs based on Legendre orthonormal polynomials and shifted Legendre orthonormal polynomials are presented in Lotfi et al. (2011, 2013), Ezz-Eldien et al. (2017) and Nemati and Yousefi (2016). The method based upon Bernoulli polynomials is presented in Keshavarz et al. (2016) for solving FOCPs. A Bezier curve method is presented in Ghomanjani (2016) for solving FOCPs and fractional Riccati differential equation.

All the above works consider FOCPs in terms of RL, Caputo, Riesz, Riesz-Caputo FDs. Formulation of FOCP results left and right FD problem. Hence, the analytical solution is difficult. Existing works in the literature present numerical solutions of FOCPs. Analytical solution of FOCP is still an open problem. In this respect this work presents FOCP in terms of CFD. The order of the FD has taken as $0 < \alpha < 1$. PI is considered in quadratic form. Both the cases of fixed final time OCP have been considered. Hamiltonian approach is used for obtaining the optimal conditions. The necessary conditions are solved using analytical method. With the best of authors' knowledge this is the first time an analytical solution of FOCP is presented in this paper. However, FOCP in the sense of CFD is already considered in Lazo and Torres (2017). The contribution of the present work is problem formulation for different end point conditions and analytical solution technique.

## 2. Mathematical preliminaries

The left conformable fractional derivative of order $\alpha \in (0,1]$ starting from $a \in \mathbb{R}$ of a unction $f : [a,b] \to \mathbb{R}$ is defined by Lazo and Torres (2017)

$$\frac{d_a^\alpha}{dt_a^\alpha} f(t) = f_a^{(\alpha)}(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon(t-a)^{1-\alpha}\right) - f(t)}{\varepsilon} \tag{1a}$$

and the right conformable fractional derivative of order $0 < \alpha \leq 1$ terminating at $b \in \mathbb{R}$ of a function $f : [a,b] \to \mathbb{R}$ is defined by

$$\frac{{}_b d^\alpha}{{}_b dt^\alpha} f(t) = {}_b f^{(\alpha)}(t) = -\lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon(b-t)^{1-\alpha}\right) - f(t)}{\varepsilon} \tag{1b}$$

The left conformable fractional integral of order $\alpha \in (0,1]$ starting from $a \in \mathbb{R}$ of a function $f \in L^1[a,b]$ is defined by Lazo and Torres (2017)

$$I_a^\alpha f(t) = \int_a^t f(\tau) d_a^\alpha \tau = \int_a^t \frac{f(\tau)}{(\tau - a)^{1-\alpha}} d\tau \tag{2a}$$

and the right conformable fractional integral of order $0 < \alpha \leq 1$ terminating at $b \in \mathbb{R}$ of a function $f \in L^1[a,b]$ is defined by

$$ {}_b I^a f(t) = \int_t^b f(\tau) \quad {}_b d^a \tau = \int_t^b \frac{f(\tau)}{(b-\tau)^{1-\alpha}} d\tau \tag{2b}$$

From the definition of CFD, we can observe that at $\alpha = 1$ the CFD reduces to integer derivative, but at $\alpha = 0$ the CFD of a function does not produce the function itself. This is the main drawback of CFD. But CFD is obeying most of the properties of integer derivative and it is having clear physical interpretations. These properties are very useful in mathematical operation and physical applications.

There are some limitations with different types of FDs. For example, Riemann-Liouville FD is not suitable for dynamic systems modelling because the solution of Riemann-Liouville FD problems depend on fractional order initial conditions. However, fractional dynamic systems defined in terms of Caputo FDs accept natural initial conditions. Moreover, these definitions do not satisfy most of the properties of integer derivative. Due to non-local properties of these FDs, the computational effort and storage requirements increase significantly as compared to local fractional derivatives. However, CFD is a local FD and it satisfies most of the properties of integer derivative. Also, it has clear physical interpretation. In this work we consider FOCP where the FD is described as CFD.

## 3. Formulation of FOCP

Fixed final time FOCP formulation is presented in this section. Here the problem is to find the control $u(t)$ which minimizes the following PI

$$J(u) = \frac{1}{2} \int_0^{t_f} \left[q(t)x^2 + r(t)u^2\right] d_0^\alpha t \tag{3}$$

subject to

$$x_0^{(\alpha)}(t) = a(t)x + b(t)u \quad (4)$$

with initial condition as $x(0) = x_0$,

where $x \in \mathbb{R}$ is the state variable, $u \in \mathbb{R}$ is the control variable, $q(t) \geq 0$ and $r(t) > 0$. Here $x_0^{(\alpha)}(t)$ represents conformable fractional derivative of $x(t)$ starting at $t = 0$. In the rest of this paper we denote $x_0^{(\alpha)}(t)$ as $x_0^{(\alpha)}$.

Introducing Lagrange multiplier $\lambda(t)$ we introduce the augmented performance index as

$$J_a(u) = \int_0^{t_f} \left[\frac{1}{2}\left(q(t)x^2 + r(t)u^2\right) + \lambda\left(a(t)x + b(t)u - x_0^{(\alpha)}\right)\right] d_0^\alpha t \quad (5)$$

The Lagrange multiplier method converts constrained optimization problem to an unconstrained optimization problem. It can be noted that $J_a(u) = J(u)$, if Eq. (4) is satisfied for any $\lambda(t)$.

We define Hamiltonian function as

$$H = \frac{1}{2}\left(q(t)x^2 + r(t)u^2\right) + \lambda(a(t)x + b(t)u) \quad (6)$$

Thus, the augmented PI $J_a(u)$ in terms of Hamiltonian function is

$$J_a(u) = \int_0^{t_f} \left[H - \lambda x_0^{(\alpha)}\right] d_0^\alpha t \quad (7)$$

The first variation of the augmented PI $J_a(u)$ is

$$\delta J_a(u) = \int_0^{t_f} \left[\frac{\partial H}{\partial x}\delta x + \frac{\partial H}{\partial u}\delta u + \frac{\partial H}{\partial \lambda}\delta\lambda - \lambda\delta x_0^{(\alpha)} - x_0^{(\alpha)}\delta\lambda\right] d_0^\alpha t \quad (8)$$

Considering the fourth term of above integral and using integration by parts we get

$$\int_0^{t_f} \lambda\delta x_0^{(\alpha)} d_0^\alpha t = \lambda\delta x|_{t_f} - \int_0^{t_f} \lambda_0^{(\alpha)}\delta x d_0^\alpha t \quad (9)$$

Substituting Eq. (9) in Eq. (8), we get

$$\delta J_a(u) = \int_0^{t_f} \left[\left(\frac{\partial H}{\partial x} + \lambda_0^{(\alpha)}\right)\delta x + \frac{\partial H}{\partial u}\delta u + \left(\frac{\partial H}{\partial \lambda} - x_0^{(\alpha)}\right)\delta\lambda\right] d_0^\alpha t - \lambda\delta x|_{t_f} \quad (10)$$

The first variation $\delta J_a = 0$, for optimum. Which requires the coefficients of $\delta x$, $\delta u$ and $\delta\lambda$ in Eq. (10) be zero. Thus, the necessary conditions are

$$\lambda_0^{(\alpha)} = -\frac{\partial H}{\partial x} = -q(t)x - a(t)\lambda \quad (11)$$

$$x_0^{(\alpha)} = \frac{\partial H}{\partial \lambda} = a(t)x + b(t)u \quad (12)$$

$$\frac{\partial H}{\partial u} = 0 \Rightarrow u = -r^{-1}(t)b(t)\lambda \quad (13)$$

Finally, the Eq. (10) becomes

$$\lambda\delta x|_{t_f} = 0 \quad (14)$$

Eqs. (11)–(13) are the necessary conditions for optimality and Eq. (14) represents a general transversality condition for the FOCP in terms of CFD. It should be mentioned here that more general necessary conditions are presented in Lazo and Torres (2017). In contrast, a general transversality condition is obtained in the present work for different end point conditions.

## 4. Analytical solution scheme for conformable FOCPs

In this section, we present an analytical solution scheme for solving conformable FOCPs. Optimal conditions are

$$\lambda_0^{(\alpha)} = -q(t)x - a(t)\lambda \quad (15)$$

$$x_0^{(\alpha)} = a(t)x + b(t)u \quad (16)$$

$$u = -r^{-1}(t)b(t)\lambda \quad (17)$$

Substituting Eq. (17) in Eq. (16), we get

$$x_0^{(\alpha)} = a(t)x - r^{-1}(t)b^2(t)\lambda \quad (18)$$

Eqs. (15) and (18) can be written in vector matrix form as

$$\begin{bmatrix} x_0^{(\alpha)} \\ \lambda_0^{(\alpha)} \end{bmatrix} = \begin{bmatrix} a(t) & -r^{-1}(t)b^2(t) \\ -q(t) & -a(t) \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \quad (19)$$

Now Eq. (19) can be written as

$$z_0^{(\alpha)} = Az, z(0) = z_0 \quad (20)$$

$$z_0^{(\alpha)} = \begin{bmatrix} x_0^{(\alpha)} \\ \lambda_0^{(\alpha)} \end{bmatrix}, A = \begin{bmatrix} a(t) & -r^{-1}(t)b^2(t) \\ -q(t) & -a(t) \end{bmatrix}, z = \begin{bmatrix} x \\ \lambda \end{bmatrix} \text{and } z(0) = \begin{bmatrix} x(0) \\ \lambda(0) \end{bmatrix}$$

Applying the operator $I_0^\alpha$ to the above equation, we obtain

$$z = z_0 + A(I_0^\alpha z) \quad (21)$$

Now we have

$$z_{n+1} = z_0 + A(I_0^\alpha z_n), \; n = 0, 1, 2, \ldots \quad (22)$$

where the "conformable fractional integral" is defined by Abdeljawad (2015)

$$(I_0^\alpha z) = \int_0^t z(\tau)\tau^{\alpha-1} d\tau \quad (23)$$

For $n = 0$,

$$z_1 = z_0 + A(I_0^\alpha z_0) = \left[I + \frac{At^\alpha}{\alpha}\right] z_0 \quad (24)$$

For $n = 1$,

$$z_2 = z_0 + A(I_0^\alpha z_1) = \left[I + \frac{At^\alpha}{\alpha} + A^2\frac{t^{2\alpha}}{\alpha(2\alpha)}\right] z_0 \quad (25)$$

and finally

$$z_n = \left[\sum_{k=0}^{n} \frac{A^k t^{k\alpha}}{\alpha^k k!}\right] z_0 \quad (26)$$

Letting

$$n \to \infty, \; z(t) = \left[\sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\alpha^k k!}\right] z_0 \quad (27)$$

Therefore, the solution of Eq. (20) can be given as Abdeljawad (2015)

$$z(t) = \left[\sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\alpha^k k!}\right] z_0 = e^{\left(\frac{At^\alpha}{\alpha}\right)} z_0 \quad (28)$$

## 5. Illustrative examples

This section presents two numerical examples to show the validity of the formulation and the solution scheme.

**Example 1.** Here, we consider the example of fixed final time-fixed final state FOCP. Minimize the PI

$$J(u) = \frac{1}{2}\int_0^{t_f} \left[x^2 + u^2\right] d_0^\alpha t \quad (29)$$

subject to

$$x_0^{(\alpha)} = -x + u \quad (30)$$

with the initial and final conditions

$$x(0) = 1 \;\; and \;\; x(1) = 0 \tag{31}$$

The necessary conditions for this problem are

$$x_0^{(\alpha)} = -x - \lambda \tag{32}$$

$$\lambda_0^{(\alpha)} = -x + \lambda \tag{33}$$

$$u = -\lambda \tag{34}$$

By solving Eqs. (32)–(34) using the solution technique discussed in Section 4, we obtain the expressions for $x(t)$ and $u(t)$ as

$$x(t) = \frac{1}{2\sqrt{2}}\left[\left(\sqrt{2}x_0 - x_0 - \lambda_0\right)e^{\frac{\sqrt{2}t^\alpha}{\alpha}} + \left(\sqrt{2}x_0 + x_0 + \lambda_0\right)e^{\frac{-\sqrt{2}t^\alpha}{\alpha}}\right] \tag{35}$$

$$u(t) = \frac{1}{2\sqrt{2}}\left[\left(x_0 - \sqrt{2}\lambda_0 - \lambda_0\right)e^{\frac{\sqrt{2}t^\alpha}{\alpha}} - \left(x_0 + \sqrt{2}\lambda_0 - \lambda_0\right)e^{\frac{-\sqrt{2}t^\alpha}{\alpha}}\right] \tag{36}$$

Using Eqs. (35) and (36) in Eq. (29), we obtain minimum PI. Thus

$$J_{min} = \frac{1}{16}\left[M\left(e^{\frac{2\sqrt{2}}{\alpha}} - 1\right) + N\left(1 - e^{\frac{-2\sqrt{2}}{\alpha}}\right)\right] \tag{37}$$

$$M = \frac{\left[\left(\sqrt{2}x_0 - x_0 - \lambda_0\right)^2 + \left(x_0 - \sqrt{2}\lambda_0 - \lambda_0\right)^2\right]}{2\sqrt{2}} \; and$$

$$N = \frac{\left[\left(\sqrt{2}x_0 + x_0 + \lambda_0\right)^2 + \left(x_0 + \sqrt{2}\lambda_0 - \lambda_0\right)^2\right]}{2\sqrt{2}}$$

Solving Eqs. (35)–(37) with the given boundary conditions for different values of $\alpha$ we get $x(t), u(t)$ *and* $J_{min}$.

**Example 2.** Here, we consider the example of fixed final time-free final state FOCP. Find the optimal control $u(t)$ which minimizes the PI in Eq. (29), subject to the system dynamics given by Eq. (30) and the initial condition $x(0) = 1$. Necessary conditions for this problem are given by Eqs. (32)–(34) and the tranversality condition is

$$\lambda(1) = 0 \tag{38}$$

Analytical expressions for $x(t)$ and $u(t)$ are given in Eqs. (35) and (36). By solving Eqs. (35) and (36) using initial and final conditions $x(0) = 1 and \lambda(1) = 0$ for different values of $\alpha$, we can get optimal state $x(t)$ and optimal control $u(t)$.

In this section, for different values of $\alpha$ we solve both the free and fixed final state FOCPs. Figs. 1–4 show the $x(t)$ and $u(t)$ for different $\alpha$. We can observe from these figures that amplitude of $x(t)$ and $u(t)$ increases like in (Agrawal, 2004; Biswas and Sen, 2011a,b) as $\alpha$ is increased. Moreover, at $\alpha = 1$ conformable FOCP results recover integer order OCP results as expected. Table 1 shows the minimum value of PI $J_{min}$ for different values of $\alpha$ for fixed final

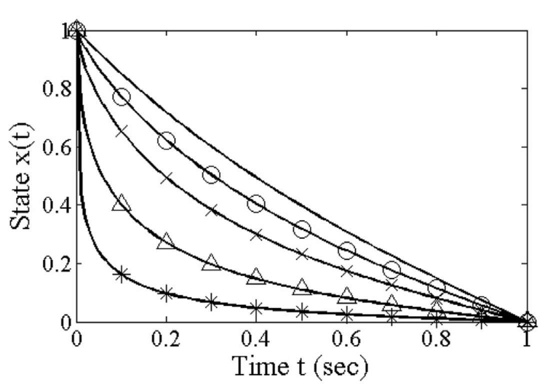


**Fig. 1.** Optimal state $x(t)$ for different $\alpha$ for the fixed final time-fixed final state problem ($* : \alpha = 0.35, \Delta : \alpha = 0.5, \times : \alpha = 0.7, O : \alpha = 0.85, - : \alpha = 1.0$).

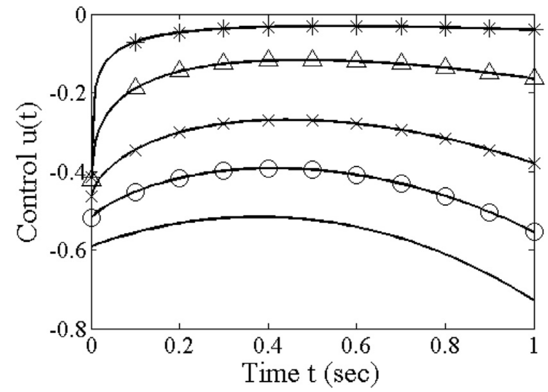


**Fig. 2.** Optimal control $u(t)$ for different $\alpha$ for the fixed final time-fixed final state problem ($* : \alpha = 0.35, \Delta : \alpha = 0.5, \times : \alpha = 0.7, O : \alpha = 0.85, - : \alpha = 1.0$).

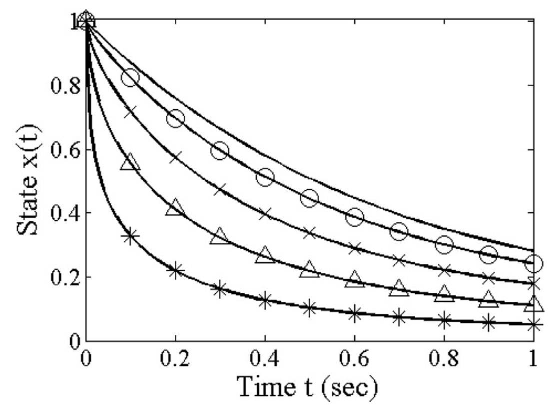


**Fig. 3.** Optimal state $x(t)$ for different $\alpha$ for the fixed final time-free final state problem ($* : \alpha = 0.45, \Delta : \alpha = 0.6, \times : \alpha = 0.75, O : \alpha = 0.9, - : \alpha = 1.0$).

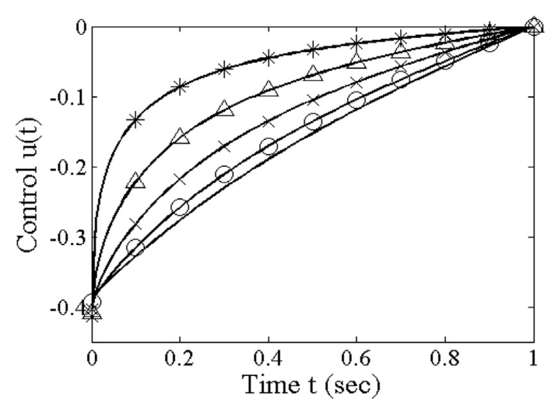


**Fig. 4.** Optimal control $u(t)$ for different $\alpha$ for the fixed final time-free final state problem ($* : \alpha = 0.45, \Delta : \alpha = 0.6, \times : \alpha = 0.75, O : \alpha = 0.9, - : \alpha = 1.0$.)

**Table 1**
Minimum value of PI $J_{min}$ for different $\alpha$ for the fixed final time-fixed final state problem.

| $\alpha$ | 0.5 | 0.7 | 0.85 | 1.0 |
|---|---|---|---|---|
| $J_{min}$ | 0.2117 | 0.2321 | 0.2594 | 0.2956 |

state problem. From this we observe that performance index decreases as $\alpha$ is decreased.

## 6. Conclusions

In this paper, a general formulation of fixed final time FOCP defined in terms of CFD, and an analytical solution scheme is presented. Quadratic PI with conformal fractional integral function is considered in this problem. Both free and fixed final state FOCPs are considered. The necessary optimality conditions and general transversality condition are obtained using Hamiltonian approach and conformable fractional calculus of variations. The necessary conditions are solved using analytical method. It can be noted that this is the first time an analytical solution of FOCP in terms of CFD has been presented in this work. The efficacy of the formulation has been demonstrated by illustrative examples. Results are produced for different values of $\alpha$. It is observed that as $\alpha$ decreases $x(t)$ is also decreased and it demands small control effort. Further, with the decreases in $\alpha$, PI value also decreases.

It should be mentioned here that existence of optimal solution has not considered in this work. Existence of optimal solution and its stability analysis will be considered in our next work.

## Appendix

*Sufficient condition*

In order to determine the nature of optimization we need to consider the second variation and examine its sign. The second variation of the functional $J_a(u)$ is given by

$$\delta^2 J_a(u) = \frac{1}{2}\int_0^{t_f}\left[\frac{\partial^2 H}{\partial x^2}\delta x^2 + 2\frac{\partial^2 H}{\partial x \partial u}\delta x \delta u + \frac{\partial^2 H}{\partial u^2}\delta u^2\right] d_0^\alpha t$$

$$= \frac{1}{2}\int_0^{t_f}[\,\delta x \quad \delta u\,]\begin{bmatrix} \frac{\partial^2 H}{\partial x^2} & \frac{\partial^2 H}{\partial x \partial u} \\ \frac{\partial^2 H}{\partial x \partial u} & \frac{\partial^2 H}{\partial u^2}\end{bmatrix}\begin{bmatrix}\delta x \\ \delta u\end{bmatrix} d_0^\alpha t$$

$$= \frac{1}{2}\int_0^{t_f}[\,\delta x \quad \delta u\,]\Pi\begin{bmatrix}\delta x \\ \delta u\end{bmatrix} d_0^\alpha t$$

For the minimum, the second variation $\delta^2 J_a$ must be positive. This means that the matrix

$\Pi = \begin{bmatrix} \frac{\partial^2 H}{\partial x^2} & \frac{\partial^2 H}{\partial x \partial u} \\ \frac{\partial^2 H}{\partial x \partial u} & \frac{\partial^2 H}{\partial u^2}\end{bmatrix}$ must be positive definite.

In most of the cases this reduces to the condition that $\frac{\partial^2 H}{\partial u^2}$ must be positive definite (negative definite) for minimum (maxium). Now using the Hamiltonian in Eq. (6) and calculating the various partials, we have

$\Pi = \begin{bmatrix} q(t) & 0 \\ 0 & r(t)\end{bmatrix}$. Since $q(t) \geq 0 and r(t) > 0$, it follows that $\Pi$ is only positive semidefinite. However, $\frac{\partial^2 H}{\partial u^2} = r(t)$ is positive definite, thus represents minimum.

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